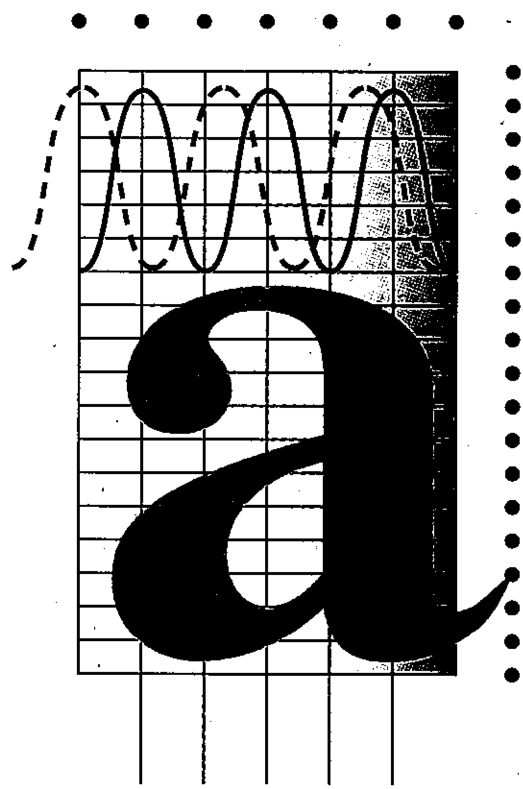
additional MATHEMATICS

6TH EDITION



PURE & APPLIED

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Preface

In this sixth edition of Additional Mathematics: Pure and Applied, the text has been reorganized. It covers all the work in Pure Mathematics and Particle Mechanics in the Cambridge Additional Mathematics syllabus.

The book is divided into three parts.

Part I deals with the basic Pure Mathematics syllabus, covering all the essential topics for Paper 1.

Part II contains materials for the further Pure Mathematics option in Paper 2.

Part III deals with Particle Mechanics which is another option in Paper 2.

There is a set of Revision Papers after each part of the book.

New features of this edition include

- more examples to illustrate the work in greater detail;
- new questions arranged in order of increasing difficulty for extensive practice;
- a revision exercise at the end of each chapter which includes questions from past Cambridge papers. Some more difficult or less direct questions are given in part B of the Revision Exercise to challenge and stimulate the more able students.

As in previous editions, the treatment is straightforward to allow rapid progress in grasping the techniques, with sufficient exercises for practice. The topics have been taken largely in the order of the syllabus for convenience, but this could be altered if desired.

We are grateful to the University of London and the Cambridge Local Examination Board for permission to reproduce questions from their past examination papers.

Contents

PART I PURE MATHEMATICS

PART I

Pure Mathematics

Coordinate Geometry

1

CARTESIAN COORDINATES

The position of a point in a plane can be given by an *ordered pair* of numbers, written as (x,y). These are called the **Cartesian coordinates** of the point. (The name comes from the French mathematician Rene Descartes (1596 – 1650)). The coordinates measure the displacement (+ or –) of the point from two perpendicular **axes,** the y-axis (Oy) and the x-axis (Ox), where O is the **origin.**

For example, in Fig.1.1, the coordinates of point A are (4,3) and the coordinates of point B are (3,4). 4 is the x-coordinate of A and 3 is its y-coordinate. (The x-coordinate is sometimes called the *abscissa* and the y-coordinate the *ordinate*).

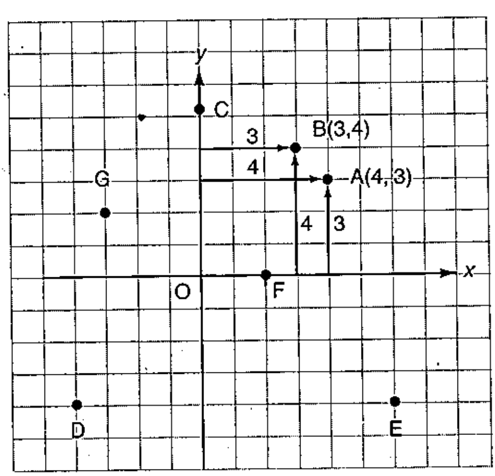


Fig.1.1

The x-coordinate is always stated first. As you can see, (4,3) is not the same point as (3,4). Now state the coordinates of the points C, D, E, F and G in Fig. 1.1.

MIDPOINT OF TWO POINTS

On graph paper, plot the points A(2,3) and B(8,7). Can you write down the coordinates of the midpoint of AB? Can you see how these are related to the coordinates of A and B? (Remember that the midpoint is *halfway* between A and B).

We can find a formula for the midpoint of AB. We could use different letters for coordinates such as (a,b), (c,d), etc. but it is neater to use *suffixes* attached to x and y for specific points. So we write the coordinates of A as (x_1,y_1) and B as (x_2,y_2).

Let the coordinates of the midpoint M be (x_M,y_M) (Fig. 1.2). AC and ME are parallel to the x-axis. MD and BC are parallel to the y-axis.

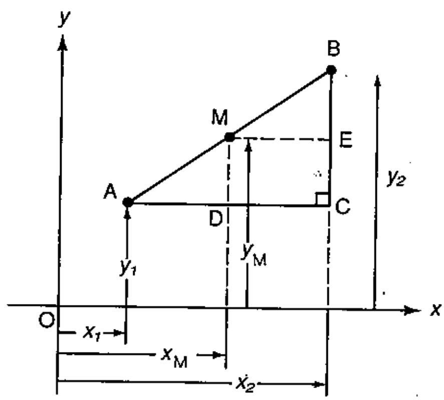


Fig. 1.2

Then AD = DC so $x_M - x_1 = x_2 - x_M$ (i)
and EC = BE so $y_M - y_1 = y_2 - y_M$ (ii)

From (i), $2x_M = x_1 + x_2$, and
from (ii), $2y_M = y_1 + y_2$.

Therefore, $x_M = \frac{x_1 + x_2}{2}$ and $y_M = \frac{y_1 + y_2}{2}$.

Midpoint of (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

The coordinates of the midpoint are the *averages* of the two x-coordinates and of the two y-coordinates of the points.

Example 1

(a) Find the midpoint of (i) (3,4) and (5,2) (ii) (–2,–1) and (4,–4).
(b) If (–2,1) is the midpoint of AB, where A is (–3,2), find the coordinates of B.

(a) (i) The midpoint is $\left(\frac{3+5}{2}, \frac{4+2}{2}\right) = (4,3)$.

(ii) The midpoint is $\left(\frac{-2+4}{2}, \frac{-1-4}{2}\right) = (1,-2\frac{1}{2})$.

(b) If (x_B,y_B) are the coordinates of B, then –2 is the average of –3 and x_B,

so $(-2) = \frac{-3+x_B}{2}$.

Hence $x_B = -1$.
Similarly $y_B = 0$.
Therefore the coordinates of B are (–1,0).

Exercise 1.1 *(Answers on page 606.)*

1 State the coordinates of the midpoints of:

(a) (0,4) and (3,–2)
(b) (–4,–2) and (–2,6)
(c) (4,–2) and (–6,9)
(d) (0,4) and (4,0)
(e) (–4,–1) and (–5,–2)
(f) (5,–3) and (–5,3)
(g) $(p,2p)$ and $(3p,-4p)$
(h) $(a+2b,b-a)$ and $(a-2b,3a+b)$
(i) $(a,a-4)$ and $(a+2,6+a)$
(j) $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$ and $\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$

2 A(1,5) and B(7,–9) are two points. AB is divided into four equal parts at C, D and E. Find the coordinates of C, D and E.

3 A$(3, 1\frac{1}{2})$, B(–5,–3) and C(7,–2) are the vertices of triangle ABC. What are the coordinates of M, the midpoint of BC and of Q, the midpoint of AM?

4 The midpoint of PQ is (2,3). If the coordinates of P are (–1,4), find the coordinates of Q.

5 A is $(a,3)$ and B is $(4,b)$. If the midpoint of AB is (3,5), find the values of a and b.

6 The points A and B are $(a,-4)$ and $(-3,b)$ respectively. If the midpoint of AB is (–2,3), find the values of a and b.

7 L is the point (–3,–2) and M is the point (5,4). N is the midpoint of LM. State the coordinates of N. P is the midpoint of NQ and the coordinates of P are $(2\frac{1}{2},4)$. Find the coordinates of Q.

8 ABCD is a parallelogram. A is the point (2,5), B is the point (8,8) and the diagonals intersect at $(3\frac{1}{2},2\frac{1}{2})$. What are the coordinates of C and D?

9 The coordinates of A and B are (–9,3) and (–3,4) respectively. B is the midpoint of AC and C is the midpoint of AD. Find the coordinates of C and of D.

10 A is the point (–1,4), B is the point (5, –2) and C is the point (–4,–5). If D is the midpoint of AB and E the midpoint of DC, find the coordinates of D and E and show that AE is parallel to the y-axis.

11 The coordinates of A and C are (–6,–3) and (–1,1) respectively.
  (a) If C is the midpoint of AB, find the coordinates of B.
  (b) BF is divided into three equal parts at D and E. If the coordinates of E are (6,–1), find the coordinates of D and F.

12 The points (x_1,y_1), (x_2,y_2), (x_3,y_3) and (x_4,y_4), in that order form a parallelogram ABCD. Show that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

13 ABCD is a quadrilateral where A is (1,7), B is (4,3), C is (–1,–3) and D is (–4,5). Is ABCD a parallelogram? If not, state new coordinates for B so that ABCD will be a parallelogram.

14 The points A(–1,4), B(4,10), C(6,–5) and D(–2,–8) form a quadrilateral ABCD. P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively. Prove that PQRS is a parallelogram.

DISTANCE BETWEEN TWO POINTS

What is the length of AB in Fig.1.3?

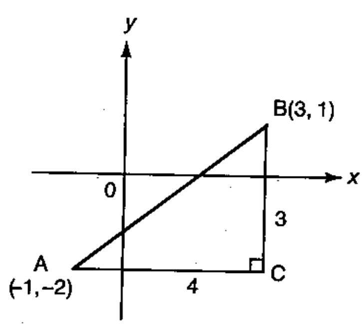


Fig. 1.3

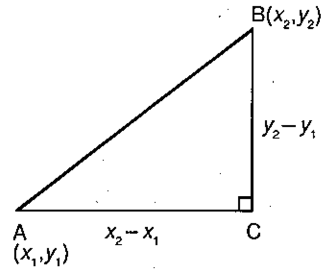
If we draw AC parallel to the x-axis and CB parallel to the y-axis, then AC = 3 – (–1) = 4 units and BC = 1 – (–2) = 3 units.
By Pythagoras' Theorem, $AB^2 = AC^2 + BC^2 = 16 + 9 = 25$ units².
Hence the length of AB = $\sqrt{25} = 5$ units.
We can generalize this to find a formula for the distance between any two given points.

Take $A(x_1,y_1)$ and $B(x_2,y_2)$ to be the two points (Fig.1.4).
Now $AC = x_2 - x_1$ and $BC = y_2 - y_1$.
Then $AB^2 = AC^2 + BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.
So the formula for the distance between the points (x_1,y_1) and (x_2,y_2) is:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: Take care with the subtractions if either or both of the coordinates are negative.

Fig. 1.4



Example 2

Find the distance between
(a) (7,13) and (2,1),
(b) (2,–3) and (–3,4).

(a) Distance = $\sqrt{(7-2)^2 + (13-1)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$ units.

(b) Distance = $\sqrt{[2-(-3)]^2 + [-3-4]^2} = \sqrt{25 + 49} = \sqrt{74} \approx 8.6$ units.

Note that this could also be done as:

distance = $\sqrt{[-3-2]^2 + [4-(-3)]^2} = \sqrt{25 + 49}$ units as before.

The coordinates can be subtracted in *either* order as the results are the same after squaring. Verify this for part (a).

Example 3

The vertices of a triangle ABC are A(–2,5), B(4,4) and C(5,–2).
(a) Which is the longest side?
(b) Is the triangle right-angled?
(c) What type of triangle is ABC?

We need only find the squares of the lengths of the sides.

$AB^2 = [-2-4]^2 + [5-4]^2 = 37$ units^2
$BC^2 = [4-5]^2 + [4-(-2)]^2 = 37$ units^2
$CA^2 = [5-(-2)]^2 + [-2-5]^2 = 98$ units^2

(a) AC is the longest side.
(b) $AC^2 \neq AB^2 + BC^2$ so the triangle is not right-angled.
(c) $AB^2 = BC^2$ so the triangle is isosceles.

Example 4

The vertices of a triangle ABC are A(1,3), B(5,11) and C(9,5). Find the lengths of the medians.

You will recall that a median is a line from a vertex to the midpoint of the opposite side.

The midpoint of BC is (7,8).
Hence the length of the median from A is $\sqrt{6^2 + 5^2} = \sqrt{61} \approx 7.8$ units.
Now find the lengths of the other two medians.
You should find that they are 7 units and $\sqrt{40}$ units.

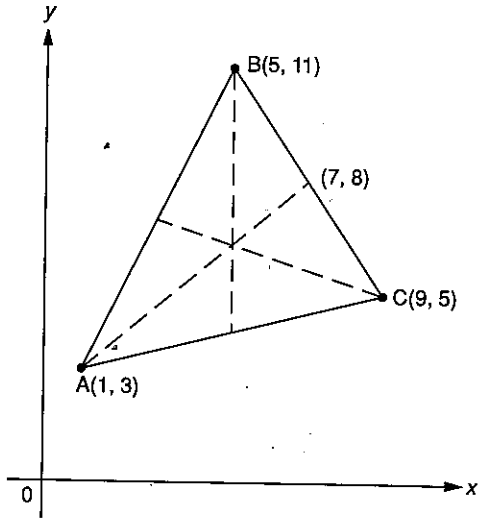


Fig 1.5

Example 5

The vertices of a triangle are A(–2,3), B(3,5) and C(0,–6) (Fig.1.5). D is the midpoint of AB and E is the midpoint of BC. Show that DE = $\frac{1}{2}$ AC.

It is simpler to work with squares of distances, so we find DE^2 and AC^2.
D is $(\frac{1}{2}, 4)$ and E is $(1\frac{1}{2}, -\frac{1}{2})$.

Then $DE^2 = (\frac{1}{2} - 1\frac{1}{2})^2 + (4 + \frac{1}{2})^2 = 1 + \frac{81}{4} = \frac{85}{4}$

$AC^2 = (-2 - 0)^2 + (3 + 6)^2 = 85$

Hence $DE^2 = \frac{1}{4}AC^2$ which means that $DE = \frac{1}{2}AC$.

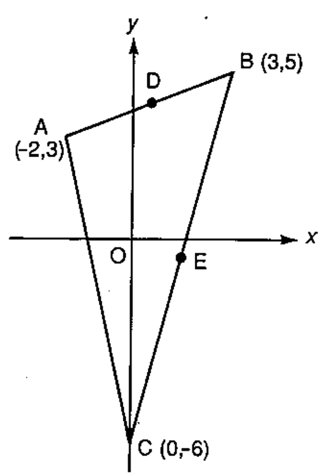


Fig 1.6

Exercise 1.2 *(Answers on page 606.)*

1 Find the distance between the following pairs of points. [Where necessary give your answer correct to 2 significant figures.]

(a) (1,2), (4,6) (b) (–1,–3), (2,1)
(c) (–4,–5), (1,7) (d) (0,–3), (4,0)
(e) (–1,–3), (–2,–5) (f) (–2,1), (4,2)
(g) (–5,0), (–7,–4) (h) (–5,–2), (0,–3)
(i) $(a,0)$, $(0,a)$ (j) $(a,a+b)$, $(a-b,b)$

2 A circle has centre at (1,2). One point on its circumference is (–3,–1). What is the radius of the circle?

3 The vertices of a triangle are A(–4,–2), B(4,2) and C(2,6).
(a) Is the triangle right-angled?
(b) If a circle is drawn round this triangle, what are the coordinates of its centre?
(c) Hence find the radius of this circle.

4 The vertices of triangle ABC are A(–1,3), B(2,7) and C(6,4).
(a) Find the squares of the lengths of the sides.
(b) Hence state completely what type of triangle ABC is.
(c) Find the area of the triangle.

5 The vertices of triangle PQR are P(3,4), Q(5,8) and R(7,4).
(a) What kind of triangle is PQR?
(b) State the coordinates of the midpoint S of side PR.
(c) Find the length of QS and deduce the area of the triangle PQR.

6 The vertices of triangle ABC are A(–4,4), B(2,6) and C(0,–6). Find the lengths of the three medians of the triangle.

7 A(–6,3), B(2,5) and C(0,–5) form a triangle. D is the midpoint of BC.
(a) State the coordinates of D.
(b) Find the values of AC^2, AB^2, AD^2 and DC^2.
(c) Hence show that $AC^2 + AB^2 = 2(AD^2 + DC^2)$.

8 The vertices of triangle ABC are A(2,3), B(4,5) and C(8,–2). P and Q are the midpoints of AB and BC respectively.
(a) State the coordinates of P and Q.
(b) Find the values of PQ^2 and AC^2.
(c) What fraction of AC is PQ?

9 Circle C_1 has centre (–3,4) and radius 2 units. Circle C_2 has centre (1,7) and radius 3 units. Find the distance between the two centres and hence show that the circles touch each other.

10 The centre of a circle is (–1,3) and its radius is 10 units. The centre of a second circle is (2,7) and its radius is 5 units. Show that the two circles touch each other and make a sketch showing the positions of the circles.

11 The vertices of triangle PQR are P(2,5), Q(4,3) and R(–2,–3). If S is the midpoint of PR, show that triangle PSQ is isosceles.

12 A circle has its centre at the origin and its radius is 3 units. P(x,y) is *any* point on the circumference. State an equation in x and y which is true for all possible positions of P.

13 A(–3,2) and B(4,3) are two fixed points. The point P(x,y) moves so that it is always equidistant from A and B (i.e. AP = PB).
(a) Describe the locus of P .
(b) Show that $(x + 3)^2 + (y - 2)^2 = (x - 4)^2 + (y - 3)^2$.
(c) Simplify this equation. (The result is called the equation of the locus of P).

AREAS OF RECTILINEAR FIGURES (Optional)

A rectilinear figure has straight line sides. The following method will be found useful but it is not essential in this Syllabus. It gives a quick way of finding the area of such a figure using the coordinates of the vertices, written in a certain way. We will start with a triangle with one vertex at the origin O (Fig. 1.7). The other vertices are A(x_1,y_1) and B(x_2,y_2). Then the area of $\triangle$OAB = area of $\triangle$OBC + area of trapezium CDAB – area of $\triangle$ODA. Verify that this is

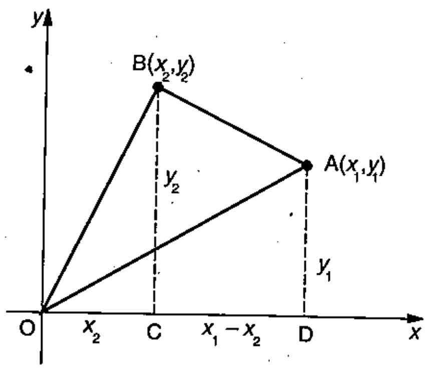


Fig 1.7

$= \frac{1}{2}(x_2y_2 + x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 - x_1y_1)$
$= \frac{1}{2}(x_1y_2 - x_2y_1)$

So, for example, the area of $\triangle$OAB, where A is (6,3) and B is (4,5) will be $\frac{1}{2}(6 \times 5 - 3 \times 4) = 9$. The vertices were taken in the order O – A – B, i.e. anticlockwise. If we take them in the order O – B – A, i.e. clockwise, the result would be –9 (check this). We now extend this to $\triangle$ABC (Fig. 1.8). Then the area of $\triangle$ABC = $\triangle$OAB – $\triangle$OAC – $\triangle$OCB (taking each triangle anticlockwise)

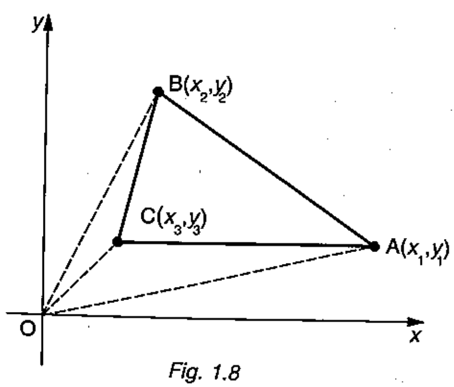
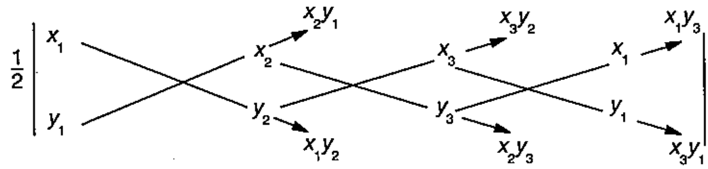


Fig. 1.8

$= \frac{1}{2}(x_1y_2 - x_2y_1) - \frac{1}{2}(x_1y_3 - x_3y_1) - \frac{1}{2}(x_3y_2 - x_2y_3)$
$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$

This result can be easily calculated by arranging the coordinate pairs as columns of a matrix, repeating the first pair at the end:

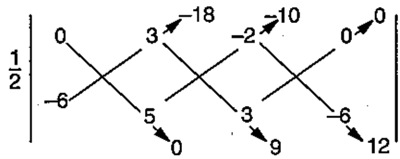


Find the products shown. The area = $\frac{1}{2}$[The sum of the DOWNWARD ↘ products – the sum of the UPWARD ↗ products].

This gives $\frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$.

Check that this is the formula given above.

For example, the area of the triangle shown in Fig. 1.6 will be

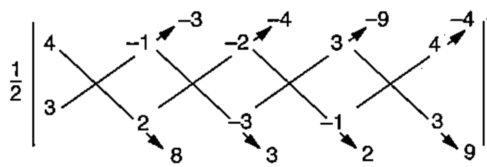


$$\text{Area} = \tfrac{1}{2}[(0 + 9 + 12) - (-18 - 10 + 0)]$$

$$= 24\tfrac{1}{2} \text{ units}^2$$

This method can be extended to give the area of a polygon, provided the vertices are taken in order anticlockwise.

For example, the area of the quadrilateral whose vertices are (4,3), (–2,–3), (–1,2) and (3,–1) is given by



Draw a sketch to make sure the vertices are taken in order.

Write the pairs as before repeating the first one at the end. Then the area = $\frac{1}{2}[(8 + 3 + 2 + 9) - (-3 -4 -9 -4)] = 21$ units^2

Optional Exercise

Find the areas of the figures whose vertices are

(a) (0,0), (3,7), (5,1)
(b) (–1,–2), (–2,3), (4,–4)
(c) (–4,2), (0,–8), (5,11)
(d) (5,3), (2,5), (10,–1), (–6,3)
(e) (–2,–4), (3,1), (–1,5), (6,–3)

GRADIENT OR SLOPE OF A STRAIGHT LINE

The rest of this Chapter deals with the coordinate geometry of straight lines. An important concept is the **gradient** or **slope** of a line. This is a measure of the *steepness* of the line relative to the x-axis. It corresponds to the slope of a path or road which we measure relative to the horizontal. Mathematically, if A and B are *any* two points on a line (Fig. 1.9) then the gradient is the value of the ratio

$$\frac{\text{vertical rise (or fall)}}{\text{horizontal distance}} \text{ i.e. } \frac{y\text{–step}}{x\text{–step}} \text{ in going from A to B.}$$

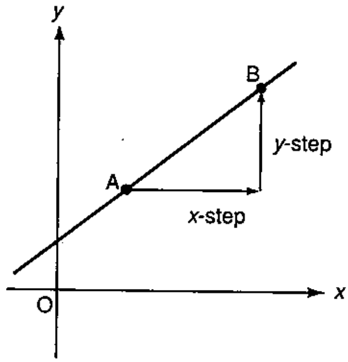


Fig. 1.9

The x-step and the y-step must be taken parallel to the x-axis and the y-axis respectively and either may be positive, negative or zero.
Then, as we shall see, a gradient can be zero, or a positive or negative number. In a special case, it may have no value.

Example 6

Find the gradient of the line through (–2,–3) and (3,5) as shown in Fig. 1.10.

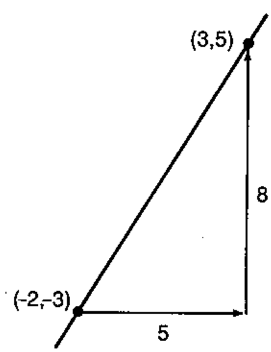


Fig. 1.10

Gradient $= \frac{5-(-3)}{3-(-2)} = \frac{8}{5}$, a **positive** gradient.

If the coordinates are taken in the reverse order, then the gradient is $\frac{-3-5}{-2-3} = \frac{-8}{-5} = \frac{8}{5}$, giving the same value. The gradient is usually left as a fraction.

Example 7

What is the gradient of the line through the points (–2,5) and (4,–2) (Fig. 1.11)?

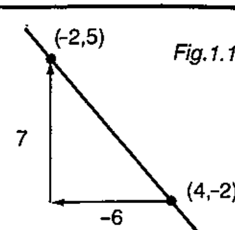


Fig.1.11

Gradient $= \frac{5-(-2)}{-2-4} = \frac{7}{-6} = -\frac{7}{6}$, a **negative** gradient.

Hence, if the coordinates of A and B are (x_1,y_1) and (x_2,y_2) respectively, as in Fig. 1.12, then the gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$, or alternatively $\frac{y_1 - y_2}{x_1 - x_2}$.
(The coordinates must be subtracted in the same order).

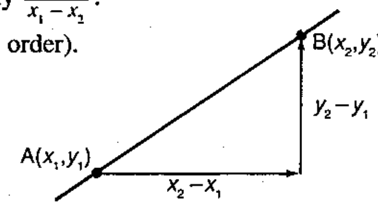


Fig. 1.12

Gradient of line through (x_1,y_1) and $(x_2,y_2) = \frac{y_2 - y_1}{x_2 - x_1}$

As the steepness of a straight line is clearly the same at all points on the line, we can take **any** two points on it to calculate its gradient.

Example 8

State the gradients of the following lines:
(a) through (–2,3) and (5,3),
(b) through (3,–4) and (3,2).

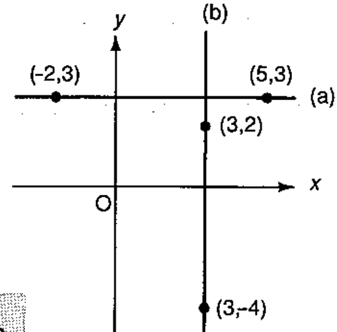


Fig. 1.13

(a) Gradient $= \frac{3-3}{5-(-2)} = \frac{0}{7} = 0$

As we see in Fig. 1.13, the line is parallel to the x-axis.

The gradient of any line parallel to the x-axis is zero.

(b) Gradient $= \frac{2-(-4)}{3-3} = \frac{6}{0}$ which is undefined as division by zero is not possible.

From Fig. 1.11, we see that the line is parallel to the y-axis.

The gradient of any line parallel to the y-axis is undefined.

Angle of Slope

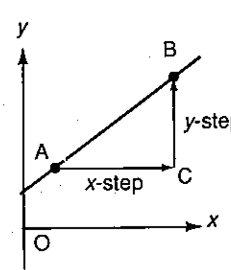


Fig. 1.14

In Fig. 1.14, the slope or gradient of the line AB is
$\frac{y\text{-step}}{x\text{-step}} = \frac{BC}{AC} = \tan \angle BAC$.
But $\angle BAC = \theta$ where θ is the angle between the line and the *positive* x-axis. So the gradient $= \tan \theta$.

θ is called the **angle of slope** and $0° \leq \theta \leq 180°$ (Fig. 1.15).

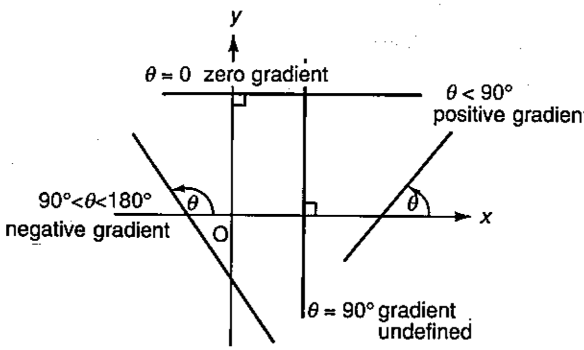


Fig. 1.15

If θ = 0°, tan θ = 0; gradient = 0. The line is parallel to the x-axis.

If 0° < θ < 90°, θ is an acute angle; tan θ is positive and the gradient is positive. The line slopes upwards from left to right. ↗

If θ = 90°, tan θ and the gradient are undefined. The line is parallel to the y-axis.

If 90° < θ < 180°, θ is an obtuse angle; tan θ is negative and the gradient is negative. The line slopes downwards from left to right. ↘

PARALLEL LINES

In Fig. 1.16, the lines AB and CD are parallel. Then the angle of slope of each line is θ. Hence they have the same gradient.

Parallel lines have equal gradients.

Lines with equal gradients are parallel.

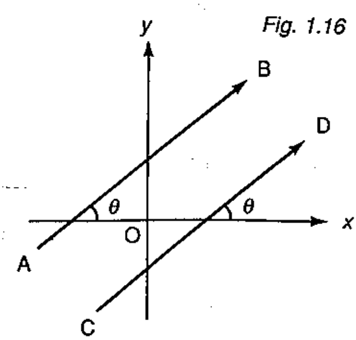


Fig. 1.16

Example 9

A(2,3), B(5,7), C(0,–1) and D(–3,–5) are four points.
(a) Which of the lines AB, BC, CA and DA are parallel?
(b) What type of quadrilateral is ABCD?

(a) The gradients of AB, BC, CA and DA are $\frac{4}{3}$, $\frac{8}{5}$, 2 and $\frac{8}{5}$ respectively. Hence BC is parallel to DA.

(b) As it has 2 parallel sides, ABCD is a trapezium.

Example 10

Two lines are drawn from A(–1,–3), one to B(4,2) and the other to C(–4,2). What are their angles of slope?

Gradient of AB $= \frac{5}{5} = 1 = \tan\theta$, so $\theta = 45°$.

Gradient of AC $= \frac{5}{-3} = \tan\theta$, so $\theta \approx 121°$.

COLLINEAR POINTS

Do the points A(–3,–5), B(0,–1) and C(3,3) lie in a straight line, i.e. are they **collinear**?

If they are, then the gradient of AB must be the same as that of BC or AC, as these will be segments of the same line.

Gradient of AB $= \frac{4}{3}$ and gradient of BC $= \frac{4}{3}$. (Check gradient of AC).

Hence the three points are collinear.

Example 11

If C(p,q) is a point on the line AB, where A is (–2,1) and B is (3,2), find a relationship between p and q.

The three points are collinear.

Hence the gradient of AC = the gradient of AB.

Then $\frac{q-1}{p+2} = \frac{1}{5}$.

Now verify that this gives $5q - p = 7$, which is the relationship required.

Exercise 1.3 *(Answers on page 606.)*

1 State the gradient of the line through the following pairs of points:

(a) (2,3), (1,5) (b) (0,3), (3,0) (c) (2,2), (5,5)
(d) (–3,–9), (1,–1) (e) (1,4), (–3,4) (f) (3,–4), (3,–1)
(g) (–1,–2), (–2,–4) (h) (–4,0), (3,–2) (i) $(a,0)$, $(0,-a)$
(j) (a,b), (b,a) (k) (p,p^2), (q,q^2)

2 A(–4,–2), B(5,–2), C(0,3) and D(1,0) are four points. State the gradients of (a) AB, (b) CD, (c) AC and (d) BD.

3 Which of the lines through the following pairs of points are parallel?

(a) (–1,3), (4,5) (b) (3,–2), (5,1) (c) (–4,–3), (1,–1)
(d) (–7,4), (2,4) (e) (0,–4), (2,–1) (f) $(a,b-1)$, $(a+5,b+1)$

4 Find the angle of slope of the line through the following pairs of points:
(a) $(-2,-1), (3,4)$ (b) $(-2,-1), (2,-5)$ (c) $(1,3), (3,7)$

5 Are the points $(-7,5)$, $(-5,8)$ and $(1,17)$ collinear?

6 A$(-6,-3)$, B$(-2,8)$, C$(0,5)$ and D$(2,2)$ are four points.
(a) Show that B, C and D are collinear.
(b) P, Q and R are the midpoints of AB, AC and AD respectively. Show that P, Q and R are also collinear.

7 If the point (a,b) lies on the line joining $(-2,3)$ and $(2,1)$, find a relationship between a and b.

8 If the points $(-2,-3)$, $(3,5)$ and $(13,p)$ are collinear, find the value of p.

9 The coordinates of a point are given as $(t-1, 2t+1)$. Show that the points where $t = 0$, 1 and 2 are collinear.

10 (a) If the line joining the points $(2,4)$ and $(5,-2)$ is parallel to the line joining $(-1,-2)$ and $(p,6)$ find the value of p.
(b) The line joining $(-1,-4)$ and $(a,0)$ is parallel to the line joining $(a,1)$ to $(11,3)$. Find the value of a.

11 (a) Show that the points $(2,-4)$, $(5,0)$ and $(8,4)$ are collinear.
(b) The point $(d,d-2)$ also lies on this line. Find the value of d.

12 If the points $(-3,-2)$, $(-1, a-2)$ and $(a, 7)$ are collinear, find the two possible values of a.

PERPENDICULAR LINES

The vertices of triangle ABC are A$(-4,-2)$, B$(4, 2)$ and C$(2, 6)$. Verify that this triangle is right-angled. Which two sides are perpendicular? Now state the gradients of these sides. If you multiply the two gradients, what result do you obtain?

The result is surprising so we investigate it further. Given the points A$(-5, -4)$, B$(-2, 3)$ and C$(-16, 9)$ show by using Pythagoras' theorem that AB and BC are perpendicular. Now find the product of their gradients. We can show that this result is true in general excluding undefined or zero gradients.

In Fig. 1.17, AB is a line with gradient m_1 and CD a line with gradient m_2. The lines intersect at right angles at T. The small triangle PQR shows that $m_1 = \frac{-a}{b}$.

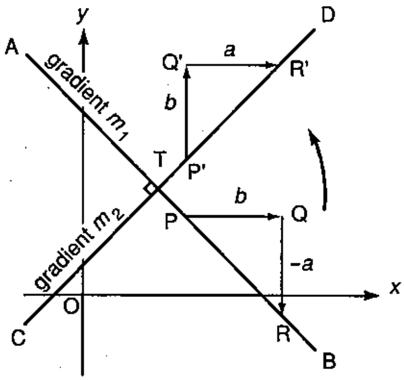


Fig. 1.17

Now imagine that line AB is rotated through 90° about T to lie along CD. Then triangle PQR takes a new position P'Q'R'.

This shows that $m_2 = \frac{b}{a}$, as a and b are now both positive.

Then $m_1 m_2 = -\frac{a}{b} \times \frac{b}{a} = -1$ and this will be true for any pair of perpendicular lines (except for lines parallel to the x- or the y-axis).

If m_1, m_2 are the gradients of two perpendicular lines,

then $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$ $(m_1 \neq 0, m_2 \neq 0)$.

Conversely, if m_1 and m_2 are the gradients of two lines $(m_1 \neq 0, m_2 \neq 0)$ and $m_1 m_2 = -1$, then the lines are perpendicular.

Example 12

The vertices of triangle ABC are A(–2,–4), B(2,–1) and C(5,–5).
(a) Show that the triangle is right-angled.
(b) State the gradient of the altitude through B.

(a) This could be done using the Pythagoras' Theorem but here we use gradients.

The gradients of AB, BC and CA are $\frac{3}{4}$, $-\frac{4}{3}$ and $-\frac{1}{7}$ respectively.

As $\frac{3}{4} \times (-\frac{4}{3}) = -1$, AB is perpendicular to BC. Hence the triangle is right-angled ($\angle B = 90°$).

(b) The altitude through B will be perpendicular to AC. Hence its gradient $= -\frac{1}{-\frac{1}{7}} = 7$.

Exercise 1.4 *(Answers on page 607.)*

1 Which of the lines through these pairs of points are perpendicular?

(a) (–4,–2), (–1,0) (b) (0,–5), (4,–2) (c) (–2,1), (1,5)
(d) (–1,–4), (2,–8) (e) (1,2), (5,–4) (f) (–2,3), (–2,7)

2 State the gradient of a line which is (a) parallel, (b) perpendicular, to AB where
(i) A is (3,–2), B is (0,4) (ii) A is (0,–1), B is (2,1)
(iii) A is (–3,–3), B is (2,4) (iv) A is (–4,1), B is (3,0)

3 Is the triangle formed by the points (–3,2), (0,4) and (4,–2) right-angled?

4 Find the gradient of a line perpendicular to the longest side of the triangle formed by A(–3,4), B(5,2) and C(0,–3).

5 (a) Show that the triangle formed by A(–2,–3), B(2,5) and C(10,1) is right-angled and isosceles.
(b) State the gradients of the three altitudes.

6 Find the angle of slope of a line with gradient $\frac{1}{2}$ and that of another line perpendicular to it.

7 Find the gradient of a line perpendicular to the line joining the points $(a,3a)$ and $(2a,-a)$.

8 CD is the perpendicular bisector of the line joining A(2,3) and B(5,7).
(a) State (i) the coordinates of the point where CD intersects AB and (ii) the gradient of CD.
(b) If the point (p,q) lies on CD, find a relationship between p and q.

9 (a) Show that the point (7,1) lies on the perpendicular bisector of the line joining (2,4) and (4,6).
(b) The point $(a,4)$ also lies on this bisector. Find the value of a.

10 A semicircle with centre O (the origin) and radius 5 units, meets Ox at A and B and the positive y-axis at C.
(a) State the coordinates of A, B and C.
(b) If a point (x,y) lies on the semicircle, show that $x^2 + y^2 = 25$.
(c) Verify that the point P(–3,4) lies on the semicircle and show by using gradients that $\angle APB = 90°$.

11 A(–1,–2), B(b,1) and C(6,–3) are three points and AB is perpendicular to BC.
(a) State, in terms of b, the gradients of AB and BC.
(b) Hence show that $(b + 1)(b - 6) = -12$.
(c) Now find the two possible values of b.

EQUATION OF A STRAIGHT LINE

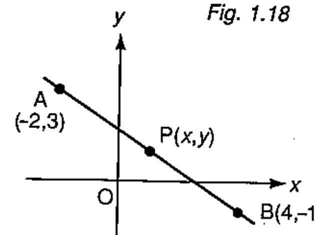


Fig. 1.18

The point P(x,y) lies on the line through A(–2,3) and B(4,–1) (Fig. 1.18). Can we find a relationship between x and y? (Note that we use the coordinates (x,y) as P is *any* point on the line).

Since the three points are collinear, the gradient of AP = gradient of AB.

Then $\frac{y-3}{x+2} = \frac{-4}{6} = -\frac{2}{3}$

i.e. $3(y - 3) = -2(x + 2)$ or $3y + 2x = 5$.

This relationship is called the **equation of the line** through A and B.

If the coordinates (x,y) of a point are substituted in the equation and both sides are equal, **then the point lies on the line.** We say the coordinates **satisfy** the equation. Conversely, if the point lies on the line, its coordinates must satisfy the equation.

For example, the point (3,–5) lies on the line $2x + 3y = -9$ because $2 \times \mathbf{3} + 3 \times (-5) = -9$. The coordinates (3,–5) satisfy the equation.

The point (2,3) does **not** lie on the line because $2 \times \mathbf{2} + 3 \times \mathbf{3} \neq -9$. The coordinates (2,3) do not satisfy the equation.

Such an equation is called a **linear** equation, as it is the equation of a straight line. Its general form is $ax + by = c$ where a, b and c are constants. For example, $2x - 3y = 1$, $y = 3x - 5$ are linear equations. Note that $y = 2$ (no x term) or $2x + 1 = 0$ (no y term) are also linear equations.

We now look at various forms of a linear equation and how to find them. The position of a line can be fixed in two ways.

1 Given one point $A(x_1,y_1)$ on the line and its gradient m.

If $P(x,y)$ is any point on the line (Fig. 1.19), then its gradient is $\dfrac{y - y_1}{x - x_1} = m$.

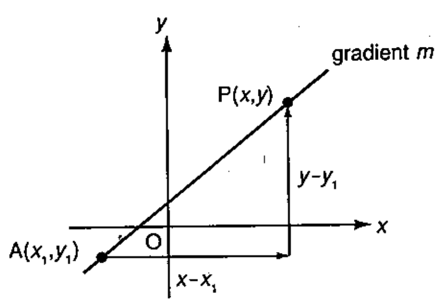


Fig. 1.19

So the equation of the line is

$$\boxed{y - y_1 = m(x - x_1)}$$ one-point, gradient form

Example 13

(a) What is the equation of the line through (3,–1) with gradient $\frac{2}{3}$?
(b) Does the point (2,3) lie on this line?
(c) Find the coordinates of the points where this line cuts the axes.

(a) Using the one-point, gradient form, the equation of the line is $y-(-1)=\frac{2}{3}(x-3)$ i.e. $3(y+1)=2(x-3)$ which simplifies to $3y=2x-9$ or $3y-2x=-9$.

(b) Substituting in the equation, $3\times3-2\times2=5$. But $5\neq-9$ so the coordinates do not satisfy the equation and hence the point (2,3) does not lie on the line.

(c) The y-coordinate of any point on the x-axis is 0. Substitute $y=0$ in the equation of the line.

Then $0=2x-9$ giving $x=4\frac{1}{2}$. The line cuts the x-axis at $(4\frac{1}{2},0)$.

Similarly, to find where the line cuts the y-axis, put $x=0$ in the equation. Verify that this gives the point (0,–3).

II Given two points $A(x_1,y_1)$ and $B(x_2,y_2)$

Let P(x,y) be any point on the line (Fig. 1.20).

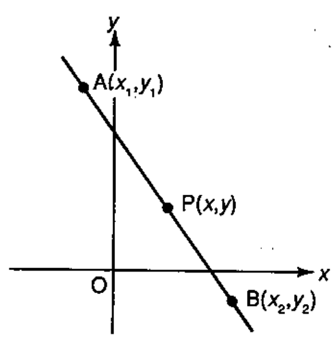


Fig. 1.20

Then by gradients, $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$.

Rewriting this in a more symmetrical form, the equation of the line is

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

two-point form
(note the order of the terms)

Example 14

Find the equation of the line through (2,–3) and (–1,4).

It does not matter which point is taken as (x_1,y_1). Take (2,–3).

Using the two-point form, the equation is $\dfrac{y-(-3)}{4-(-3)} = \dfrac{x-2}{-1-2}$

i.e. $\dfrac{y+3}{7} = \dfrac{x-2}{-3}$.

Now remove the fractions to get $-3(y+3) = 7(x-2)$, which simplifies to $3y + 7x = 5$.

Lines Parallel to the *x*- or *y*-axis

Equations for these lines are special cases.

Example 15

Find the equation of the line through
(a) (–3,2) and (5,2),
(b) (3,–1) and (3,5).

(a) If we use the two-point form, we get $\dfrac{y-2}{2-2} = \dfrac{x+3}{5+3}$ which is not defined. We can see however that the line is parallel to the x-axis (Fig. 1.21). Every point of the line will have coordinates of the form $(x,2)$ so its equation will be $y = 2$ as y is always $= 2$, whatever the value of x.

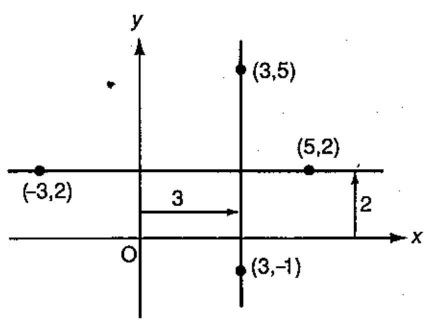


Fig. 1.21

(b) Similarly this line is parallel to the y-axis. Every point will have coordinates of the form $(3,y)$. So the equation is $x = 3$. Hence, if k is a constant, then

> $y = k$ is the equation of a line parallel to the x-axis
>
> $x = k$ is the equation of a line parallel to the y-axis

Exercise 1.5 *(Answers on page 607.)*

1 Find, in its simplest form, the equation of the line
   (a) through (2,3) with gradient 1
   (b) through (–1,–1) with gradient $\frac{3}{4}$
   (c) through (1,3) with gradient $-\frac{1}{5}$
   (d) through (1,0) and (–2,3)
   (e) through (0,1) and (–1,3)
   (f) through (3,–2) and (7,–2)
   (g) through (–2,4) parallel to the y-axis
   (h) through (1,2) and parallel to a line with gradient 2
   (i) through (–3,–1) and perpendicular to a line with gradient $-\frac{1}{3}$
   (j) through (–1,2) and (–1,7)
   (k) through (0,–3) and (0,5)

2 Find the coordinates of the points where each of the lines in Question 1 cut the axes.

3 A line cuts the x-axis at (3,0) and the y-axis at (0,–2). Find the equation of the line.

4 P(0,9) and Q(6,0) are two points. A line is drawn from the origin perpendicular to PQ. Find the equation of this line.

5 Find the equations of the lines through (–1,–4) which are (a) parallel and (b) perpendicular to another line with gradient $-\frac{2}{3}$.

6 The gradient of a line is 2 and it cuts the y-axis at (0,3). Find its equation and the coordinates of the point where it cuts the x-axis.

7 Find the equations of the sides of triangle ABC where A is (–2,3), B is (0,5) and C is (3,–1).

8 The points A(4,4), B(–2,0) and C(6,–2) form a triangle.
   (a) Find the equations of the medians of this triangle.
   (b) If AD is an altitude of the triangle, find the equation of AD.

9 From the point (2,5), a perpendicular is drawn to the line joining (–1,–4) and (5,2). Find the equation of this perpendicular.

10 ABCD is a parallelogram where A is (2,–1), B is (6,2) and C is (11,–2).
   (a) State the coordinates of the midpoint of AC.
   (b) Hence find the coordinates of D.
   (c) Find the equations of the diagonals of the parallelogram.

11 A(–1,2) and C(3,4) are opposite vertices of a rhombus ABCD. Find
   (a) the coordinates of the point where the diagonals intersect,
   (b) the gradient of AC,
   (c) the equation of the diagonal BD.

GRADIENT-INTERCEPT FORM

Suppose the equation of a line is $2x - 3y = 5$. How can we find its gradient?

To do this we convert the equation to a special form – the **gradient-intercept** form. Fig. 1.22 shows a line with gradient m which cuts the y-axis at C(0,c). c is called the y-intercept of the line. Let P(x,y) be any point on this line. Then the gradient of the line $= \frac{y-c}{x} = m$ so $y - c = mx$.

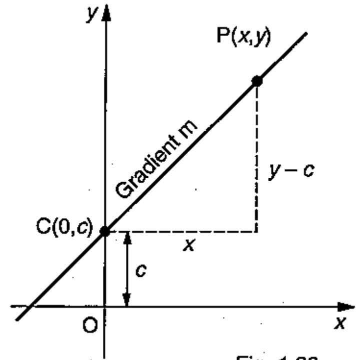


Fig. 1.22

i.e. $\underset{\text{gradient}}{\underset{\uparrow}{y = mx}} + \underset{\text{y-intercept}}{\underset{\uparrow}{c}}$ gradient-intercept form

Hence, if an equation is written in this form, the gradient is given by the **coefficient** of x and the y-intercept by the **constant term.**

To verify this, suppose the equation of line is $y = 2x - 3$ (gradient-intercept form). This line cuts the y-axis where $x = 0$, so $y = -3$ (the constant term). The points (2,1) and (5,7) lie on the line (check this). The gradient is $\frac{6}{3} = 2$ which is the coefficient of x.

Example 16

Find the gradients of the lines (a) $2x - 3y = 5$, (b) $2y + x = -4$.

(a) Convert to the gradient-intercept form, $y = mx + c$:

$$-3y = -2x + 5$$

Then $y = \underset{\text{gradient}}{\underset{\uparrow}{\tfrac{2}{3}x}} - \underset{\text{y-intercept}}{\underset{\uparrow}{\tfrac{5}{3}}}$ (dividing by -3)

So the gradient is $\frac{2}{3}$ (and the y-intercept is $-\frac{5}{3}$).

(b) $2y + x = -4$ i.e. $2y = -x - 4$ so $y = -\frac{1}{2}x - 2$

The gradient is $-\frac{1}{2}$.

It is useful to practise this conversion, i.e. making y the subject of the equation. The gradient is then obtained quickly.

Equations of Parallel and Perpendicular Lines

Example 17

Find the equations of the lines through the point (1,2) which are (a) parallel, (b) perpendicular, to the line $2x - 3y = 4$.

(a) The gradient of the line $2x - 3y = 4$ is $\frac{2}{3}$. So the gradient of any parallel line is also $\frac{2}{3}$. Hence its equation will be $y = \frac{2}{3}x + c$.

To find c, we substitute the coordinates (1,2) in the equation as (1,2) lies on the line. Then $2 = \frac{2}{3} + c$ giving $c = \frac{4}{3}$.

The equation is $y = \frac{2}{3}x + \frac{4}{3}$ i.e. $3y = 2x + 4$.

(b) The gradient of any perpendicular line will be $-\frac{3}{2}$ so its equation is $y = -\frac{3}{2}x + c$. Substitute (1,2) to find c and verify that the required equation is $2y = -3x + 7$.

Exercise 1.6 *(Answers on page 607.)*

1 State the gradients of the following lines:

(a) $x + y = 2$	(b) $x - y = -1$	(c) $y - 2x = 3$
(d) $2x + y = 1$	(e) $3x + 2y = 6$	(f) $5x - 2y = 5$
(g) $y = 4$	(h) $x - 2y = 0$	(i) $2x + 3y = 1$
(j) $2x - 3y = 4$	(k) $4x = 3y - 2$	(l) $5x - 2y = 10$
(m) $tx - y = t$	(n) $py + x = 2p$	(o) $ax + by = 1$

2 Find the equation of the line which is
- (a) parallel to $x - y = 1$ and passes through (2,3)
- (b) parallel to $2x + y = 3$ and passes through (0,1)
- (c) perpendicular to $2x + y = 0$ and passes through (–1,–2)
- (d) perpendicular to $3x + y = 5$ and passes through (–2,–1)
- (e) parallel to $y = 4$ and passes through (0,1)
- (f) perpendicular to $x - 3y = 1$ and passes through (–3,0)
- (g) perpendicular to $x = 2$ and passes through (–2,3)

3 Find the equations of the lines parallel and perpendicular to
- (a) $x + y = 3$ passing through (–1,2)
- (b) $2x - y = 4$ passing through (0,3)
- (c) $4x + 3y = 1$ passing through (0,–2)
- (d) $x - 3y = 1$ passing through (–1,–1)

4 A line is drawn through the point (–1,2) parallel to the line $y + 5x = 2$. Find its equation and that of the perpendicular line through the same point.

5 The side BC of a triangle ABC lies on the line $2x - 3y = 4$. A is the point (2,3). Find the equation of the altitude through A.

INTERSECTION OF LINES

At what point do the lines $2x - 3y = -7$ and $3x + 8y = 2$ intersect? This point lies on **both** these lines so its coordinates must satisfy **both** equations. Hence its coordinates will be the solution of the simultaneous equations

$$2x - 3y = -7 \qquad \text{(i)}$$

and $$3x + 8y = 2 \qquad \text{(ii)}$$

These can be solved by any of the methods you have learnt previously. We use the elimination method here.

Multiply (i) by 3: $6x - 9y = -21$

Multiply (ii) by 2: $6x + 16y = 4$

Subtract: $-25y = -25$ so $y = 1$

Substitute in (i): $2x - 3 = -7$ so $x = -2$

The point is $(-2,1)$.

Suppose the lines were $2x - 3y = -7$ and $4x - 6y = 3$. What happens in the solution? Explain this.

Example 18

From the point $P(-1,3)$, a perpendicular PQ is drawn to the line joining $A(-4,-8)$ and $B(4,4)$. Find

(a) the equations of AB and the perpendicular,

(b) the coordinates of the point where they intersect,

(c) the distance of P from the line AB.

A sketch diagram should always be drawn to help in such questions (Fig. 1.23).

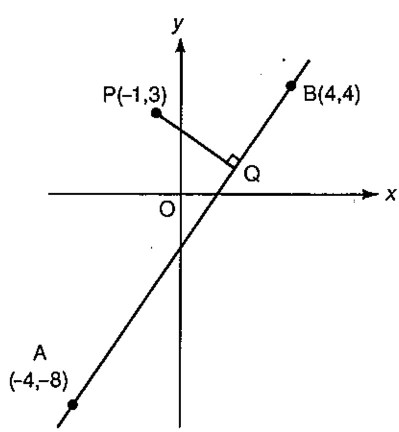


Fig. 1.23

(a) The equation of AB is $\frac{y+8}{12} = \frac{x+4}{8}$ i.e. $3x - 2y = 4$.

Now check that the equation of PQ is $2x + 3y = 7$.

(b) Solving the equations $3x - 2y = 4$ and $2x + 3y = 7$, we get (2,1) as the coordinates of Q.

(c) The distance of P from AB is PQ.

$PQ^2 = (-1-2)^2 + (3-1)^2 = (-3)^2 + 2^2 = 13$ so $PQ = \sqrt{13}$

Example 19

ABCD is a rectangle where A is (–3,2), D is (2,5) and B lies on the y-axis. Find

(a) the equation of AD,

(b) the equation of AB,

(c) the coordinates of B,

(d) the coordinates of C.

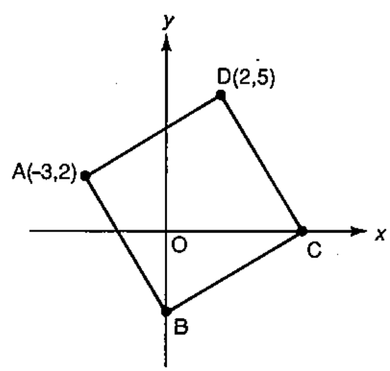


Fig. 1.24

Fig. 1.24 shows the facts given

(a) Equation of AD is $\frac{y-2}{3} = \frac{x+3}{5}$ i.e. $3x - 5y = -19$.

(b) AB is perpendicular to AD. The gradient of AD is $\frac{3}{5}$ so the gradient of AB is $-\frac{5}{3}$. Knowing the gradient and the point A, verify that the equation of AB is $3y = -5x - 9$.

(c) AB meets the y-axis where $x = 0$. Hence $y = -3$. The coordinates of B are (0,–3).

(d) Let the diagonals meet at M. M is the midpoint of BD, so M is (1,1). As M is also the midpoint of AC, therefore C is (5,0).

Example 20

The line $2x + 3y = 6$ meets the y-axis at A and the x-axis at B. C is the point such that AB = BC. CD is drawn perpendicular to AC to meet the line through A parallel to $5x + y = 7$ at D.

(a) Find the coordinates of A, B and C.

(b) State the equations of CD and AD, and hence find the coordinates of D.

(c) Calculate the area of the triangle ACD.

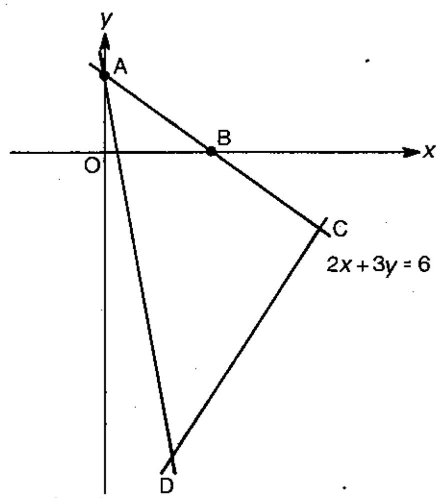


Fig. 1.25

(a) The line meets the y-axis where $x = 0$, $y = 2$, so A is (0,2).
It meets the x-axis where $y = 0$, $x = 3$, so B is (3,0).
Since B is the midpoint of AC, then C must be (6,–2).

(b) CD is perpendicular to AC. Therefore its gradient is $\frac{3}{2}$ and it passes through C.
Verify that the equation of CD is $3x - 2y = 22$.

AD is parallel to $5x + y = 7$. Therefore its gradient is –5 and it passes through A.
Verify that the equation of AD is $5x + y = 2$.

Solving these two equations gives $x = 2$ and $y = -8$. So the coordinates of D are (2,–8).

(c) As ACD is a right-angled triangle,

$$\text{its area} = \tfrac{1}{2} \times \text{AC} \times \text{DC} = \tfrac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26 \text{ units}^2.$$

Exercise 1.7 *(Answers on page 607.)*

1 The line $4x - 3y = 12$ meets the axes at A and B. Find the length of AB.

2 Find the equation of the line through the point of intersection of $2x + 3y = 5$ and $3x - y = 2$, and which is parallel to $4y - x = 14$.

3 Through A(2,3) two lines are drawn with gradients –1 and 2. These lines meet the line $x - 2y = 5$ at B and C. Find
(a) the equations of AB and AC,
(b) the coordinates of B and C.

4 The lines $x + 3y = 1$ and $2x - 5y = -9$ intersect at A. Find the equation of the line through A and the point (–1,–2).

5 A line through A(5,2) meets the line $3x + 2y = 6$ at right angles at B. Find the coordinates of B and calculate the length of AB.

6 (a) Find the equation of the perpendicular bisector of the line joining A(–3,3) and B(1,–5).
(b) If this bisector meets the x-axis at C, find the coordinates of C.

7 The sides of a triangle lie on the lines $y = -1$, $2x + y = 1$ and $4x - 3y = -13$. Find the coordinates of the vertices and show that the triangle is isosceles.

8 The intersections of the lines $5x + 6y = 36$, $x - 2y = 4$ and $7x + 2y = 12$ are the vertices of a triangle.
(a) Find the coordinates of these vertices.
(b) Obtain the equation of the altitude drawn to the longest side.

9 OABC is a parallelogram where O is the origin and B is the point (5,7). C lies on the line $x - 2y = 0$ and A lies on the line $2x - y = 0$. Calculate the coordinates of A and C.

10 The sides of a triangle lie on the lines $y = 1$, $x + y = 6$ and $3x - y = 2$.
(a) Calculate the coordinates of the vertices of the triangle.
(b) Find the equations of the three altitudes.
(c) Show that these altitudes intersect at a point and find the coordinates of this point.

11 A(3,1) and B(0,6) are two points. BC is perpendicular to AB and meets the x-axis at C. Find
(a) the equation of BC,
(b) the coordinates of C,
(c) the area of triangle ABC.

12 The diagonals of a rhombus meet at the point (–1,5) and one of them is parallel to the line $2x - 5y = 3$.
(a) Find the equations of the diagonals.
(b) If two of the vertices of the rhombus are (–3,10) and (9,9), find the coordinates of the other two.

13 A is the point (–1,6). Lines are drawn through A with gradients 3 and –2, meeting the x-axis at B and C respectively. BD is perpendicular to AB and CD is perpendicular to AC.
(a) Find the coordinates of B and C.
(b) State the equations of BD and CD.
(c) Find the coordinates of D.
(d) Calculate the ratio BD:CD.

14 A(1,2) and C(5,4) are two vertices of the rectangle ABCD. AB and CD are parallel to the line $y - x = 5$.
(a) Find the equations of AB and BC.
(b) Find the coordinates of B and D.
(c) Hence find the area of the rectangle.

15 ABCD is a rectangle where A is (1,3) and D is (5,5). AC lies on the line $3y = 4x + 5$. Find
(a) the equation of DC,
(b) the coordinates of C,
(c) the coordinates of B,
(d) the area of ABCD.

16 The point B(a,b) is the reflection of A(5,–2) in the line $2x - 3y = 3$.
(a) Find the equation of AB and show that $3a + 2b = 11$.
(b) State the coordinates of the midpoint of AB in terms of a and b and show that $2a - 3b = -10$.
(c) Hence find the values of a and b.

SUMMARY

- Midpoint of (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Distance between (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- Gradient of line through (x_1,y_1) and (x_2,y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$.
- Parallel lines have equal gradients.
- Three points A, B and C are collinear if the gradient of AB equals the gradient of BC.
- If m_1 and m_2 are the gradients of perpendicular lines, $m_1m_2 = -1$. If m_1 and m_2 ($m_1 \neq 0$, $m_2 \neq 0$) are the gradients of two lines and $m_1m_2 = -1$, then the lines are perpendicular.
- Equation of line through (x_1,y_1) with gradient m is $y - y_1 = m(x - x_1)$.
- Equation of line through (x_1,y_1) and (x_2,y_2) is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.
- The form $y = mx + c$ gives the gradient (m) and the y-intercept (c).

REVISION EXERCISE 1 *(Answers on page 607.)*

A

1 Find the equation of the line
(a) through (–2,3) with gradient $-\frac{1}{2}$,
(b) through the points (–3,2) and (–1,–5),
(c) through (–1,–1) perpendicular to the line $3x - 2y = 1$.

2 A and B are the points (–2,–1) and (4,1) respectively. BC is perpendicular to AB.
(a) Find the equation of BC.
(b) If the gradient of AC is 1, find the equation of AC and the coordinates of C.
(c) Hence find the area of triangle ABC.

3 A(–1,1) and B(3,4) are two vertices of triangle ABC. If the area of the triangle is 15 units^2, find the distance of C from AB.

4 The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line are such that AB = BC. The line through B perpendicular to AC passes through the point D(–1,6). Find
(a) the equation of BD,
(b) the coordinates of B,
(c) the coordinates of C. (C)

5 (a) The line $\frac{x}{4} - \frac{y}{3} = 1$ meets the axes at A and B. Find the coordinates of the midpoint of AB and the length of AB.
(b) A circle is drawn with its centre at the origin. If the point P(4,3) lies on this circle, find the equation of the tangent to the circle at P.

6 Fig. 1.26 shows a triangle ABC with A(1,1) and B(–1,4). The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.
(a) Find the value of m.
(b) Find the coordinates of C.
(c) Show that AC = 2AB. (C)

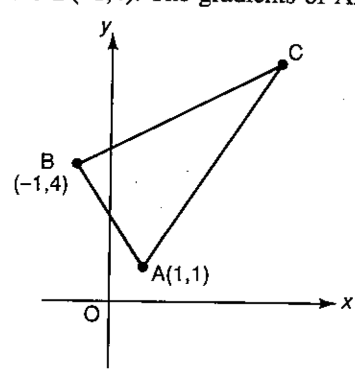


Fig. 1.26

7 A(–3,4) and C(4,–10) are opposite vertices of the parallelogram ABCD. The gradients of the sides AB and BC are $-\frac{1}{2}$ and 3 respectively. Find
(a) the equations of AB and BC,
(b) the coordinates of B and D.

8 Three points have coordinates A(1,–3), B(5,5) and C(5,9). Find the equation of the perpendicular bisector of (a) AB, (b) BC. Hence find the coordinates of the point equidistant from A, B and C. (C)

9 (a) Find the equation of the perpendicular bisector of AB, given that A is (2,7) and B is (6,–1).
(b) The bisector meets the y-axis at C. Find the coordinates of C and the area of triangle ABC.

10 A(0,6), B(1,3) and C(4,6) are three points. D is the foot of the perpendicular from A to BC. Find
(a) the coordinates of D,
(b) the length of AD.

11

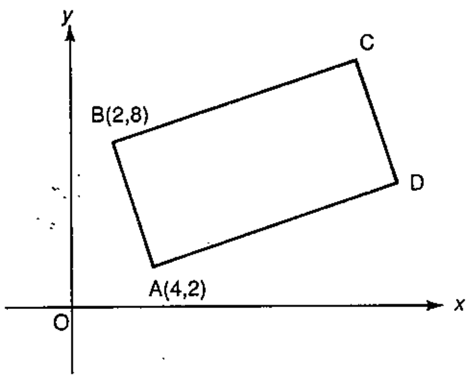


Fig. 1.27

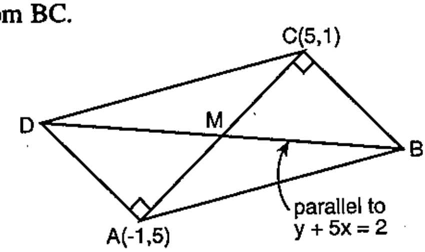
In Fig. 1.27, ABCD is a rectangle, and A and B are the points (4,2) and (2,8) respectively. Given that the equation of AC is $y = x - 2$, find
(a) the equation of BC,
(b) the coordinates of C,
(c) the coordinates of D,
(d) the area of the rectangle ABCD. (C)

12 Two points have coordinates A(1,3) and C(7,7). Find the equation of the perpendicular bisector of AC.

B is the point on the y-axis equidistant from A and C and ABCD is a rhombus. Find the coordinates of B and D.

Show the area of the rhombus is 52 units^2 and hence calculate the perpendicular distance of A from BC. (C)

13



ABCD is a parallelogram, lettered anticlockwise, such that A and C are the points (–1,5) and (5,1) respectively. Find the coordinates of the midpoint of AC.

Given that BD is parallel to the line whose equation is $y + 5x = 2$, find the equation of BD.

Given that BC is perpendicular to AC, find the equation of BC. Calculate (i) the coordinates of B, (ii) the coordinates of D, (iii) the area of ABCD. (C)

14 A(–2,2) and C(4,–1) are opposite vertices of a parallelogram ABCD whose sides are parallel to the lines $x = 0$ and $3y = x$.
(a) Find the coordinates of B and D.
(b) If P and Q are the feet of the perpendiculars from D and B respectively to AC, find the coordinates of P and Q and show that PQ = $\frac{1}{3}$ AC.

15 Fig. 1.28 shows a quadrilateral ABCD in which A is (2,8) and B is (8,6). The point C lies on the perpendicular bisector of AB and the point D lies on the y-axis. The equation of BC is $3y = 4x - 14$ and angle DAB = 90°. Find
(a) the equation of AD,
(b) the coordinates of D,
(c) the equation of the perpendicular bisector of AB,
(d) the coordinates of C.
Show that the area of triangle ADC is 10 units^2 and find the area of the quadrilateral ABCD.
(C)

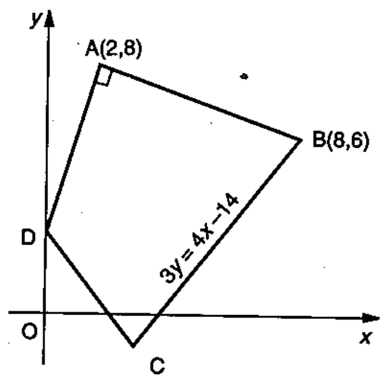


Fig. 1.28

16 The line $x + y = 3$ meets the y-axis at A and the x-axis at B. AC is perpendicular to AB and the equation of BC is $y = 3x - 9$.
(a) Find the equation of AC and the coordinates of C. AD is parallel to CB where D lies on the x-axis.
(b) Find the coordinates of D.
(c) Hence find the area of the trapezium ACBD.

17 Fig. 1.29 shows the quadrilateral OABC. The coordinates of A are $(k,2k)$ where $k > 0$, and the length of OA is $\sqrt{80}$ units.
(a) Calculate the value of k.
AB is perpendicular to OA and B lies on the y-axis.
(b) Find the equation of AB and the coordinates of B.
The point C lies on the line through O parallel to $y + 3x = 5$ and also on the perpendicular bisector of AB.
(c) Calculate the coordinates of C.
Calculate the area of the quadrilateral OABC.
(C)

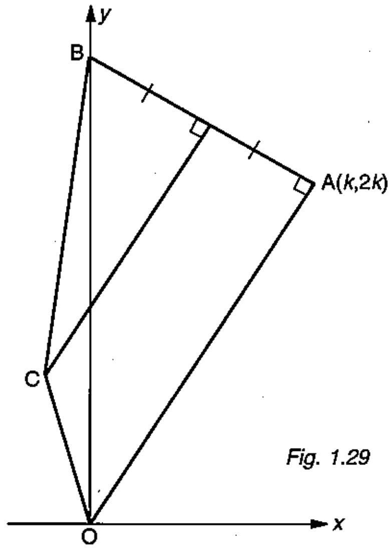


Fig. 1.29

18 The vertices of a triangle are $(-3,5)$, $(4,-2)$ and $(6,2)$.
  (a) Find the equations of the perpendicular bisectors of the sides.
  (b) Show that they meet at the same point and find the coordinates of this point.
  (c) Find the radius of the circle passing through the vertices.

19 A and B are the points $(2,4)$ and $(4,0)$ respectively.
  (a) Find the equation of the perpendicular bisector of AB.
  (b) The bisector meets the line through B parallel to the y-axis at C. Find the coordinates of C.
  (c) Calculate the radius of the circle which passes through A and touches the x-axis at B.

20 The sides AB, BC and CA lie on the lines $2y = x - 4$, $x + y = 5$, and $y = mx$ respectively. If the origin O is the midpoint of AC, find the value of m.

B

21 A(h,k) lies on the line $y + 3x = -10$. B lies on the line $x + y = 4$. If the origin is the midpoint of AB, find the value of h and of k.

22 A$(1,5)$ lies on the line $y = 2x + 3$. P lies on the perpendicular to that line through A.
  (a) Show that the coordinates of P can be written as $(11 - 2a, a)$.
  (b) If OP $= \sqrt{34}$, where O is the origin, find the possible values of a.

23 A line with gradient m passes through the point P$(3,2)$ and meets the y-axis at A. A line perpendicular to the first also passes through P and meets the x-axis at B.
  (a) Express the coordinates of A and B in terms of m.
  (b) If AB $= \sqrt{65}$, find the possible values of m.

24 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x– and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and R is the point whose x-coordinate is $2t$, where t is positive. Express the y-coordinate of R in terms of t and evaluate t given that the area of triangle PQR is 21 units^2. (C)

25 A line through (3,1) has gradient m ($> \frac{1}{3}$). It meets the x-axis at A and the y-axis at B. From A and B, perpendiculars to the line are drawn to meet the y-axis at C and the x-axis at D respectively. Show that the gradient of CD is $\frac{1}{m^3}$.

26 $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4,y_4)$ are the vertices of a parallelogram ABCD.

(a) Show that $x_1 + x_3 = x_2 + x_4$ and $y_1 + y_3 = y_2 + y_4$.

(b) If ABCD is a rhombus show that $(x_1 - x_3)(x_2 - x_4) + (y_1 - y_3)(y_2 - y_4) = 0$.

(c) If however ABCD is a rectangle show that $x_1x_3 + y_1y_3 = x_2x_4 + y_2y_4$.

Simultaneous Equations

2

Two linear equations, say $3x + 4y = -5$ and $2x - 3y = 8$, can be solved to find values of x and y which satisfy **both** equations simultaneously. As we have seen, this solution gives the coordinates of the point of intersection of the two lines represented by the equations.

In this Chapter we consider two simultaneous equations where one of them is not a linear equation but is an equation of the second degree such as $xy = 8$ or $x^2 + y^2 = 10$, etc. These are the equations of *curves*.

Example 1

Solve the following equations:

$$x + y = 9 \qquad (i)$$

$$xy = 8 \qquad (ii)$$

Equation (i) represents a straight line but equation (ii) is the equation of a **hyperbola,** a curve with two branches (Fig.2.1).

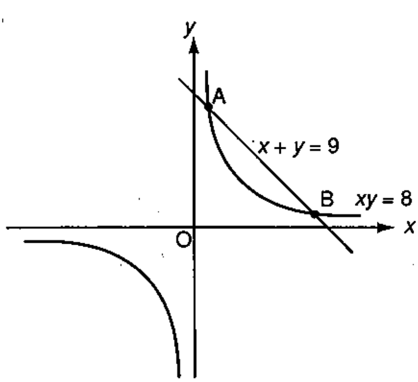


Fig.2.1

The line meets the curve at two different points (A and B) so we expect to obtain **two** solutions, giving the coordinates of A and B.

The usual method is to eliminate one of the variables. Make one variable the subject of the linear equation and then substitute this in the other (non-linear) equation. This will lead to a quadratic equation, which can usually be solved by factorization.

From (i), $x = 9 - y$.

Then substituting for x in (ii),

$$(9 - y)y = 8$$

i.e. $y^2 - 9y + 8 = 0$ or $(y - 8)(y - 1) = 0$.

Hence $y = 8$ or 1.

Now find the corresponding values of x from (i).

When $y = 8, x = 1$;
when $y = 1, x = 8$.

So the solutions are $x = 1, y = 8$ (coordinates of A)
or $x = 8, y = 1$ (coordinates of B).

Example 2

Find the coordinates of the points where the line

$$2x + 3y = -1 \qquad (i)$$

meets the curve $$x(x - y) = 2 \qquad (ii)$$

We use the same method but the algebra will be more complicated as neither x nor y in (i) has a coefficient of 1.

Choosing y as the subject, we obtain from (i)

$$y = \frac{-1 - 2x}{3}.$$

Then substituting for y in (ii),

$$x\left(x - \frac{-1 - 2x}{3}\right) = 2 \text{ or } x\left(\frac{3x + 1 + 2x}{3}\right) = 2$$

which simplifies to $x(5x + 1) = 6$ or $5x^2 + x - 6 = 0$.

Hence $(5x + 6)(x - 1) = 0$ giving $x = -\frac{6}{5}$ or 1.

From (i), when $x = -\frac{6}{5}$, $-\frac{12}{5} + 3y = -1$ so $y = \frac{7}{15}$,

and when $x = 1$, $2 + 3y = -1$ so $y = -1$.

Hence the coordinates of the two points are $(-\frac{6}{5}, \frac{7}{15})$ and $(1,-1)$.

Example 3

Solve the equations

$$x + 2y = 7 \qquad (i)$$

and $$x^2 - 4x + y^2 = 1 \qquad (ii)$$

What is the geometrical meaning of your answer?

From (i), choosing x as the subject for simplicity, $x = 7 - 2y$.

Then substituting for x in (ii),

$$(7 - 2y)^2 - 4(7 - 2y) + y^2 = 1$$

i.e. $49 - 28y + 4y^2 - 28 + 8y + y^2 = 1$

which reduces to $y^2 - 4y + 4 = 0$

or $(y - 2)(y - 2) = 0$ giving $y = 2$ or 2.

Then $x = 3$ or 3.

We obtain two *equal* solutions. This means that the line is a tangent to the curve. It *touches* the curve, which is a circle, at the point (3,2) as shown in Fig.2.2.

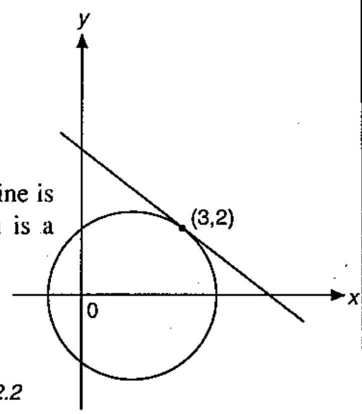


Fig.2.2

Example 4

A straight line through (0,–1) meets the curve $x^2 + y^2 - 4x - 2y + 4 = 0$ at the point (3,1). Find the coordinates of the second point where this line meets the curve.

First we find the equation of the straight line:

$$\frac{y+1}{1+1} = \frac{x-0}{3-0}$$

which gives $2x = 3y + 3$ or $x = \frac{3y+3}{2}$.

Then substitute for x in the equation of the curve:

$$\left(\frac{3y+3}{2}\right)^2 + y^2 - 4\left(\frac{3y+3}{2}\right) - 2y + 4 = 0$$

i.e. $\dfrac{9y^2 + 18y + 9}{4} + y^2 - 6y - 6 - 2y + 4 = 0.$

Clearing the fraction,

$$9y^2 + 18y + 9 + 4y^2 - 24y - 24 - 8y + 16 = 0$$

and so $13y^2 - 14y + 1 = 0$ or $(13y - 1)(y - 1) = 0$.

Hence $y = \frac{1}{13}$ or $y = 1$.

The corresponding values of x are then $\frac{21}{13}$ or 3.

So the second point is $\left(\frac{21}{13}, \frac{1}{13}\right)$.

Example 5

If the line $3x - 5y = 8$ meets the curve $\frac{3}{x} - \frac{1}{y} = 4$ at A and B, find the coordinates of the midpoint of AB.

First remove the fractions from the equation of the curve:

$$3y - x = 4xy \qquad \text{(i)}$$

From the linear equation, $x = \frac{8 + 5y}{3}$.

Substituting for x in (i),

$$3y - \frac{8 + 5y}{3} = 4y\left(\frac{8 + 5y}{3}\right)$$

Clearing the fraction, we have $9y - 8 - 5y = 4y(8 + 5y) = 32y + 20y^2$.
or $20y^2 + 28y + 8 = 0$,
i.e. $5y^2 + 7y + 2 = 0$ or $(5y + 2)(y + 1) = 0$

Hence $y = -\frac{2}{5}$ or -1.
Then $x = 2$ or 1.

The coordinates of A and B are $(2, -\frac{2}{5})$ and $(1, -1)$ and the coordinates of the midpoint are therefore $(1\frac{1}{2}, -\frac{7}{10})$.

Example 6

If the sum of two numbers is 4 and the sum of their squares minus three times their product is 76, find the numbers.

Suppose the numbers are x and y.
The sum of the numbers is $x + y$.

Then $\quad x + y = 4 \qquad$ (i)

The (sum of the squares) – (3 × the product) is $x^2 + y^2 - 3xy$.

Then $\quad x^2 + y^2 - 3xy = 76 \qquad$ (ii)

We solve these equations.

From (i), $x = 4 - y$
Substituting in(ii):

$(4 - y)^2 + y^2 - 3y(4 - y) = 76$ which is then expanded.
$16 - 8y + y^2 + y^2 - 12y + 3y^2 = 76$
or $\quad 5y^2 - 20y - 60 = 0$

Hence $y^2 - 4y - 12 = 0$ which gives $(y - 6)(y + 2) = 0$ and $y = 6$ or -2.
Then from (i), the corresponding values of x are -2, and 6.
Therefore the two numbers are 6 and –2.

Arithmetically, there is only one solution. Geometrically, the line $x + y = 4$ meets the curve given by equation (ii) in two points $(6,-2)$ and $(-2,6)$.

Exercise 2.1 *(Answers on page 608.)*

1 Solve the following pairs of simultaneous equations:

(a) $x + y = 5$, $xy = x + 3$

(b) $x - y = 2$, $x(y + 2) = 9$

(c) $2x + y = 5$, $x^2 + y^2 = 10$

(d) $x - 2y = 2$, $x^2 + xy = 20$

(e) $2x + 3y = 5$, $y(y - x) = 5$

(f) $3x - 2y = 7$, $x^2 + y^2 = 10$

(g) $3x - y = 7$, $x^2 + xy - y^2 = 1$

(h) $x + 3y = 1$, $x^2 - xy + y^2 = 21$

(i) $3x + 4y = 2$, $x^2 - 3y^2 = 1$

(j) $3x + 2y = 13$, $3x^2 + y^2 = 31$

(k) $\frac{x}{3} - \frac{y}{2} = 1$, $\frac{3}{x} + \frac{2}{y} = \frac{3}{2}$

(l) $\frac{x}{4} - \frac{y}{3} = 1$, $\frac{16}{x} + \frac{3}{y} = 3$

(m) $3x - 2y = 11$, $(x - 1)(y + 3) = 4$

2 The line $y = x + 2$ meets the curve $y^2 = 4(2x + 1)$ at A and B. Find the coordinates of the midpoint of AB.

3 Show that the line $x + y = 6$ is a tangent to the curve $x^2 + y^2 = 18$ and find the coordinates of the point of contact.

4 A line through (2,1) meets the curve $x^2 - 2x - y = 3$ at A(–2,5) and at B. Find the coordinates of B.

5 What is the relationship of the line $3x - 2y = 4$ to the curve $y = x - \frac{2}{x}$?

6 The perimeter of a rectangle is 22 cm and its area is 28 cm^2. Find its length and breadth.

7 The line through (1,6) perpendicular to the line $x + y = 5$ meets the curve $y = 2x + \frac{4}{x}$ again at P. Find the coordinates of P.

8 A(3,1) lies on the curve $(x - 1)(y + 1) = 4$. A line through A perpendicular to $x + 2y = 7$ meets the curve again at B. Find the coordinates of B.

9 The difference between two numbers is 2 and the difference of their squares is 28. Find the numbers.

10 Fencing is used to make 3 sides of a rectangle: two pieces each of length a m and one piece of length b m. The total length of fencing used is 30 m and the area enclosed is 100 m^2. What are the values of a and b?

11 The line $x - y = 7$ meets the curve $x^2 + y^2 - x = 21$ at A and B. Find the coordinates of the midpoint of AB.

12 The line through (–3,8) parallel to $y = 2x - 3$ meets the curve $(x + 3)(y - 2) = 8$ at A and B. Find the coordinates of the midpoint of AB.

SUMMARY

- To solve simultaneous equations, one linear, the other of the second degree:
  (a) make one of the variables the subject of the linear equation,
  (b) substitute in the second degree equation,
  (c) simplify and then solve the quadratic equation obtained,
  (d) find the corresponding values of the second variable.

  If two equal solutions are obtained, the line is a tangent to the curve given by the second degree equation.

REVISION EXERCISE 2 *(Answers on page 608.)*

A

1 Solve the simultaneous equations $4x - 3y = 11$ and $16x^2 - 3y^2 = 61$.

2 The line $y - 2x - 8 = 0$ meets the curve $y^2 + 8x = 0$ at A and B. Find the coordinates of the midpoint of AB. (C)

3 A straight line through the point $(0,-3)$ intersects the curve $x^2 + y^2 - 27x + 41 = 0$ at $(2,3)$. Calculate the coordinates of the point at which the line again meets the curve. (C)

4 Calculate the coordinates of the points of intersection of the straight line $2x + 3y = 10$ and the curve $\frac{2}{x} + \frac{3}{y} = 5$. (C)

5 Solve the simultaneous equations $2x + 3y = 6$ and $(2x + 1)^2 + 6(y - 2)^2 = 49$. (C)

6 The perimeter of the shape shown in Fig.2.3 is 90 cm and the area enclosed is 300 cm^2. All corners are right-angled. Find the values of x and y.

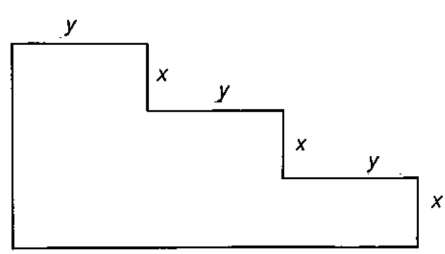


Fig.2.3

7 The point $A(0,p)$ lies on the curve $y = (x - 2)^2$. A line through A perpendicular to $y = x + 3$ meets the curve again at B. Find
(a) the value of p,
(b) the coordinates of B.

8 Two quantities u and v are connected by the equation $u + 2v = 7$. A third quantity P, is given by $P = u(v - 3)$. Find the values of u and v when $P = -3$.

9 The hypotenuse of a right-angled triangle is $(2y - 1)$ cm long. The other two sides are x cm and $(y + 5)$ cm in length. If the perimeter of the triangle is 30 cm, find the possible values of x and y.

10 Solve the simultaneous equations $2x + 4y = 9$ and $4x^2 + 16y^2 = 20x + 4y - 19$. (C)

B

11 Solve the simultaneous equations $3x - 2y = 11$ and $x^2 + xy + y^2 = 7$.

12 A(3,4) and B(7,8) are two points. P(a,b) is equidistant from A and B such that AP = $\sqrt{26}$.

(a) Show that $a + b = 11$.

(b) Find the values of a and b.

13 In Fig.2.4, ABE is an isosceles triangle and BCDE is a rectangle. The total length round ABCDEA is 22 cm and the area enclosed is 30 cm^2.

(a) State the distance of A from BE in terms of x.

(b) Find the possible values of x and y.

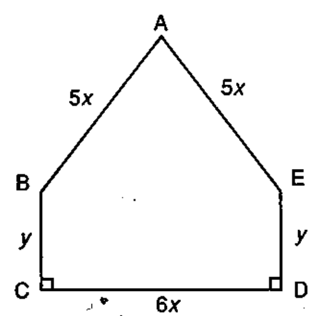


Fig.2.4

14 Solve the simultaneous equations $x + y = 6$ and $\frac{1}{x-1} = \frac{3}{y} + \frac{1}{4}$.

15 The point P(a,b) lies on the line through A(–1,–2) and B(3,0) and PA = $\sqrt{125}$. Find the values of a and b.

16 A circle has centre (4,2) and radius $\sqrt{5}$ units. P(x,y) is any point on the circumference.

(a) Show that $x^2 + y^2 - 8x - 4y + 15 = 0$.

(b) Find the coordinates of the ends of the diameter which, when extended, passes through the origin.

(c) Find the coordinates of the ends of the perpendicular diameter.

Functions

3

RELATIONS AND FUNCTIONS

A **relation** links the members of two sets together. Relations can be of many kinds, e.g. "is the father of", "is a divisor of", "is the same age as", "is the square of" etc. Fig. 3.1 illustrates the relation "is the father of" linking the set of men {A, B, C, D} and the set of children {p, q, r, s, t, u, v}. An arrow identifies the relation between a father and child. The diagram shows that A has two children (p and q), B has 1 child (t), C has 3 children (r, s and u) and D 1 child (v). So 2 arrows leave from A, 1 from B, 3 from C and 1 from D.

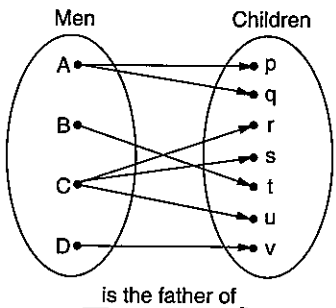


Fig. 3.1

In our work the relation will usually be some mathematical operation. Fig. 3.2 shows the relation "$y = 1 + x^2$" where the starting values (the **inputs**) are chosen values of x. These are linked to the values of y produced by the relation (the **outputs**), i.e. the set {1, 2, 5, 26}.

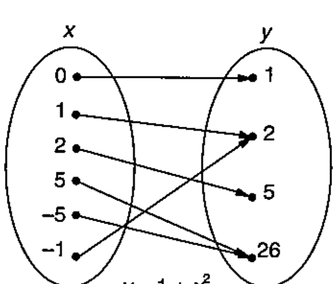


Fig. 3.2

Note that only ONE arrow leaves each input, unlike the relation in Fig. 3.1. In Fig. 3.2 each input produces a **unique** output. This is a special type of relation called a **function**, one of the most important concepts in Mathematics. The relation in Fig. 3.1 is NOT a function.

In Fig. 3.3, each member of set A is *squared* to produce the set of outputs B. As each input

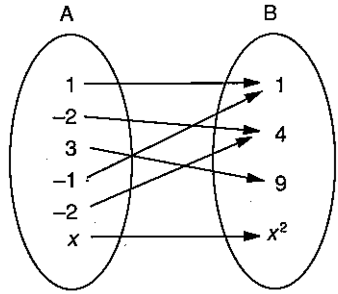


Fig. 3.3

has a unique square, Fig. 3.3 illustrates a function f. f is "square the input". So, if x is the input, the output is x^2.
A function is also called a **mapping** and we say that x is **mapped onto** x^2 *by the function* f. We symbolize this as

$$f: x \longmapsto x^2$$

Read this as 'f is the function which maps x onto x^2'.
f operates on the input x to produce x^2 so we write $f(x) = x^2$. Hence the image of 2 is $f(2) = 2^2 = 4$. The image of –3 is $f(-3) = (-3)^2 = 9$. The image of a is $f(a) = a^2$ and so on. What is the image of 5? What is f(6), $f(-x)$ and $f(2x)$? If $f(x) = 49$, what is the value of x?
Now look at the relation illustrated in Fig. 3.4(a).

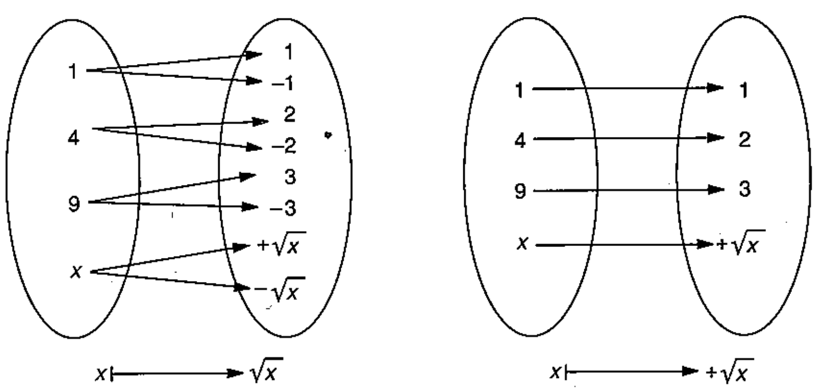


Fig. 3.4 (a) (b)

Is this a function? As you can see, each input has *two* outputs (2 arrows from each input). So this operation (taking the square root, $x \longmapsto \sqrt{x}$) is NOT a function. It does not produce a unique image as x has 2 square roots $+\sqrt{x}$ and $-\sqrt{x}$.

However, if we defined $\sqrt{\ }$ to mean the **positive** root only, then $f(x) = +\sqrt{x}$ would be a function (Fig. 3.4(b)).

Summarizing,

- a function f is a process or operation which takes an input x and maps it onto a **unique** output $f(x)$, the image of x;
- $f: x \longmapsto f(x)$;
- to define f, we write, for example, $f(x) = x^2$ or $f(x) = +\sqrt{x}$ or $f(x) = \sin x$ etc.

f and x are the usual letters for the function and the input respectively, but other letters can be used e.g. $F(x)$, $g(x)$ or $A(r)$, etc.

A function need not be defined algebraically. It may be stated in words, such as the function 'Y is the father of x', or given in the form of a table such as a table of sines.

Example 1

A function f *is given by* $f : x \longmapsto x^2 - x + 1$. *Find*
(a) $f(2)$, *(b)* $f(-3)$, *(c) the image of* -2, *(d)* $f(r)$, *(e)* $f(\frac{x}{2})$.

$f(x) = x^2 - x + 1$

(a) $f(2) = 2^2 - 2 + 1 = 3$

(b) $f(-3) = (-3)^2 - (-3) + 1 = 13$

(c) The image of -2 is $f(-2) = (-2)^2 - (-2) + 1 = 7$.

(d) $f(r) = r^2 - r + 1$

(e) $f(\frac{x}{2}) = (\frac{x}{2})^2 - (\frac{x}{2}) + 1 = \frac{x^2 - 2x + 4}{4}$

Example 2

The function h *is given by* $h(x) = \frac{x+1}{x-1}, x \neq 1$. *Find*
(a) $h(2)$, *(b)* $h(\frac{1}{2})$, *(c)* $h(x + 1)$

(a) $h(2) = \frac{2+1}{2-1} = 3$

(b) $h(\frac{1}{2}) = \frac{\frac{1}{2}+1}{\frac{1}{2}-1} = -3$

(c) $h(x + 1) = \frac{x+1+1}{x+1-1} = \frac{x+2}{x}, x \neq 0$

Note: A function may not produce an image for certain values of the input. In this example, $x \neq 1$. If $x = 1$, $h(1) = \frac{1+1}{1-1}$ which is impossible as division by zero is undefined. Hence 1 has no image under this function.

Example 3

$F(x) = x^2 + x - 1$. If $F(x) = 5$, find the values of x.

$F(x) = 5$ is the equation $x^2 + x - 1 = 5$ i.e. $x^2 + x - 6 = 0$ which we can solve for the values of x.

$x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$

Hence $x = -3$ or $x = 2$.
These are the two values of x which have an image of 5.
Check by finding $F(-3)$ and $F(2)$.

DOMAIN AND RANGE

There are special names for the sets of inputs and outputs. The set of inputs is called the **domain** and the set of outputs the **range**.

Fig. 3.5 shows the domain and range for the function $f(x) = (1 - x)^2$. The domain is the set $\{-1, 0, 2, 4\}$ and the range is the set $\{1, 4, 9\}$.

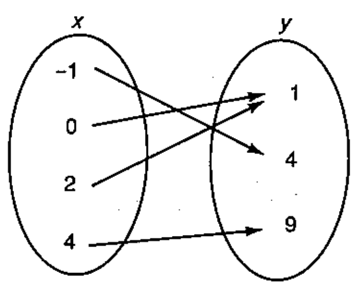


Fig. 3.5

The domain can be any set of numbers which have images. It could be just a few selected numbers or all positive numbers or all real numbers, etc. If it is not specified it is taken to be all real numbers. However, as we saw in *Example 2*, some numbers may have no image and these must be stated. They are excluded from the domain.

Example 4

State the domain for the function $f(x) = \frac{1}{x}$.

Every real number will have an image under this function **except** $x = 0$. So the domain will be {all real values of x, $x \neq 0$}.

This is often briefly stated as $f(x) = \frac{1}{x}$, $x \neq 0$.

Example 5

State the domain for $f(x) = \sqrt{x}$ *(positive root).*

Every positive number and 0 will have a square root but negative numbers will not. These must be excluded. So the domain is {all positive numbers and 0} or just $x \geq 0$.

Example 6

What values of x must be excluded from the domain of the function

$$f(x) = \frac{x+1}{x^2+x-2}\ ?$$

This function will always produce an image except when $x^2 + x - 2 = 0$ or $(x + 2)(x - 1) = 0$ i.e. when $x = -2$ or $x = 1$.

These values must be excluded from the domain.
Hence the domain is {all real values of x, $x \neq -2$ or 1}

Exercise 3.1 *(Answers on page 608.)*

1 For each of the following functions, find the images of –3, –1, 0, 1, 2, 4:

(a) $f(x) = x^2 - x - 5$ (b) $g(x) = (x + 1)^2$

(c) $h(x) = \frac{x-1}{x+1}$ (d) $F(x) = (x + 1)(x - 2)$

2 What value of x must be excluded from the domain of the function in Question 1 part (c)?

3 State the values of x which must be excluded from the domains of the following functions:

(a) $f(x) = \frac{2}{x-2}$ (b) $g(x) = \frac{x-2}{2x-3}$

(c) $h(x) = \frac{5}{x^2-x-2}$ (d) $F(x) = 3 - \frac{2}{x+3}$

4 f is the function 'square x and add 2'.

(a) Write f in the form $f(x) = ...$

(b) Find f(1), f(–1), f(0).

(c) If $f(x) = 27$, find the values of x.

5 F is the function 'add 2 to x and then square'.

(a) Write F in the form $F(x) = ...$

(b) Find F(1), F(–1), F(0).

(c) If $F(x) = 25$, find the values of x.

(d) Is this the same function as f in Question 4?

6 If $f(x) = 3x + 2$, what is the value of x which is mapped onto 8?

7 A function such as $f(x) = 5$ is a *constant* function. State the values of f(0), f(–1) and f(5).

8 The function E, where $E(x) = 2^x$, is an exponential function.
(a) Find the values of E(1), E(2) and E(5).
(b) If $E(x) = 16$, state the value of x.

9 $f(x) = \frac{x+3}{x-1}$
(a) What value of x must be excluded from the domain of this function?
(b) Find the positive value of x for which $f(x) = x$.

10 If $f(x) = \frac{x+3}{x+2}$, find the values of x for which $f(x) = 2x$ to 2 decimal places.

11 Given that $g(x) = x^2 - 4x - 6$ solve the equation $g(x) = x$.

12 Given that $f(x) = x^2 - 4x + 1$ solve the equations
(a) $f(x) = x - 3$, (b) $f(2x) = 13$.

13 For the linear function $f(x) = ax + b$, where a and b are constants, $f(-2) = 7$ and $f(2) = -1$. Find the values of a and b.

14 $f(x) = ax^2 + bx + c$, where a, b and c are constants. If $f(0) = 7$, what is the value of c? Given also that $f(1) = 6$ and $f(-1) = 12$, find the value of a and of b.

15 For the function $f(x) = px^2 + qx + r$, where p, q and r are constants, $f(0) = 4$, $f(-1) = 8$ and $f(-2) = 18$. Find the values of p, q and r.

16 $F(x) = x^2 - 2x$. What values of x have an image of 15?

17 The function h is given by $h(t) = 7t - 2t^2$. Find the values of t whose image is 5.

18 $f(x) = \frac{x-1}{2x^2 - x - 3}$
(a) Find f(2) and $f(\frac{1}{2})$.
(b) Find x if $f(x) = 0$.
(c) What values of x must be excluded from the domain?

19 The number of diagonals in a polygon with n sides is given by the function $D(n) = \frac{n(n-3)}{2}$.
(a) State the domain of this function.
(b) Find the number of diagonals in polygons with 4, 5 and 10 sides.
(c) If $D(n) = 20$, find the value of n.

20 The domain for the function $f(x) = 2x^2 + 1$ is $\{-2, -1, 0, 1, 2\}$.
Find the range.

21 The domain of the function $f(x) = \frac{2x+1}{x-1}$ is $\{0, 2, 4\}$.
Find the range of the function.

22 If the range for the function $g(x) = x^2 - 2$ is $\{-2, -1, 7\}$, find the domain.

23 The range of the function $f(x) = 1 - \frac{2}{x}$ is $\{-1, 2, 4\}$. Find the domain.

24 S is the function $S: x \longmapsto \sin x°$, $0 \le x \le 180$.
(a) Find (correct to 2 decimal places) S(30), S(50), S(120).
(b) If $S(x°) = 1$, what is the value of x?
(c) State the range of S.

25 Functions f and g are given as $f(x) = x^2 - x$ and $g(x) = 2x - 3$.
(a) Find f(0), f(−1), g(0) and g(−1).
(b) If $f(x) + g(x) = 3$, find x.
(c) If $f(p) + g(-p) = 1$, find p.
(d) If $f(z) = g(z) + 1$, find z.

26 Given that $f(x) = x^2 - 3x + 6$ and that $g(x) = x + 6$, solve the equations
(a) $f(x) = 2g(x)$, (b) $f(x) = g(2x)$, (c) $f(2x) = g(x) - 3$.

27 If $f(x) = \frac{x+1}{x^2 - x + 1}$, find the value of k (other than $k = 1$) such that $f(k) = f(1)$.

28 Given the function $f(x) = x^2 - 3x - 2$, express $f(2a) - f(a)$ in its simplest form in terms of a.

29 $f: x \longmapsto x^2 - x + 3$
Find $f(p)$, $f(-2p)$ and $f(p - 1)$ in their simplest forms.

30 If $f(x) = 3x + 1$, find $f(a)$, $f(b)$ and $f(a + b)$.
Is $f(a + b) = f(a) + f(b)$?

31 If $f(x) = x^2 + x - 3$, find $f(x + h)$ where h is a constant.
Hence express $\frac{f(x+h) - f(x)}{h}$ in its simplest form.

32 If $f(-x) = f(x)$, f is called an *even* function, but if $f(-x) = -f(x)$, f is called an *odd* function. Which of the following functions are even, which are odd and which are neither?
(a) $2x$ (b) $3x^2$ (c) x^3
(d) $1 - x$ (e) $\frac{1}{x}$ $(x \neq 0)$ (f) $x - \frac{1}{x}$ $(x \neq 0)$

GRAPHICAL REPRESENTATION OF FUNCTIONS

A simple way of illustrating a function graphically is to use two parallel number lines, one for values of the domain, the other for the range. Fig. 3.6 shows the function $f(x) = x - 2$, $x = -1, 0, 1, 2, 3, 4$. An arrowed line joins x in the domain to $f(x)$ in the range.

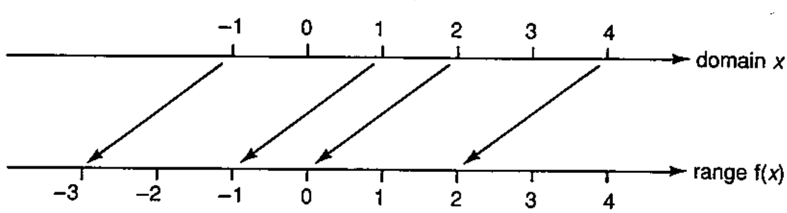


Fig. 3.6

Example 7

Illustrate the function f(x) = $x^2 - x + 2$ *on two number lines for the domain* $\{-2, -1, 0, 1, 2, 3\}$.

Verify that the range is {2, 4, 8}. Fig. 3.7 shows the result.

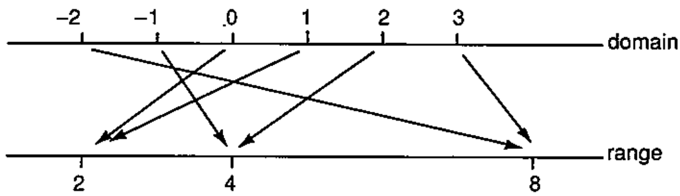


Fig. 3.7

This method is only suitable if the domain consists of a few values. If the domain was all real numbers for example, it would be impossible to show all the arrowed lines. Furthermore, the pattern of the arrowed lines gives no idea of the type of function.

A far better method is to use a **Cartesian graph**, with which you are already familiar. Here we use two perpendicular lines, the x-axis and the y-axis (Fig. 3.8). Values of the domain are placed on the x-axis and the range on the y-axis. Then x and its image f(x) give the coordinates (x,y) of a point. If sufficient points are plotted and joined up, we have the **graph** of the function. y = f(x) is the Cartesian equation of the curve.

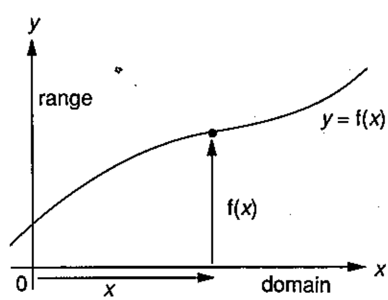


Fig. 3.8

Using this method of representing a function, we find that the graphs of various kinds of functions have characteristic shapes. Hence functions can be recognized from their graphs.

Common Functions And Their Graphs

Fig. 3.9 shows the graphs of some common functions.

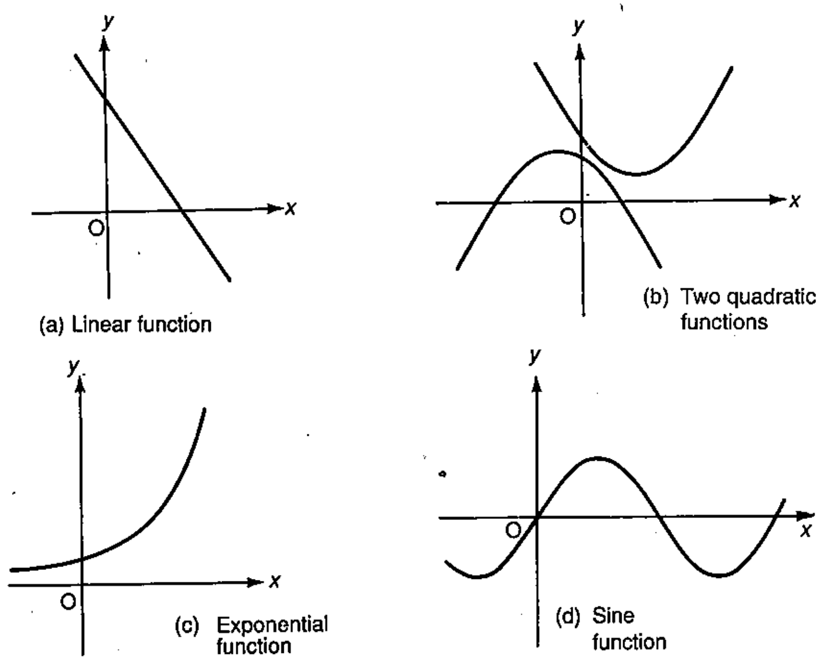
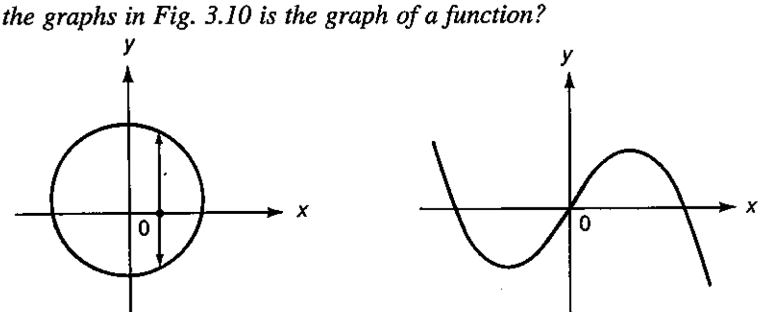


Fig. 3.9

(a) is a linear function such as $y = -3x + 4$.
(b) shows two quadratic functions such as $y = x^2 - x + 4$ (upper graph) and $y = 2 - x - x^2$.
(c) is an exponential function such as 2^x.
(d) is the graph of $y = \sin x$ (see Chapter 7).

Example 8

Which of the graphs in Fig. 3.10 is the graph of a function?



(a) (b)

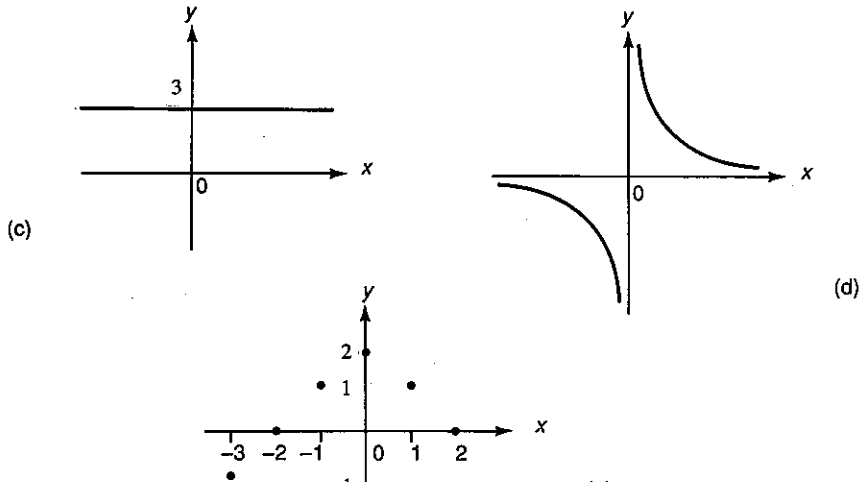


Fig. 3.10

For a function, each value of x in the domain must give just one and only one value of y. If there is more than one value of y for the same value of x in the domain, the graph does not represent a function.

(a) is not the graph of a function, as there are 2 values of y for each value of x.
(b) is the graph of a function.
(c) is the graph of a constant function $y = 3$. The domain is the set of all real numbers but the range is just 3.
(d) is the graph of a function provided $x = 0$ is excluded from the domain.
(e) is the graph of a function for the domain $\{-3,-2,-1,0,1,2\}$. The graph consists only of the points marked and these must not be joined up. The range is $\{2,1,0,-1\}$.

Example 9

Fig. 3.11 shows part of the graph of a function $y = f(x)$. Sketch the corresponding parts of the functions (a) $y_1 = -f(x)$, (b) $y_2 = f(-x)$, (c) $y_3 = 2 + f(x)$, (d) $y_4 = 3 - f(x)$, (e) $y_5 = f(x + 1)$, (f) $y_6 = f(x - 2)$.

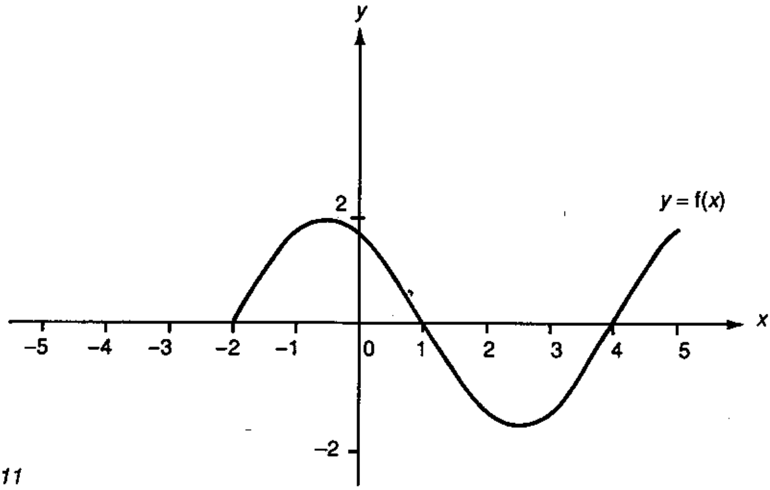


Fig. 3.11

(a) For each value of x, $y_1 = -y$. So the graph of of y_1 is the reflection of $y = f(x)$ in the x-axis (Fig. 3.12(a)).
Points where $y = f(x)$ meets the x-axis are unchanged.

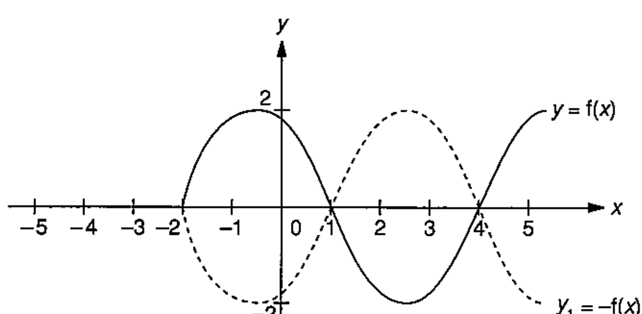


Fig. 3.12(a)

(b) When $x = a$, $y = f(a)$ and $y_2 = f(-a)$.
Now $f(-a)$ is the value of y when $x = -a$. For example when $x = 2$, the value of y_2 is the same as the value of y when $x = -2$.
So the graph of y is reflected in the y-axis to produce the graph of y_2 (Fig. 3.12(b)).
Points where $y = f(x)$ meets the y-axis will be unchanged.

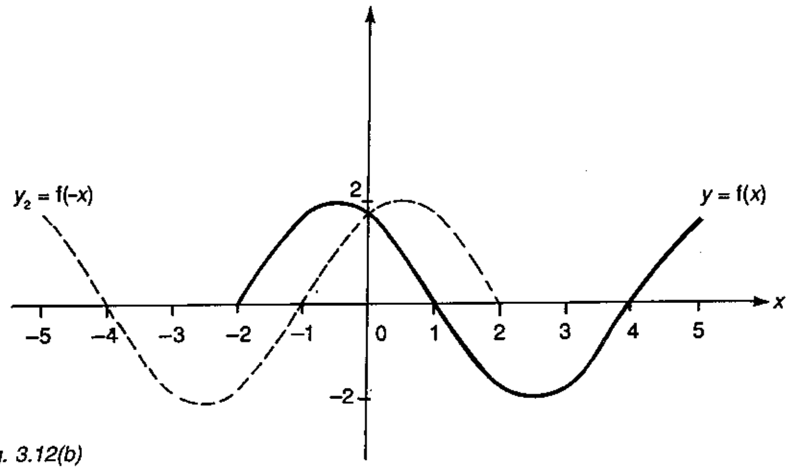


Fig. 3.12(b)

(c) Here 2 is added to each value of y.
So the original graph is shifted upwards through 2 units (Fig. 3.12(c)).

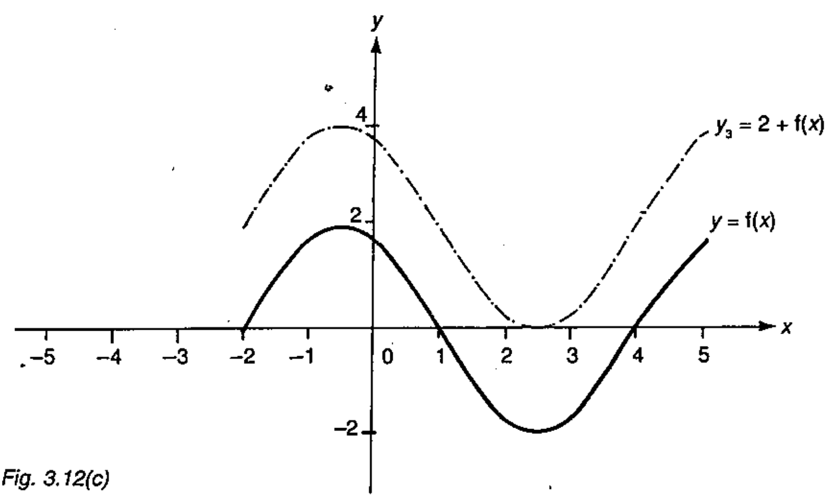


Fig. 3.12(c)

(d) $y_4 = 3 - f(x) = 3 + (-y) = 3 + y_1$.
So the graph of y_1 is shifted upwards through 3 units to obtain the graph of y_4 (Fig. 3.12(d)).

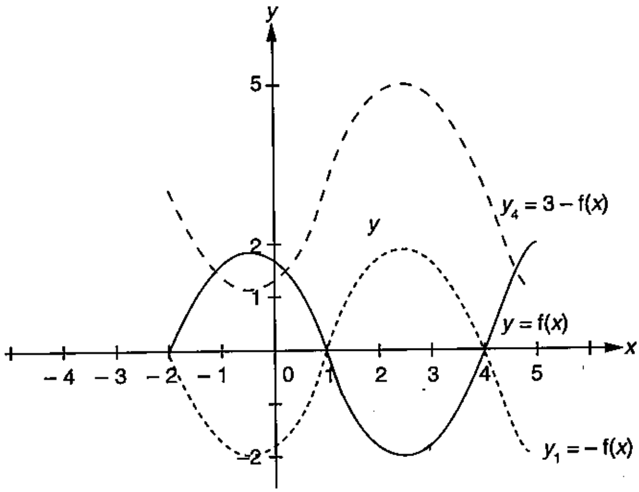


Fig. 3.12(d)

(e) Suppose $x = 1$. Then $y_5 = f(1 + 1) = f(2)$, which is the value of y when $x = 2$. Again when $x = 3$, $y_5 = f(3 + 1) = f(4)$, which is the value of y when $x = 4$. All the values of y have been shifted 1 unit **to the left** to obtain y_5 (Fig. 3.12(e)).

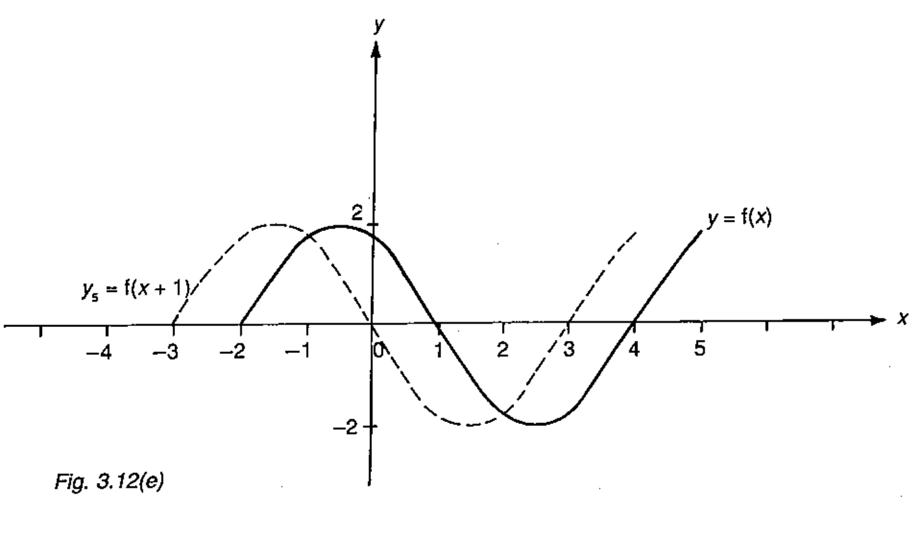


Fig. 3.12(e)

(f) You will be able to work out that y_6 is the original curve shifted 2 units **to the right** (Fig. 3.12(f)).

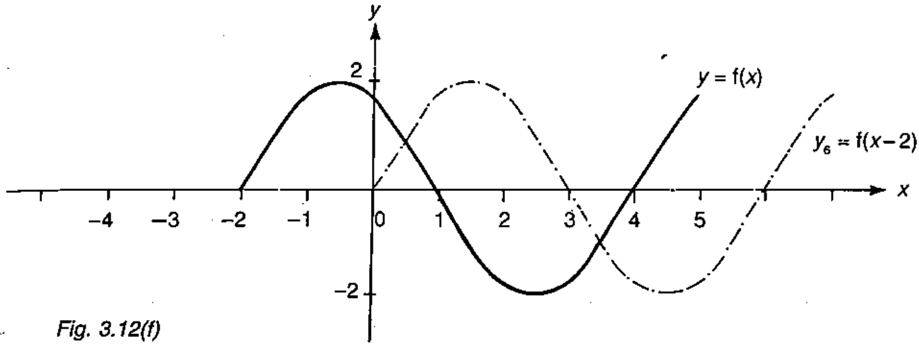


Fig. 3.12(f)

It would be useful to summarize such transformations of the graph of a function $y = f(x)$.

- $y = -f(x)$ is the reflection in the x-axis.
- $y = f(-x)$ is the reflection in the y-axis.
- $y = a + f(x)$ shifts the graph through a units **upwards** if a is **positive**, and **downwards** if a is **negative.**
- $y = f(x + a)$ shifts the graph through a units to the **left** if a is **positive**, but to the **right** if a is **negative.**

Exercise 3.2 *(Answers on page 609.)*

1 Which of the following are graphs of functions?

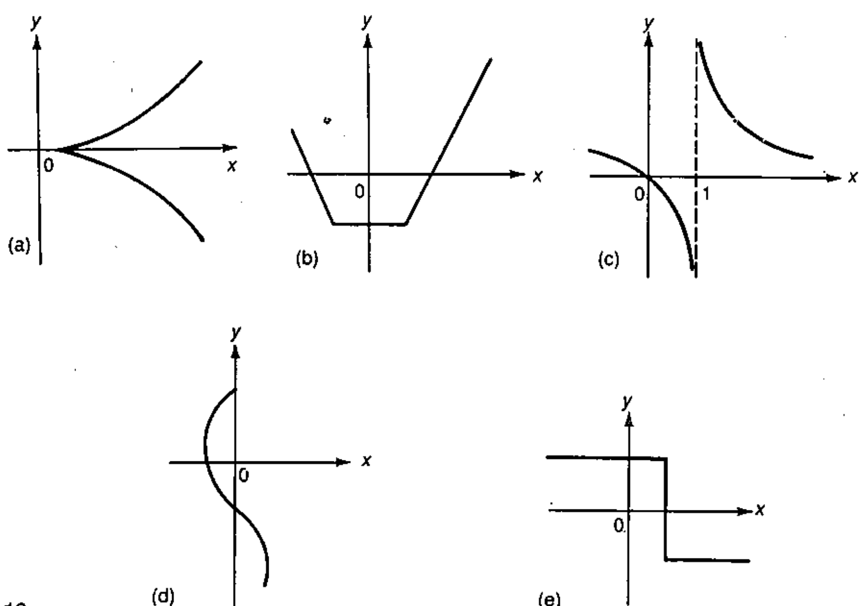
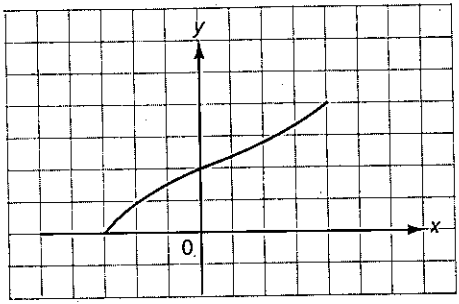


Fig. 3.13

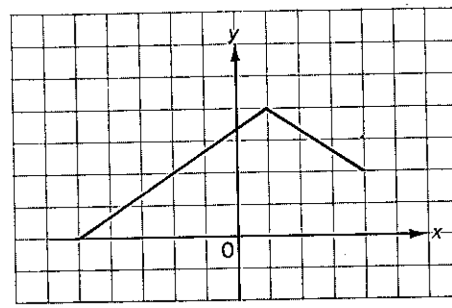
2 Each of the diagrams in Fig. 3.14 shows part of the graph of a function f(x). Copy each diagram and sketch the corresponding parts of

(i) $y_1 = \mathrm{f}(-x)$ (ii) $y_2 = \mathrm{f}(x - 1)$

(iii) $y_3 = \mathrm{f}(x + 1)$ (iv) $y_4 = 1 + \mathrm{f}(x + 1)$



(a)



(b)

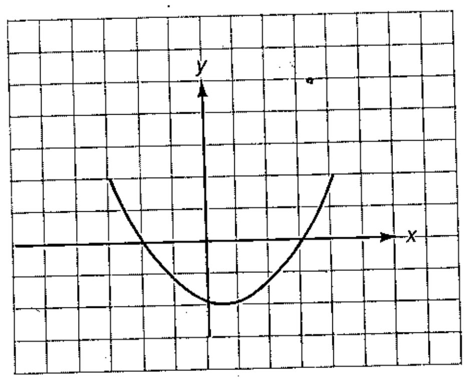


Fig. 3.14 (c)

3 On another copy of the diagrams in Fig. 3.14, sketch the corresponding parts of

(i) $y_5 = \mathrm{f}(x - 2)$

(ii) $y_6 = 2 - \mathrm{f}(x - 2)$

(iii) $y_7 = \mathrm{f}(1 - x)$

4 Fig. 3.15 shows part of the graph of $y = \mathrm{f}(x)$ with three graphs derived from it. State y_1, y_2 and y_3 in terms of f(x).

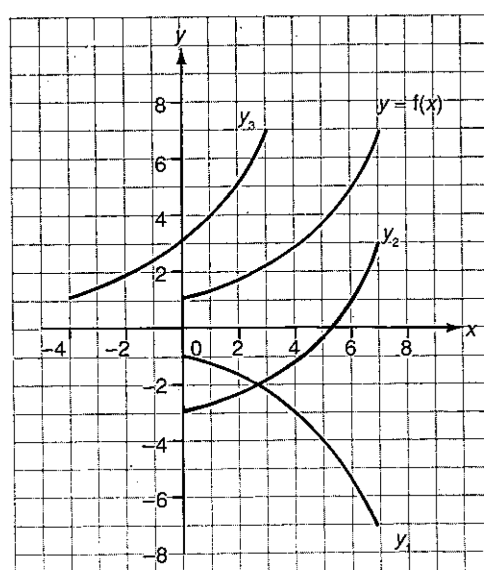


Fig. 3.15

5 The domain of a function f(x) is –1 to 4 inclusive. What would be the corresponding domain for the following?

(a) $y_1 = f(-x)$ (b) $y_2 = f(x-2)$

(c) $y_3 = f(x+1)$

6 The range of the function $y = f(x)$ is 0 to 5 inclusive. What is the corresponding range for the following?

(a) $y_1 = -f(x)$ (b) $y_2 = 1 + f(x)$

(c) $y_3 = f(x-3)$ (d) $y_4 = f(x) - 3$

THE MODULUS OF A FUNCTION

If $y = x$, the values of y are negative when x is negative. They can be converted to positive values by using the **modulus** $y = |x|$, read as 'y = mod x'. $|x|$ gives the **numerical** or **absolute** value of x. For example $|-3.5| = 3.5$. It does not alter 0 or any positive number: $|0| = 0$, $|2| = 2$ etc. $|x|$ is *never negative.*

So we define the modulus of x as

$$|x| = \begin{cases} -x \text{ if } x < 0 \\ x \text{ if } x \geq 0 \end{cases}$$

Similarly the modulus of a function f(x) written $|f(x)|$ is the numerical value of f(x).

Example 10

State the values of $|1-x|$ for $x = -3, 2, 4$.

When $x = -3$, $|1-x| = |1+3| = 4$.

When $x = 2$, $|1-x| = |1-2| = 1$.

When $x = 4$, $|1-x| = |1-4| = 3$.

Example 11

f(x) = $x^2 - x - 6$. Find the values of $|f(x)|$ for $x = -1, 0, 2, 4$.

x	–1	0	2	4
f(x)	–4	–6	–4	6
$\lvert f(x) \rvert$	4	6	4	6

Example 12

What is the least value of x if $|2x-3| = 2x-3$?

$|2x-3|$ will be equal to $2x-3$ if $2x-3$ is 0 or greater than 0. Hence the least value of x will be when $2x-3=0$, i.e. when $x = 1\frac{1}{2}$.

Example 13

Draw the graph of $y = |x-1|$ for the domain $-2 \le x \le 3$ and state the range of y.

$-2 \le x \le 3$ means that x can take any value between -2 and 3 (inclusive).

We make a table for the integer values of x:

x	-2	-1	0	1	2	3
$x-1$	-3	-2	-1	0	1	2
$y = \|x-1\|$	3	2	1	0	1	2

Plotting the points given by x and y, the graph is seen to consist of the two lines AB and AC (Fig. 3.16). The range is $0 \le y \le 3$.

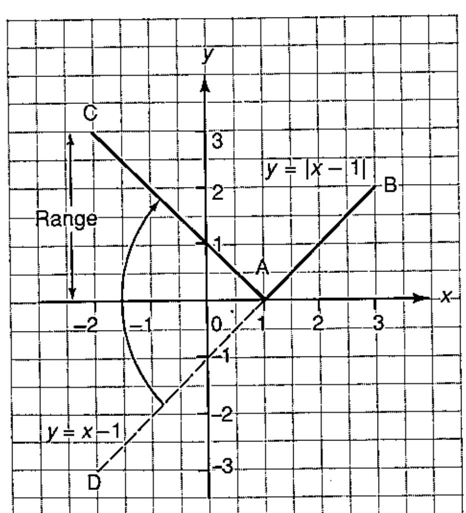


Fig. 3.16

However, if we extend BA to D (shown dotted) where D is $(-2, -3)$ we see that the part AC is the reflection of AD in the x-axis. So a quicker method of drawing the graph is to draw $y = x - 1$ for the given domain first and then reflect any negative part in the x-axis.

To draw a graph of the type $y = |\,f(x)\,|$, draw $y = f(x)$ first and then reflect any negative part in the x-axis.

Example 14

Draw the graph of $y = |2 - x|$ for the domain $-1 \le x \le 3$ and state the range of y.

Draw the line $y = 2 - x$ first (Fig. 3.17). (The negative part is dotted).

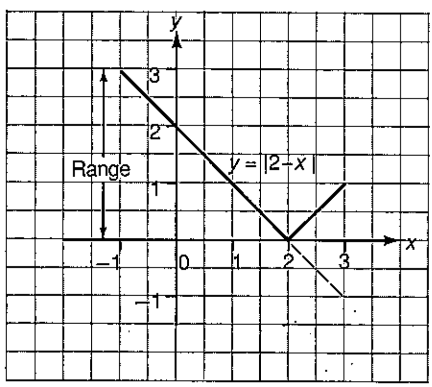


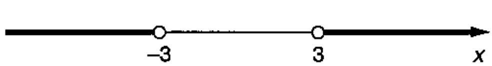
Fig. 3.17

Then reflect the negative part in the x-axis.
The graph consists of two lines meeting on the x-axis where $x = 2$.
The range is $0 \le y \le 3$.

MODULAR INEQUALITIES

Suppose we know that $|\,x\,| > 3$. Suggest some values that x could take to satisfy the inequality.

From the definition of a modulus, $|\,x\,| > 3$ means that either $x > 3$ or $-x > 3$. $-x > 3$ means that $x < -3$ (dividing by -1 and reversing the inequality sign). So the range of x is $x < -3$ **or** $x > 3$. We can show the range on a number line:



x must lie on the thick lined parts. o means this value is excluded.

So if $|x| > k$ then $x < -k$ or $x > k$.

Next suppose $|x| < 3$. Then $x < 3$ or $-x < 3$ i.e. $x > -3$. Hence x lies between -3 and 3 (not inclusive) and we write $-3 < x < 3$.

On a number line we have 

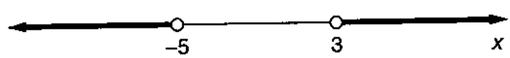
So if $|x| < k$ then $-k < x < k$.

These rules apply also to linear and quadratic functions.

Example 15

Find and show on a number line, the range of values of x if (a) $/x + 1/ > 4$, (b) $/1 - 2x/ \leq 5$.

(a) From the above, if $|x + 1| > 4$ then $x + 1 < -4$ or $x + 1 > 4$. Hence $x < -5$ or $x > 3$.



(b) If $|1 - 2x| \leq 5$ then $-5 \leq 1 - 2x \leq 5$. Taking each part, $-5 \leq 1 - 2x$ gives $-6 \leq -2x$ or $3 \geq x$ i.e. $x \leq 3$; $1 - 2x \leq 5$ gives $-2x \leq 4$ i.e. $x \geq -2$.

So $-2 \leq x \leq 3$ 

In this case the ● means the value *is* included (due to the $\leq$ sign)

Exercise 3.3 *(Answers on page 610.)*

1 State the values of
(a) $|-6|$ (b) $|-\frac{1}{2}|$ (c) $|\cos 120°|$ (d) $|3^2 - 6^2|$

2 By testing with $x = -3, 0, 2$ verify that $|1 - x| = |x - 1|$.

3 What is the least value of x for which $|2x - 1| = 2x - 1$?

4 For what domain will the graph of $y = |3 - x|$ be the same as the graph of $y = x - 3$?

5 Find and show on a number line the range of values of x which satisfy the inequalities:
(a) $|2x - 3| > 5$ (b) $\left|\frac{x-3}{2}\right| \leq 4$
(c) $\left|\frac{2x-1}{3}\right| \geq 2$ (d) $\left|1 - \frac{x}{3}\right| < 3$

6 For the domain $-3 \leq x \leq 4$, draw the graphs of

(a) $y = |x|$ (b) $y = |x+1|$ (c) $y = -|x-2|$

(d) $y = |2x-1|$ (e) $y = |3-x|$

7 State the range for each of the functions in Question 6.

8 Using the graph you have drawn for part (a) in Question 6, add the graph of $y = -|x|$.

9 On the same piece of graph paper, draw the graphs of $y = |3x|$ and $y = |x-3|$ for the domain $-2 \leq x \leq 3$. Hence solve the equation $|3x| = |x-3|$.

10 By drawing two graphs, solve the equation $|x-1| = |2x-5|$. (Take $0 \leq x \leq 7$ as domain).

11 The range of the function $y = |x-1|$ is $0 \leq y \leq 3$. Find a possible domain. What is the widest possible domain?

12 The domain of the function $y = |2x-3|$ ends where $x = 2$. If the upper limit of the range is 7, what is the least value of the domain?

13 Draw the graph of $y = |x-1|$ for the domain $-1 \leq x \leq 2$. Now add the graph of $y = 2 - |x-1|$ for the same domain. State the range of this function.

THE INVERSE OF A FUNCTION

Fig. 3.18 shows the mapping of the domain {–3, 0, 1, 2} by the function $f : x \longmapsto 3x - 2$. Verify that the range is {–11, –2, 1, 4}.

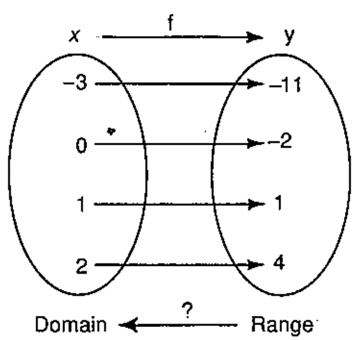


Fig. 3.18

Is there a function that will map the range back to the domain?

The function f in Fig.3.18 mapped x onto y where $y = 3x - 2$. Now we wish to start with y and return to x. If $3x - 2 = y$, then $x = \frac{y+2}{3}$.

So this new function will map y onto $\frac{y+2}{3}$.

Testing this with $y = -11$, we get $\frac{-11+2}{3} = -3$ which is the original value of x. Check the other values.

Such a function, *if it exists*, is called the **inverse** function of f and is written as f^{-1}. (Read this as 'inverse f'.) It is usual to take x as the 'starting' letter so we have

$f^{-1} : x \longmapsto \frac{x+2}{3}$.

Summarizing,

function $f{:}x \longmapsto 3x - 2$ and inverse $f^{-1}{:}x \longmapsto \frac{x+2}{3}$.

It then follows that the inverse of f^{-1} is f.

Example 16

Find the inverse function to $f{:}x \longmapsto \frac{x-3}{2}$.

f maps x onto y where $y = \frac{x-3}{2}$.

Make x the subject of this equation.

$\frac{x-3}{2} = y$ so $x - 3 = 2y$ and $x = 2y + 3$.

Hence $f^{-1} : y \longmapsto 2y + 3$.

Changing to the usual letter x, $f^{-1} : x \longmapsto 2x + 3$.

Suppose -4 was a value in the original domain. Then f will map this onto $3\frac{1}{2}$. f^{-1} will now map this value onto $2(-3\frac{1}{2}) + 3 = -4$, which is the original value. Repeat this check with other values of x, say 0, 1 and 5.

Example 17

Given the function $f : x \longmapsto \frac{x+p}{x-3}$ *($x \neq 3$), where p is a constant,*

(a) find the value of p if $f(5) = 1\frac{1}{2}$,
(b) find f^{-1} *in a similar form,*
(c) state the value of x for which f^{-1} *is undefined.*

(a) $f(5) = \frac{5+p}{5-3} = 1\frac{1}{2}$

Then $5 + p = 3$ and $p = -2$.

(b) From (a), $f(x) = \frac{x-2}{x-3}$ i.e. $y = \frac{x-2}{x-3}$

or $yx - 3y = x - 2$, and $x(y - 1) = 3y - 2$.

Hence $x = \frac{3y-2}{y-1}$.

Therefore $f^{-1} : x \longmapsto \frac{3x-2}{x-1}$.

(c) f^{-1} is undefined for $x = 1$. (This means that there is no value of x in the original domain which had an image of 1. So 1 does not exist in the range and therefore cannot be used).

Example 18

Find the inverse of f : $x \longmapsto 3 - x$.

f maps x onto y where $y = 3 - x$.
So $x = 3 - y$ and the inverse function will be $f^{-1} : x \longmapsto 3 - x$, which is the same function as f.
Check this by taking $x = 3, -1$ and 5.
Such a function f is called **self-inverse**, i.e. it is its own inverse.

Functions With No Inverse

Some functions do not have an inverse. Take the function f : $x \longmapsto x^2$ (Fig. 3.19).

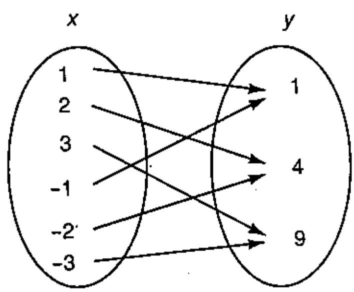


Fig. 3.19

Two arrows arrive at 1 in the range. An inverse would have two paths to return from 1 to the domain and so could not be a function. There is no inverse function.

An inverse function can only exist if the original function is a **one-to-one** function (Fig. 3.20(a)), i.e. there is only one arrow reaching each member of the range. There will be no inverse if the function is a **many-to-one** function (Fig. 3.20(b)), i.e. more than one arrow reaches some members of the domain.

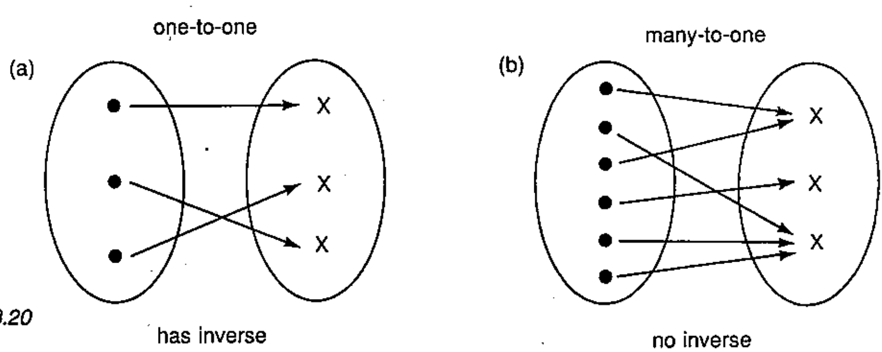


Fig. 3.20

The inverse function f^{-1} exists only if f is one-to-one.

Graphical Illustration of an Inverse Function

Verify that the inverse of $f : x \longmapsto 2x - 3$ is $f^{-1} : x \longmapsto \frac{x+3}{2}$.

Now draw the lines

$$y = 2x - 3 \qquad \text{(i)}$$

and

$$y = \frac{x+3}{2} \qquad \text{(ii)}$$

on graph paper (Fig. 3.21). Add the line $y = x$ (shown dotted).

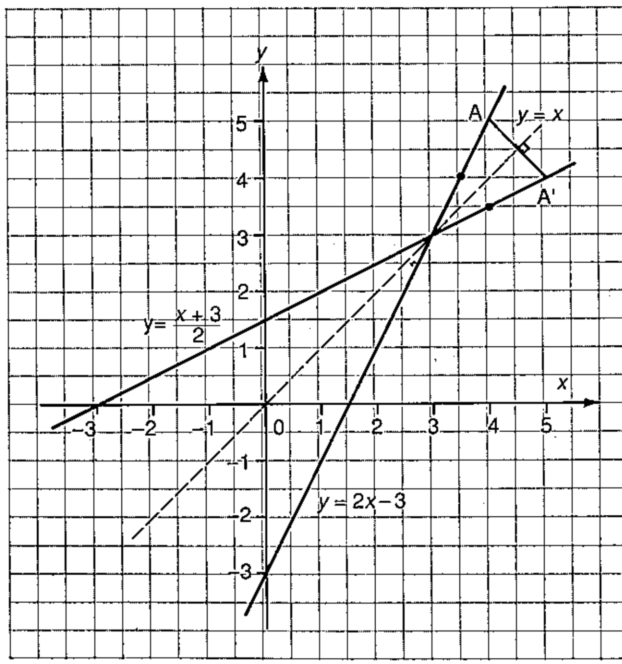


Fig. 3.21

How do the two lines (i) and (ii) appear in relation to the line $y = x$?

Consider the point where $x = 4$ (point A) on (i). The image of 4 from f is 5, so the coordinates of A are (4,5).

Now if we take $x = 5$, its image in f^{-1} will be 4. This gives point A'(5,4) which lies on line (ii).

The gradient of AA' is -1 so AA' is perpendicular to the line $y = x$ and the midpoint of AA' $(4\frac{1}{2}, 4\frac{1}{2})$ lies on the line $y = x$. Hence A and A' are reflections of each other in the line $y = x$.

We can repeat this for any other point. The coordinates will be interchanged by the inverse function, so the two points are reflections of each other. Hence lines (i) and (ii) are reflections of each other in the line $y = x$. You can also test this by folding the graph paper along the line $y = x$.

The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.

Exercise 3.4 *(Answers on page 611.)*

1 Find the inverses of the following functions in the same form:

(a) $f : x \longmapsto x$ (b) $f : x \longmapsto x + 2$

(c) $f : x \longmapsto 2x - 1$ (d) $g : x \longmapsto 3x + 4$

(e) $f : x \longmapsto \frac{x+1}{3}$ (f) $f : x \longmapsto 9 - x$

(g) $f : x \longmapsto 2x - 5$ (h) $f : x \longmapsto 8 - 2x$

(i) $f : x \longmapsto \frac{x}{3} - 1$ (j) $h : x \longmapsto \frac{8}{x}$ $(x \neq 0)$

(k) $f : x \longmapsto \frac{5}{x+1}$ $(x \neq -1)$ (l) $F : x \longmapsto \frac{3}{x} + 2$ $(x \neq 0)$

(m) $f : x \longmapsto \frac{x+1}{x-2}$ $(x \neq -2)$ (n) $h : x \longmapsto \frac{2x-1}{x-3}$ $(x \neq 3)$

2 Which of the functions in Question 1 are self-inverse?

3 Given $f^{-1} : x \longmapsto 2x - 3$, find f in the same form.

4 If $f^{-1} : x \longmapsto \frac{x+3}{2}$, find f in the same form.

5 $f : x \longmapsto a - x$, where a is a constant, is a self-inverse function. Given that $f^{-1}(4) = 3$, find the value of a.

6 Given the function $h : x \longmapsto \frac{1-x}{x-4}$ $(x \neq 4)$, find the value of $h^{-1}(-3)$.

7 Given the function $g : x \longmapsto \frac{x+3}{x+2}$ $(x \neq -2)$, find $g^{-1}(-1)$.

8 Given the function $f : x \longmapsto \frac{x+d}{x-1} (x \neq 1)$ and that $f(2) = 5$, find (a) the value of d, (b) f^{-1}.
What can be said about this function?

9 $f : x \longmapsto \frac{x+r}{x+s}$, where r and s are constants and $f(4) = 6$, $f(-1) = -\frac{1}{4}$. Find

(a) the values of r and s,

(b) the value of x for which f is undefined,

(c) f^{-1} in the same form,

(d) the value of x for which f^{-1} is undefined.

10 Fig. 3.22 shows part of the graph of a function $y = f(x)$. Copy the diagram and sketch the graph of f^{-1}.

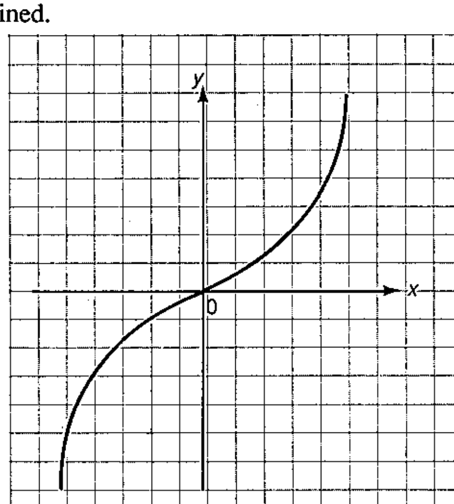


Fig. 3.22

11 On graph paper, draw the graph of $f : x \longmapsto 3 - x$. Construct the reflection of this graph in $y = x$. Explain your result.

12 (a) Find the inverse of $f : x \longmapsto \frac{6 - 3x}{2}$.

(b) On graph paper, draw the graph of $y = \frac{6 - 3x}{2}$.

(c) Construct the reflection of the graph in part (b) in $y = x$. Show that this is the graph of f^{-1}.

13 The function f is defined as $f : x \longmapsto \begin{cases} x + 3 \text{ for } x \geq 0 \\ 2x + 3 \text{ for } x < 0 \end{cases}$

Sketch the graphs of f and f^{-1}.

14 Copy Fig. 3.23 and sketch the inverse of the function $y = f(x)$.

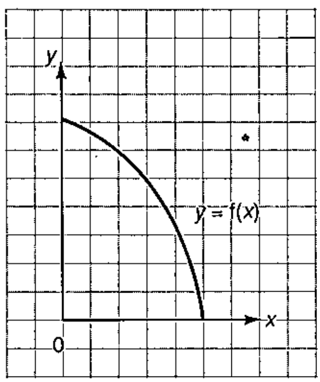


Fig. 3.23

15 (a) If $f(x) = 3 - \frac{2}{x}$, solve the equation $f(x) = x$.

(b) Draw the graph of $f(x)$ for $\frac{1}{2} \leq x \leq 2$.

(c) Add a sketch of the graph of $f^{-1}(x)$ for $-1 \leq x \leq 2$.

Composite Functions

Consider the function $f: x \longmapsto 2x - 3$ (Fig. 3.24). 4 is mapped onto 5.

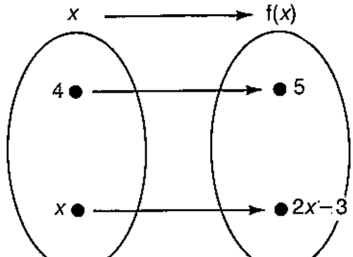


Fig. 3.24

Let g be another function such that $g : x \longmapsto x + 1$.
We now use g on f(4) to obtain g[f(4)] = 6. So 4 has been mapped onto 6 by f followed by g (Fig. 3.25).

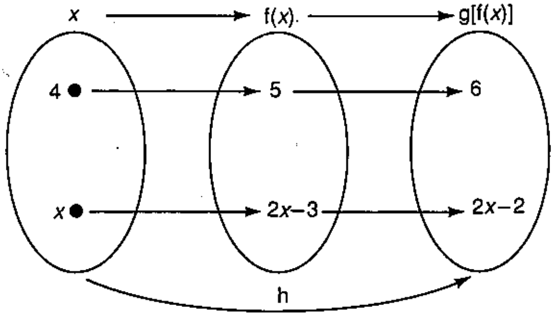


Fig. 3.25

Can we find a single function h which combines f and g?

x is mapped onto $2x - 3$ by f and this is the starting value for g. So g maps $2x - 3$ onto $(\mathbf{2x - 3}) + 1 = 2x - 2$. Hence $h : x \longmapsto 2x - 2$. If $x = 4$, the final result is 6 as we have seen. h is called the **composite** (or **combined**) function g[f(x)] which we write briefly as $\mathbf{h} = \underset{\text{second}}{\mathbf{g}}\underset{\text{first}}{\mathbf{f}}$.

Note carefully that the *first* function is written on the *right*.

Now suppose we do g first, followed by f, i.e. fg.

$$\underbrace{x \xmapsto{g} x + 1 \xmapsto{f} 2(\boldsymbol{x} + \mathbf{1}) - 3 = 2x - 1}_{fg}$$

The result is different. **fg is not the same function as gf**. We say the combination of functions is **not commutative**, i.e. the order in which they are done is important and cannot (in general) be interchanged. However for some values of x, fg may be equal to gf.

NB: Take care! fg does **NOT** mean f × g when dealing with functions.

Example 19

If $f : x \longmapsto 2x - 3$, *find (a)* f^{-1} *and (b)* $f^{-1}f$.

(a) $y = 2x - 3$ so $x = \frac{y + 3}{2}$

Therefore $f^{-1} : x \longmapsto \frac{x + 3}{2}$

(b) $f^{-1}f$ means that we do f first, f^{-1} second,

$$x \xmapsto{f} 2x - 3 \xmapsto{f^{-1}} \frac{\mathbf{2x - 3 + 3}}{2} = x$$

So $f^{-1}f : x \longmapsto x$
i.e. $f^{-1}f(x) = x$

Verify that ff^{-1} gives the same result.

Part (b) above is an example of an important result:

$$f^{-1}f(x) \equiv ff^{-1}(x) \equiv x$$

This follows from the definition of the inverse function. f maps x onto the range giving $f(x)$. f^{-1} operates on $f(x)$ to return to the original element x. So $f^{-1}f(x) = x$. Similarly, if we start from the range, $ff^{-1}(x) = x$.

Example 20

If $f : x \longmapsto x^2$ *and* $g : x \longmapsto x - 1$*, find in a similar form, (a)* fg *and (b)* gf.

(a) fg is g first, f second.

$x \xmapsto{g} x - 1 \xmapsto{f} (x - 1)^2$

So the combined function fg is $fg : x \longmapsto (x - 1)^2$.

(b) gf is f first, g second.

$x \xmapsto{f} x^2 \xmapsto{g} x^2 - 1$

The combined function gf is $x^2 - 1$.

Note that fg $\neq$ gf.

Example 21

Functions f *and* g *are defined as*

$$f : x \longmapsto \frac{2}{x+1} \text{ and } g : x \longmapsto 3x - 2.$$

Find (a) fg, *(b)* gf, *(c)* $(fg)^{-1}$, *(d)* $(gf)^{-1}$.
(e) For what value(s) of x *is* gf = fg?

(a) $x \xmapsto{g} 3x - 2 \xmapsto{f} = \dfrac{2}{(3x-2)+1} = \dfrac{2}{3x-1}$

Hence $fg : x \longmapsto \dfrac{2}{3x-1}, x \neq \dfrac{1}{3}$.

(b) $x \xmapsto{f} \dfrac{2}{x+1} \xmapsto{g} 3\left(\dfrac{2}{x+1}\right) - 2 = \dfrac{6-2x-2}{x+1} = \dfrac{4-2x}{x+1}$

Hence $gf : x \longmapsto \dfrac{4-2x}{x+1}, x \neq -1$.

(c) $(fg)^{-1}$ is the inverse of the combined function fg.

Now fg maps x onto $y = \dfrac{2}{3x-1}$ from (a).

So $3xy - y = 2$

i.e. $3xy = y + 2$ giving $x = \dfrac{y+2}{3y}$.

Hence $(fg)^{-1} : x \longmapsto \dfrac{x+2}{3x}, x \neq 0$.

(d) $(gf)^{-1}$ is the inverse of gf.

Verify that $(gf)^{-1} : x \longmapsto \frac{4-x}{x+2}, x \neq -2.$

(e) If gf = fg, then $\frac{4-2x}{x+1} = \frac{2}{3x-1}, x \neq -1, \frac{1}{3}$.

So $2x + 2 = (3x - 1)(4 - 2x) = -6x^2 + 14x - 4$ or $6x^2 - 12x + 6 = 0$.

Then $x^2 - 2x + 1 = 0$ or $(x - 1)(x - 1) = 0$ giving $x = 1$.

This is the only value of x for which gf = fg.

Example 22

Using the functions f *and* g *in Example 21, find* f^{-1} *and* g^{-1}. *Show that* $(fg)^{-1} = g^{-1}f^{-1}$. *Suggest and test a similar result for* $(gf)^{-1}$.

Verify that $f^{-1} : x \longmapsto \frac{2-x}{x}$ *and* $g^{-1} : x \longmapsto \frac{x+2}{3}$

From (c) in Example 21, $(fg)^{-1} : x \longmapsto \frac{x+2}{3x}$.

$g^{-1}f^{-1}$ is given by $x \xrightarrow{f^{-1}} \frac{2-x}{x} \xrightarrow{g^{-1}} \frac{\frac{2-x}{x}+2}{3} = \frac{2+x}{3x}$

Hence $(fg)^{-1} = g^{-1}f^{-1}$.

So the inverse of fg is the inverse of f followed by the inverse of g. This suggests that $(gf)^{-1} = f^{-1}g^{-1}$. Show that this is correct using $(gf)^{-1}$ from Example 21.

The results of Example 22 are true in general:

[redacted]

Example 23

Given that $f : x \longmapsto \frac{x}{x+2}$ $(x \neq -2)$, *find in a similar form (a)* f^2, *(b)* f^3, *(c)* f^4 *and deduce an expression for* f^5.

(a) f^2 means ff, i.e. f done twice in succession.

So $x \xrightarrow{f} \frac{x}{x+2} \xrightarrow{f} \frac{\frac{x}{x+2}}{\frac{x}{x+2}+2} = \frac{x}{x+2x+4} = \frac{x}{3x+4}, x \neq -2, -\frac{4}{3}$

(b) f^3 means f^2 followed by f.

So $f^3(x) = f[f^2(x)] = \frac{\frac{x}{3x+4}}{\frac{x}{3x+4}+2} = \frac{x}{7x+8}, x \neq -2, \frac{4}{3}, -\frac{8}{7}$

(c) $f^4(x) = f[f^3(x)] = \frac{\frac{x}{7x+8}}{\frac{x}{7x+8}+2} = \frac{x}{15x+16}, x \neq -2, -\frac{4}{3}, -\frac{8}{7}, -\frac{16}{15}$

Studying the pattern, the numerator is always x. The denominators are $3x + 4$, $7x + 8$, $15x + 16$ so the next denominator will be $31x + 32$.

f^5 is $x \longmapsto \frac{x}{31x + 32}$, $x \neq -2, -\frac{4}{3}, -\frac{8}{7}, -\frac{16}{15}, -\frac{32}{31}$.

Example 24

If $f : x \longmapsto ax + b$ *(a > 0) and* $f^2 : x \longmapsto 9x - 8$, *find (a) the values of a and b,* *(b)* f^3, (c) f^4.
(d) Deduce f^5

(a) We first find f^2 in terms of a and b.

$x \xrightarrow{f} ax + b \xrightarrow{f} a(ax + b) + b = a^2x + ab + b$

But this is $9x - 8$.

Hence $a^2 = 9$ which gives $a = 3$ (since $a > 0$) and $ba + b = -8$ so $b = -2$.

(b) $f^3(x) = f[f^2(x)] = 3(9x - 8) - 2 = 27x - 26$

(c) $f^4(x) = f^2[f^2(x)] = 9(9x - 8) - 8 = 81x - 80$

(d) The pattern in these results is:

$f^2 : \; 9x - 8 \quad = 3^2x - (3^2 - 1)$
$f^3 : \; 27x - 26 = 3^3x - (3^3 - 1)$
$f^4 : \; 81x - 80 = 3^4x - (3^4 - 1)$
so we can deduce that $f^5 = 3^5x - (3^5 - 1) = 243x - 242$.

Example 25

Given $f : x \longmapsto 2x - 5$, *find a function* g *such that* $fg : x \longmapsto 6x - 1$.

Clearly g must be a linear function as no squares appear in fg.
Take g as $x \longmapsto ax + b$.

Then $fg : x \longmapsto 2(ax + b) - 5 = 2ax + 2b - 5$.
But this must be identical to $6x - 1$.
Then $2a = 6$, giving $a = 3$ and $2b - 5 = -1$, giving $b = 2$.
Hence $g : x \longmapsto 3x + 2$.

Example 26

Express in terms of the functions f : $x \longmapsto x + 3$ *and* g : $x \longmapsto x^2$,
(a) $x^2 + 3$, *(b)* $x^2 + 6x + 9$, *(c)* $x + 6$, *(d)* $x^2 + 6x + 12$, *(e)* $x^2 - 6x + 9$.

(a) This is fg.

(b) Note that $x^2 + 6x + 9 = (x + 3)^2$. f gives $(x + 3)$. g gives the square. So this is gf.

(c) Here g is not involved as there is no square. Try ff.

(d) Note that $x^2 + 6x + 12 = (x + 3)^2 + 3$. We get $(x + 3)^2$ from gf.
If we now use f, we obtain the result. The answer is therefore fgf (first f, then g and lastly f again).

(e) $x^2 - 6x + 9 = (x - 3)^2$. Now f does not produce $(x - 3)$ but $f^{-1} : x \longmapsto x - 3$.
Hence the answer is gf^{-1}.

Exercise 3.5 *(Answers on page 613.)*

1 Using the functions f : $x \longmapsto x + 2$ and g : $x \longmapsto x - 3$, find in the same form (a) fg, (b) gf, (c) ff, (d) gg.

2 S : $x \longmapsto \sin x°$ and T : $x \longmapsto 2x$ are two functions.
Find (a) ST(20), (b) TS(20).

3 Taking f : $x \longmapsto x + 2$ and g : $x \longmapsto 3x - 1$, find (a) fg, (b) gf, (c) f^{-1}, (d) g^{-1}, (e) f^{-1}, (f) $g^{-1}f$.

4 If f : $x \longmapsto x + 1$, find (a) f^2, (b) f^3 and deduce (c) f^4, (d) f^5. (e) f^n.

5 Taking the function f as f : $x \longmapsto \frac{x-3}{x+2}$, $x \neq -2$, find (a) f^{-1}, (b) f^2, (c) $(f^2)^{-1}$. In each case, state the values of x which must be excluded from the domain.

6 If f : $x \longmapsto x^2 - 2$ and g : $x \longmapsto x + 3$, find (a) fg, (b) gf.
For what value of x is fg = gf?

7 Given that g : $x \longmapsto x + 2$ and h : $x \longmapsto x^2 - 3$, find the value of x for which gh = hg.

8 For the functions f : $x \longmapsto x - 4$ and g : $x \longmapsto 3x - 2$, find similarly (a) f^{-1}, (b) g^{-1}, (c) fg^{-1}, (d) $(fg)^{-1}$.

9 Functions f and g are defined by $x \longmapsto 2x + 1$ and $x \longmapsto 1 - 3x$ respectively. For what value of x is $fg^{-1} = f^{-1}g$?

10 Functions f and g are defined as f : $x \longmapsto \frac{x-1}{2}$ and g : $x \longmapsto \frac{1}{x}$ $(x \neq 0)$ respectively. Find similarly (a) fg, (b) $g^{-1}f$, (c) $f^{-1}g^{-1}$. In each case, state the values of x which must be excluded from the domain. (d) For what values of x is $g^{-1}f = f^{-1}g^{-1}$?

11 The functions f and g are defined as f : $x \longmapsto 3x + 2$ and g : $x \longmapsto \frac{1}{x}$ $(x \neq 0)$. Find similar expressions for (a) fg, (b) gf, (c) $f^{-1}g$, (d) gf^{-1}. In each case, state the values of x which must be excluded from the domain.
Find the value(s) of x for which (e) fg = gf, (f) $f^{-1}g = gf^{-1}$.

12 Given $f : x \longmapsto 1 - \frac{1}{x}$, $x \neq 0$, and $g : x \longmapsto \frac{1}{1-x}$, $x \neq 1$, find $fg(x)$ and $gf(x)$. Hence state the inverses of f and g.

13 f is the function that maps x onto $\frac{x+1}{x-1}$ $(x \neq 1)$.
(a) Show that f is self-inverse.
(b) Find f^2.
(c) Show that $f^3 = f$.

14 $f : x \longmapsto ax + b$ (a, b constants) and $g : x \longmapsto 2x + 3$ are two functions.
(a) If $fg = gf$, find a relation between a and b.
(b) Given that $f^{-1}(7) = -1$, find the values of a and b.

15 If f maps x onto $5 - \frac{x}{2}$ and g maps x onto $2x + 1$, show that fg and gf are both self-inverse.

16 $f : x \longmapsto \frac{x-2}{x+1}$, $x \neq -1$
(a) Find f^2. State the value of x which must be excluded from the domain.
(b) If $f^2(x) = -1$, find the value of x.

17 If $f(x) = \frac{x-1}{x+2}$, $x \neq -2$, find f^2 and f^3. In each case, state the values of x which must be excluded from the domain. Solve the equation $f^3(x) = 1$.

18 $f : x \longmapsto 3x + 1$. Find a function g so that $gf : x \longmapsto 3x + 2$.

19 If $f : x \longmapsto 2x + 3$, find a function g so that $fg : x \longmapsto 2x - 1$.

20 Express the following in terms of the functions $g : x \longmapsto x + 2$ and $h : x \longmapsto 3x$.
(a) $x \longmapsto 3x + 2$
(b) $x \longmapsto 3x + 6$
(c) $x \longmapsto x + 4$
(d) $x \longmapsto 3x + 12$
(e) $x \longmapsto 9x$
(f) $x \longmapsto 9x + 2$
(g) $x \longmapsto x - 2$
(h) $x \longmapsto 3x - 6$

21 Given $f : x \longmapsto x + 3$ and $g : x \longmapsto x^2 - 1$, state the following in terms of f and g.
(a) $x \longmapsto x^2 + 2$
(b) $x \longmapsto x^2 + 6x + 8$
(c) $x \longmapsto x + 6$
(d) $x \longmapsto x^2 + 12x + 35$
(e) $x \longmapsto x^2 - 6x + 8$
(f) $x \longmapsto x^2 - 4$

22 Given $f : x \longmapsto \sqrt{x}$ (positive root) and $g : x \longmapsto x + 2$, express the following in terms of f and g:
(a) $x \longmapsto \sqrt{x+2}$
(b) $x \longmapsto \sqrt{x} + 2$
(c) $x \longmapsto x + 4$
(d) $x \longmapsto \sqrt{x+4}$
(e) $x \longmapsto \sqrt{x-2}$
(f) $x \longmapsto x^2 + 4x + 4$
(g) $x \longmapsto x^2 - 4x + 4$
(h) $x \longmapsto x^2 + 8x + 16$

23 If $f : x \longmapsto x - 3$, what is the function g which makes $gf : x \longmapsto x^2 - 6x + 10$?

24 $f : x \longmapsto 2 + \frac{3}{x-1}$, $x \neq 1$, and $g : x \longmapsto x + 4$. Find the inverse of fg in a similar form.

25 f is given by $f : x \longmapsto \frac{x}{x-3}$ $(x \neq 3)$. Find (a) f^2, (b) f^3, (c) f^4. Deduce f^5.
In each case, state the values of x that must be excluded from the domain.

SUMMARY

- A function f maps an input x (domain) onto a unique image y (range).
  $f : x \longmapsto y = f(x)$
- $y = f(x)$ is the equation of the graph of the function.
  $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis.
  $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis.
  $y = a + f(x)$ shifts the graph upwards through a units if $a > 0$, and downwards if $a < 0$.
  $y = f(x + a)$ shifts the graph through a units to the left if $a > 0$, and to the right if $a < 0$.
- Modulus of x : $|x| = x$ for $x \geq 0$,
  $= -x$ for $x < 0$.
- If $|x| > k$, then $x < -k$ or $x > k$;
  if $|x| < k$, then $-k < x < k$
- To draw the graph of $y = |f(x)|$, first draw the graph of $y = f(x)$ and then reflect any negative part in the x-axis.
- If f is one-to-one, the inverse function f^{-1} exists.
  $ff^{-1}(x) = f^{-1}f(x) = x$
- If $f = f^{-1}$, f is self-inverse.
- The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each other in the line $y = x$.
- Functions may be combined, but the order is important.

$$\underbrace{g}_{\text{second}}\underbrace{f}_{\text{first}} : x \xmapsto{f} f(x) \xmapsto{g} g[f(x)]$$

$$fg : x \xmapsto{g} g(x) \xmapsto{f} f[g(x)]$$

- f^2 means ff, and so on.
- $(fg)^{-1} = g^{-1}f^{-1}$; $(gf)^{-1} = f^{-1}g^{-1}$

REVISION EXERCISE 3 *(Answers on page 614.)*

A

1 $f : x \longmapsto 2x - 3$. Find the domain of x if $-5 \leq f(x) \leq 3$.

2 f is a function given by $f : x \longmapsto \frac{2x+1}{x-3}$, $(x \neq 3)$.
(a) Find f^{-1}.
(b) State the value of x for which f^{-1} is undefined.

3 (a) Solve these inequalities and show the results on a number line for each one:
(i) $|4x-3| \geq 2$ (ii) $\left|1-\frac{3x}{4}\right| < 4$

(b) Given that $|ax+b| \leq 5$ where a and b are constants and that $-4 \leq x \leq 1$, find the value of a and of b.

4 On the same diagram, sketch the graphs of
(a) $y = |x-2|$,
(b) $y = 2|x-2|$,
(c) $y = 2 - |x-2|$ for the domain $-2 \leq x \leq 4$.

5 On graph paper, sketch the graphs of
(a) $y = |x+1|$,
(b) $y = |3-x|$.
Hence solve the equation $|x+1| = |3-x|$.

6 Fig. 3.26 shows part of the graph of $y = f(x)$. Copy the diagram and add the graphs of
(a) $y_1 = f(-x)$,
(b) $y_2 = f(x-1)$,
(c) $y_3 = f(1-x)$.

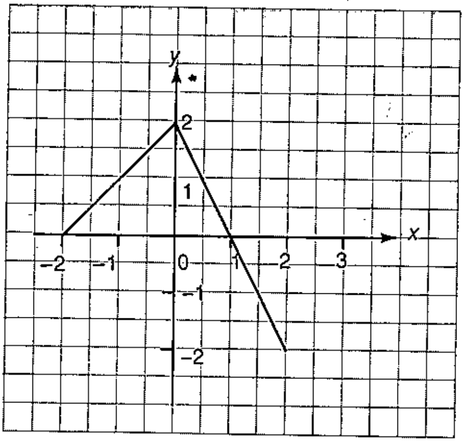


Fig. 3.26

7 Given the function $f : x \longmapsto 3x - \frac{10}{x}$, $x \neq 0$, find the value of f(2) and the values of x whose image under f is 1.

8 $g : x \longmapsto \frac{2x+3}{x-2}$, $x \neq 2$. Show that $gg(x) = x$ for all values of x except $x = 2$.

9 For the functions $f : x \longmapsto x^2 - 4$ and $g : x \longmapsto 2x + 3$, find in a similar form
(a) fg, (b) gf.
(c) Find the values of x for which fg = gf.

10 The function R maps x onto the remainder when 16 is divided by x. If the domain is $\{2, 3, 5, 7\}$, state the range. Does R^{-1} exist?

11 A function f is defined as $f : x \longmapsto \frac{x}{x+1}$, $x \neq -1$. Prove that
$f^2 : x \longmapsto \frac{x}{2x+1}$, $x \neq -1, \frac{1}{2}$. Obtain a similar expression for f^3 and hence suggest a possible expression for f^n. (C)

12 Given that $f : x \longmapsto x + 2$ and $gf : x \longmapsto x^2 + 4x + 2$, find the function g.
Hence express $x \longmapsto x^2 - 4x + 2$ in terms of f and g.

13 (a) The function $f : x \longmapsto 3x + a$ is such that $ff(6) = 10$. Find the value of a and of $f^{-1}(4)$.

(b) Functions f and g are defined by

$f : x \longmapsto \frac{8}{x-3}, x \neq 3, \; g : x \longmapsto 2x - 3$

(i) Find expressions for f^{-1}, fg and gf.

(ii) Find the value of x for which $fg(x) = gf(x)$.

(c) The function $f : x \longmapsto 2x - 5$ is defined for the domain $x \geq 1$. State the range of f and the corresponding range of ff (C)

14 Fig. 3.27 illustrates part of the function $f : x \longmapsto y$, where $y = ax + b$. Calculate the value of a and of b.

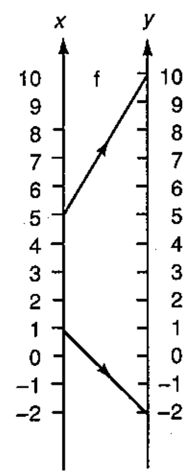


Fig. 3.27

Find the end-points of the shortest arrow that can be drawn for this function.

15 (a) Functions f and g are defined by $f : x \longmapsto 3x - 2$ and $g : x \longmapsto \frac{12}{x} - 4$ $(x \neq 0)$. Find an expression for the function (i) ff, (ii) fg, (iii) g^{-1}.

(b) The function $h : x \longmapsto x^3 + ax + b$ is such that the equation $h(x) = x$ has solutions of $x = 2$ and $x = 3$. Find the value of a and of b. (C)

16 The functions f and g are defined over the positive integers by $f : x \longmapsto 6 - 2x$ and $g : x \longmapsto \frac{2}{x}, x \neq 0$.

Express in similar form (a) fg, (b) gf, (c) f^{-1}, (d) g^{-1}, (e) $(fg)^{-1}$.
Find the value of x for which $ff(x) = gg(x)$.

17 Express in terms of the functions $f : x \longmapsto \sqrt{x}, x \geq 0$ and $g : x \longmapsto x + 5$

(a) $x \longmapsto \sqrt{x+5}, x \geq -5$ (b) $x \longmapsto x - 5$

(c) $x \longmapsto x + 10$ (d) $x \longmapsto \sqrt{x} + 10, x \geq 0$

(e) $x \longmapsto x^2 + 5$ (C)

18 Fig. 3.28 shows part of the mapping of x to y by the function $f : x \longmapsto 9x - a$ and part of the mapping of y to z by the function $g : y \longmapsto \frac{b}{12 - y}$, $y \neq 12$.

(a) Find the values of a and b.

(b) Express in similar form the function which maps an element x to an element z.

(c) Find the element x which is unchanged when mapped to z.

(C)

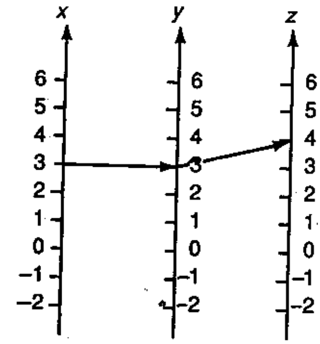


Fig. 3.28

19 Given that $f : x \longmapsto \frac{x + p}{x - 3}$ $(x \neq 3)$ and that $f(4) = 9$, find

(a) the value of p,

(b) $f^{-1}(-3)$.

(c) Obtain a similar expression for f^2.

(d) Find the value(s) of x which have the same image under f^2 and f^{-1}.

20 The function P maps x onto $\frac{18}{ax + b}$, $x \neq -\frac{b}{a}$.

(a) Given that $P(3) = 2$ and $P(-3) = -6$, find the values of a and b.

(b) Find the value of x whose image under P is $\frac{6}{5}$.

(c) Obtain a similar expression for P^{-1}.

21 (a) Given the functions $f : x \longmapsto 2x - 5$ and $g : x \longmapsto \frac{3}{x}$ (for $x \neq 0$), find in a similar form (i) fg, (ii) gf.
Hence solve the equation $fg(x) = g(x)$.

(b) Functions p and q are defined as $p : x \longmapsto \frac{1}{x + 3}$, $x \neq -3$, and $q : x \longmapsto \frac{x}{4}$.
Find in a similar form (i) $p^{-1}q$ and (ii) pq^{-1}.

(c) The function h is defined by $h : x \longmapsto \frac{tx + 9}{1 - x}$ $(x \neq 1)$.

Find the value of t for which the equation $h(x) = x$ has the solution $x = 3$.

22 The function f is defined as $f : x \longmapsto \begin{cases} 2 \text{ for } x \geq 0 \\ x + 2 \text{ for } x < 0 \end{cases}$

Sketch the graphs of f and f^{-1}.

23 If $f(x) = 3 + \frac{2}{x}$, $x \neq 0$, sketch the graph of $f(x)$ for $1 \leq x \leq 4$.
Now add a sketch of the graph of $f^{-1}(x)$ for $3\frac{1}{2} \leq x \leq 5$.

24 (a) Given that $f : x \longmapsto \frac{1}{x-2}$ $(x \neq 2)$ find $f^{-1}(x)$ and $f^2(x)$. Hence solve the equation $f^2(x) + 2f^{-1}(x) = 5$.

(b) If $g : x \longmapsto \frac{a}{x-2}$ $(x \neq 2)$, find the values of a if $g^2(-1) + 2g^{-1}(-1) = -3$.

B

25 For the domain $-3 \leq x \leq 3$, sketch the graph of $y = |\,[x]\,|$, where $[x]$ means the greatest integer less than or equal to x (for example, $[3.4] = 3$, $[-3.4] = -4$ etc). State the range of this function for this domain.

26 Draw the graph of $y = |\,1 - |\,2 - x\,|\,|$ for the domain $-3 \leq x \leq 5$.

27 Fig. 3.29 illustrates the function $y = f(x)$ over the domains $-1 \leq x \leq 0$ and $0 < x \leq 3$. The function is undefined for all other values of x. Sketch the functions given by

(a) $y_1 = f(x) + 1$,
(b) $y_2 = f(x + 1)$.

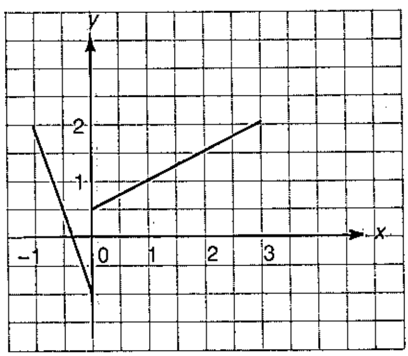


Fig. 3.29

28 f, g and h are functions defined by $f : x \longmapsto \sqrt{x}$, $g : x \longmapsto \frac{x}{2}$ and $h : x \longmapsto x + 1$. Express in terms of f, g and h:

(a) $x \longmapsto \sqrt{\frac{x+1}{2}}$ (b) $x \longmapsto \frac{(x+1)^2}{2}$

(c) $x \longmapsto 2(x + 1)^2$ (d) $x \longmapsto 2x^2 + 1$

29 The functions f and g are defined by

$f : x \longmapsto$ remainder when x^2 is divided by 7,
$g : x \longmapsto$ remainder when x^2 is divided by 5.

(a) Show that $f(5) = g(3)$.
(b) If n is an integer, prove that $f(7n + x) = f(x)$ and state the corresponding result for g. (C)

30 The function T maps (x,y) onto $(x + y, x - 2y)$.

(a) A is the point (2,1). T maps A onto B and B onto C. Find the coordinates of B and C.

(b) The point D is mapped onto E(1,7) by T. Find the coordinates of D.

(c) Another point F is mapped onto G(0,9) by T^2. Find the coordinates of F.

(d) Express T^{-1} in the same form as T.

31 Given that the range of $y = f(1 - x) - 1$ is $-2 \leq y \leq 3$, find the range of (a) $f(x)$, (b) $f(x + 1) + 1$.

The Quadratic Function

4

You have solved quadratic equations such as $x^2 - 4x - 5 = 0$ in previous work. In this Chapter we study the **quadratic function**

$$\mathbf{f}: x \longmapsto ax^2 + bx + c \quad (a \neq 0)$$

First we review some essential techniques for solving quadratic equations. These will always give two solutions or **roots**, though sometimes they may be equal.

SOLVING QUADRATIC EQUATIONS

I By factorization

This is the simplest method if it is possible. For example $x^2 - 4x - 5 = 0$ gives $(x - 5)(x + 1) = 0$ so $x = 5$ or -1. However, certain quadratic equations, like $x^2 - 4x - 4 = 0$ for example, cannot be factorized.

It is useful to remember that the **equation with roots** α and β is

$$(x - \alpha)(x - \beta) = 0.$$

II By completing the square

To solve $x^2 - 4x - 4 = 0$, we can complete the square i.e. we make the $x^2 - 4x$ part into a square.

Rewrite $x^2 - 4x$ as $(x - 2)^2 - 4$. (Check by expanding this.)

Then $x^2 - 4x - 4 = 0$ becomes $(x - 2)^2 - 8 = 0$ or $(x - 2)^2 = 8$.

Now take the square root of each side: $x - 2 = \pm\sqrt{8}$ and $x = 2 \pm \sqrt{8}$ giving $x = 4.83$ or -0.83 (correct to 2 decimal places).

III By formula

We can derive a formula for the roots of any equation as follows.

$$ax^2 + bx + c = 0$$

so $\quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Then $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

Completing the square: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$

So $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

and then $\quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

giving

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$. D is called the *discriminant*. You will find out why later.

When using the formula note carefully that it begins with $-b$ and that the denominator is $2a$.

Note: The formula is the preferred method but it is essential to know the technique of completing the square for later use.

Example 1

Solve $2x^2 - 3x - 1 = 0$.

Check that the left hand side does not factorize.
Using the formula, $a = 2$, $b = -3$, $c = -1$.

Then $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} = \frac{3 \pm \sqrt{17}}{4}$

giving $x = 1.78$ or -0.28 (2 decimal places).

Example 2

Solve $2x^2 - 3x + 4 = 0$.

$$x = \frac{3 \pm \sqrt{9 - 4(2)(4)}}{4} = \frac{3 \pm \sqrt{-23}}{4}$$

But -23 has no (real) square root. Hence the equation has no real roots. We shall see the significance of this later. Such an equation is said to have **complex** roots. We shall not however use complex numbers in our work.

GRAPH OF THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$

As you will have noticed in drawing such graphs, the graph of a quadratic function, $y = ax^2 + bx + c$, has a characteristic shape. It is a curve called a **parabola** (Fig.4.1).

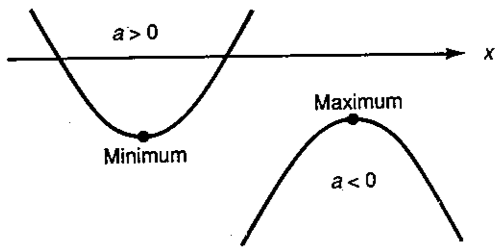


Fig.4.1

When $a > 0$, as in $2x^2 - 3x - 1$, the parabola has a **minimum** value at the bottom of the curve.

When $a < 0$, as in $1 - x - 2x^2$, the graph has a **maximum** value at the top of the curve.

The position of the curve relative to the x-axis depends on the type of the roots of the equation $f(x) = 0$. These roots are the values of x where the curve meets the x-axis.

TYPES OF ROOTS OF $ax^2 + bx + c = 0$

The roots are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$ where $D = b^2 - 4ac$.

1 If D is negative ($D < 0$ i.e. $b^2 < 4ac$), then there is no value of $\sqrt{D}$. The equation has **no real roots** and the curve does not meet the x-axis (Fig.4.2).

Fig.4.2

Example 3

What type of roots does the equation $5x^2 - 3x + 1 = 0$ have?

Using the formula, $a = 5$; $b = -3$, $c = 1$.

Then $D = (-3)^2 - 4(5)(1) = -11$.

As $D < 0$, the equation has no real roots.

II If D is positive ($D > 0$ i.e. $b^2 > 4ac$), then $\sqrt{D}$ has two values. The equation has **two different real roots** and the curve meets the x-axis at two points (Fig.4.3).

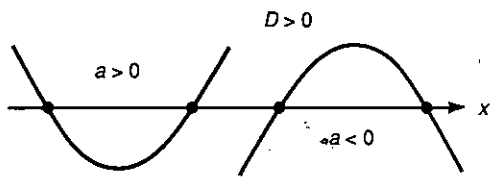


Fig.4.3

Example 4

For what values of p will the equation $x^2 + px + 9 = 0$ have two real roots?

Using the formula, $a = 1$, $b = p$, $c = 9$.

Then $D = p^2 - 36$.

For real roots, D must be > 0.

So $p^2 - 36 > 0$ i.e. $p^2 > 36$.

It follows that p must be numerically greater than 6, i.e. $p > 6$ or $p < -6$. (We could also write this as $|p| > 6$).

If D is a perfect square, $\sqrt{D}$ will be an integer. Then the roots will be *rational numbers*, i.e. fractions and whole numbers.

Example 5

What type of roots does the equation $2x^2 + 3x - 5 = 0$ have?

$D = 3^2 - 4(2)(-5) = 49$

As D is positive, the equation has two different real roots.

The roots are $\frac{-3 \pm 7}{4} = 1$ or $-\frac{5}{2}$.

The equation could have been solved by factorization.

III If D = 0 (i.e. $b^2 = 4ac$), then $x = \frac{-b}{2a}$. This means that the **roots are equal** (also called repeated or coincident roots). The curve *touches* the x-axis with the two roots merging into one (Fig.4.4).

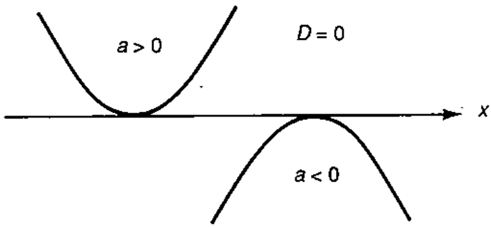


Fig.4.4

Example 6

(a) For what values of k will the x-axis be a tangent to the curve $y = kx^2 + (1 + k)x + k$*?*

(b) With these values, find the equations of the curves.

(a) On the x-axis, $y = 0$. So the roots of $kx^2 + (1 + k)x + k = 0$ must be equal if the x-axis is to be a tangent. Then $b^2 = 4ac$ where $a = k$, $b = 1 + k$ and $c = k$.

Therefore $(1 + k)^2 = 4kk = 4k^2$.

So $1 + 2k + k^2 = 4k^2$ or $3k^2 - 2k - 1 = 0$.

Solving this we get $(3k + 1)(k - 1) = 0$ giving $k = 1$ or $-\frac{1}{3}$.

(b) If $k = 1$, the equation is $y = x^2 + 2x + 1$.

If $k = -\frac{1}{3}$, the equation is $y = -\frac{x^2 - 2x + 1}{3}$.

Summarizing the conditions for the various types of roots of the equation $ax^2 + bx + c = 0$:

$D < 0$	$D \geq 0$	
$b^2 < 4ac$	$b^2 \geq 4ac$	
No real roots	Real roots	
	Different	Equal
	if $b^2 > 4ac$	if $b^2 = 4ac$

As we have learnt, D is called the **discriminant**: it discriminates between the types of roots.

Example 7

The equation $px^2 - 2(p + 3)x + p - 1 = 0$ has real roots. What is the range of values of p?

For real roots, $b^2 \geq 4ac$.

Here $a = p$, $b = -2(p + 3)$ and $c = p - 1$.

Then $[-2(p + 3)]^2 \geq 4p(p - 1)$.

Simplifying, $4(p^2 + 6p + 9) \geq 4p^2 - 4p$

or $\quad 6p + 9 \geq -p$.

Hence $7p \geq -9$ and $p \geq -\frac{9}{7}$.

Example 8

Find the range of values of p for which the line $2x - y = p$ (i)

meets the curve $x(x - y) = 4$. (ii)

The line may meet the curve at two points or touch the curve. The coordinates of these points will be the solutions of the simultaneous equations (i) and (ii).

From (i), $y = 2x - p$.

Substituting in (ii), $x(x - 2x + p) = 4$

which simplifies to $x^2 - px + 4 = 0$.

The roots of this equation are the x-coordinates of the point(s) where the line meets the curve. These must be real. So $b^2 \geq 4ac$ where $a = 1$, $b = -p$ and $c = 4$.

Then $(-p)^2 \geq 4(1)(4)$ or $p^2 \geq 16$ which gives $p \geq 4$ or $p \leq -4$.

Example 9

(a) Find the relation between m and k if the line $y = mx + k$ is a tangent to the curve $y^2 = 8x$.

(b) If $m = \frac{1}{2}$, find the equation of the tangent and the coordinates of its point of contact.

(c) Find the equations of the two tangents to this curve which pass through the point $(-3,-5)$.

(a) As in Example 8, we solve the simultaneous equations.
Substituting $y = mx + k$ in the equation of the curve:

$$(mx + k)^2 = 8x$$

Then $m^2x^2 + 2mkx + k^2 - 8x = 0$ i.e. $m^2x^2 + (2mk - 8)x + k^2 = 0$

Now this equation must have equal roots as the line is a tangent.

Then $b^2 = 4ac$ where $a = m^2$, $b = (2mk - 8)$ and $c = k^2$, so $(2mk - 8)^2 = 4m^2k^2$ or $4m^2k^2 - 32mk + 64 = 4m^2k^2$ which gives $mk = 2$, the relation required.

(b) If $m = \frac{1}{2}$, then $k = 4$. The equation of the tangent is therefore $y = \frac{x}{2} + 4$.

To find the coordinates of the point of contact, we solve this equation with that of the curve.

Then $(\frac{x}{2} + 4)^2 = 8x$ i.e. $\frac{x^2}{4} + 4x + 16 = 8x$

or $x^2 - 16x + 64 = 0$.

Hence $(x - 8)^2 = 0$ giving $x = 8$.

The corresponding value of y is $\frac{8}{2} + 4 = 8$.

Hence the coordinates of the point of contact are (8,8).

(c) As $mk = 2$, the equation of any tangent is $y = mx + \frac{2}{m}$.

If $(-3,-5)$ lies on the tangent, then $-5 = -3m + \frac{2}{m}$

which simplifies to $3m^2 - 5m - 2 = 0$.

Solving this, $(3m + 1)(m - 2) = 0$ giving $m = 2$ or $-\frac{1}{3}$.

Hence the equations are $y = 2x + 1$ and $y = -\frac{x}{3} - 6$ i.e. $x + 3y = -18$.

Exercise 4.1 *(Answers on page 616.)*

1 Without solving these equations, state the type of roots they have i.e., real, real and equal or not real:

(a) $x^2 - 10x + 25 = 0$
(b) $x^2 - 6x + 10 = 0$
(c) $x^2 = 4x + 7$
(d) $2x^2 - x + 2 = 0$
(e) $3x^2 + x = 1$
(f) $4x^2 - 20x + 25 = 0$
(g) $\frac{1}{x} + \frac{1}{x-1} = 2$
(h) $\frac{2}{x} + 1 = \frac{2}{x+1} - 1$
(i) $2x^2 = px + p^2$
(j) $ax^2 - x = a$ $(a > 0)$

2 Find the values of k if the equation $x^2 + (k - 2)x + 10 - k = 0$ has equal roots.

3 What is the largest value m can have if the roots of $3x^2 - 4x + m = 0$ are real?

4 For what values of p does the equation $x^2 - 2px + (p + 2) = 0$ have equal roots?

5 The equation $x^2 - 2x + 1 = p(x - 3)$ has equal roots. Find the possible values of p.

6 Show that the equation $a^2x^2 + ax + 1 = 0$ can never have real roots.

7 Find the values of k if the line $x + y = k$ is a tangent to the circle $x^2 + y^2 = 8$.

8 The equation $kx^2 + 2(k+a)x + k = a$ has equal roots. Express k in terms of a. Show that the line $y = k(x-3)$ is a tangent to the curve $y = k(x^2-3x+1)$ for any value of k except 0.

9 (a) Find the range of values of m for which the line $y = mx + 5$ meets the curve $y = x^2 + 9$ in two distinct points.

(b) If this line is to be a tangent, find the two possible equations of the tangent and the coordinates of the points of contact.

10 The line $y = mx + 1$ is a tangent to the curve $y^2 = 2x - 3$. Find the values of m.

11 The line $y = 2x + p$ is a tangent to the curve $x(x+y) + 12 = 0$. Find the possible values of p.

12 (a) Find the relation between m and c if the line $y = mx + c$ is a tangent to the curve $y^2 = 2x$.

(b) Hence find the equations of the two tangents to this curve which pass through the point $(2, 2\frac{1}{2})$.

13 What is the range of values of c if the line $y = 2x + c$ is to meet the curve $x^2 + 2y^2 = 8$ in two distinct points?

14 The equation $(p+3)x^2 + 2px + p = 1$ has real roots. Find the range of values of p.

15 If the equation $x^2 - (p-2)x + 1 = p(x-2)$ is satisfied by only one value of x, what are the possible values of p?

16 If the equation $x^2 - 2kx + 3k + 4 = 0$ has equal roots, find the possible values of k and solve the two equations.

17 Find the values of k for which the line $x + y = k$ is a tangent to the curve $x(x-y) + 2 = 0$.

MAXIMUM AND MINIMUM VALUES OF A QUADRATIC FUNCTION

The maximum or minimum values of the function $f(x) = ax^2 + bx + c$ are the values of $f(x)$ at the top or bottom of the curve. These are also called the **turning points** of the curve.

By completing the square, we find that $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$, where $a > 0$. Now the least or **minimum** value of this expression will be when the squared term is 0 (it cannot be negative) as the other terms are fixed. This occurs when $x = -\frac{b}{2a}$.

Hence the minimum value of $f(x) = ax^2 + bx + c$ $(a > 0)$, i.e. at the bottom of the curve, is $f(-\frac{b}{2a})$.

If $a < 0$, the turning point will be a **maximum** (the top of the curve) where $x = -\frac{b}{2a}$. This can be proved in a similar way and is illustrated in Example 11.

Example 10

What is the minimum value of $3x^2 - 4x + 1$ and for what value of x does it occur?

$f(x) = 3x^2 - 4x + 1$, $a = 3$, $b = -4$.

As $a > 0$, the minimum value of $f(x)$ occurs when $x = -\frac{b}{2a} = -\frac{-4}{6} = \frac{2}{3}$.

The minimum value $= f(\frac{2}{3}) = 3 \times \frac{4}{9} - \frac{8}{3} + 1 = -\frac{1}{3}$.

This is illustrated in Fig.4.5. The line $x = \frac{2}{3}$ through the turning point is called the **axis** of the curve and the curve is symmetrical about this line.

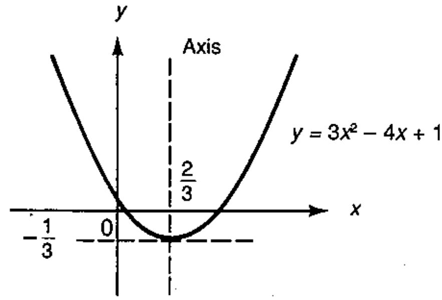


Fig. 4.5

Example 11

Express $5 - x - 2x^2$ in the form $a - b(x + c)^2$ and hence or otherwise find its maximum value and the value of x where this occurs.

$$5 - x - 2x^2 = 5 - 2\left(x^2 + \frac{x}{2}\right)$$

$$= 5 - 2\left[\left(x + \frac{1}{4}\right)^2 - \frac{1}{16}\right] \text{ by completing the square}$$

$$= 5 - 2\left(x + \frac{1}{4}\right)^2 + \frac{1}{8} = 5\frac{1}{8} - 2\left(x + \frac{1}{4}\right)^2.$$

Now the least value of $\left(x + \frac{1}{4}\right)^2$ is 0 when $x = -\frac{1}{4}$ so the **maximum** value of the expression is $5\frac{1}{8}$ when $x = -\frac{1}{4}$.

Alternatively as the question allows us to use another method (otherwise) we can use the rule stated above. Here $a = -2$, $b = -1$. Verify that the same result is obtained. This is illustrated in Fig. 4.6. The equation of the axis of summetry is $x = -\frac{1}{4}$.

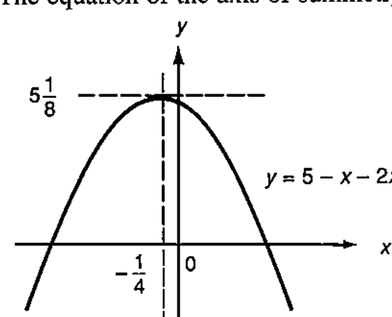


Fig. 4.6

SUMMARY

To find the maximum/minimum value of $f(x) = ax^2 + bx + c$, rewrite $f(x)$ as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and complete the square on $x^2 + \frac{bx}{a}$.

$f(x)$ is then converted to $a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$.

The turning point of $f(x)$ is a $\begin{matrix}\text{maximum if } a < 0\\ \text{minimum if } a > 0\end{matrix}$ and occurs where $x = -\frac{b}{2a}$.

The $\begin{matrix}\text{max}\\ \text{min}\end{matrix}$ value is $f(-\frac{b}{2a})$.

SKETCHING THE GRAPH OF A QUADRATIC FUNCTION

To *draw* the graph, we need a table of values. For a *sketch*, we need only know:

(1) the shape of the curve;

(2) where it cuts the y-axis. This is given by $f(0)$;

(3) the positions of the roots (if any). If $f(x)$ factorizes, the roots are easily found; otherwise, approximate values will be sufficient;

(4) the position of the turning point. Remember that the curve is symmetrical about the axis through this position.

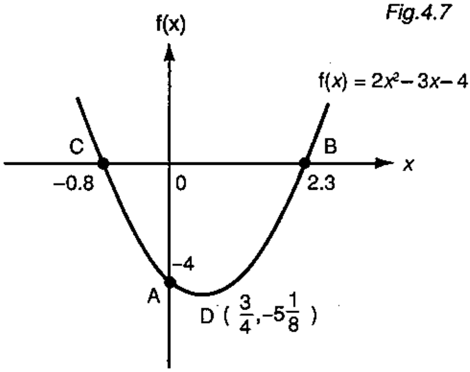
Example 12

Sketch the graph of $f(x) = 2x^2 - 3x - 4$.

(1) As $a > 0$, the shape is $\cup$.

(2) $f(0) = -4$.
This is the point A (Fig.4.7).

Fig.4.7



(3) $f(x)$ does not factorize.
The roots of $f(x) = 0$ are given by $x = \frac{3 \pm \sqrt{41}}{4} \approx \frac{3 \pm 6}{4} \approx 2.3$ and -0.8 (points B and C respectively).

(4) $f(x) = 2(x^2 - \frac{3x}{2} - 2)$
$= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16} - 2\right]$
$= 2\left(x - \frac{3}{4}\right)^2 - \frac{41}{8}$.

So the minimum is at point $D\left(\frac{3}{4}, -\frac{41}{8}\right)$.

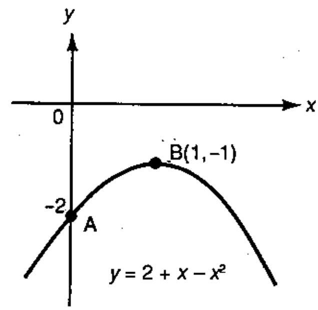
The curve can now be sketched through these points.

Example 13

Sketch the curve $y = -2 + 2x - x^2$.

(1) $a < 0$ so the shape is $\frown$.
(2) When $x = 0$, $y = -2$
(point A in Fig. 4.8).
(3) $b^2 < 4ac$ so the curve
does not meet the x-axis.
(4) $f(x) = -(x^2 - 2x + 2)$
$= -[(x-1)^2 - 1 + 2]$
$= -(x-1)^2 - 1$
So the maximum is at (1,–1) (point B).

Fig. 4.8



Example 14

Sketch the graph of $f(x) = |x^2 - x - 2|$.

To sketch this graph, we use the same method as before.
First sketch $f(x) = x^2 - x - 2$ and then reflect the negative part in the x-axis.

$x^2 - x - 2 = (x-2)(x+1)$ so the graph meets the x-axis at –1 and 2 (Fig.4.9).

It meets the y-axis at –2 and the minimum is at $\left(\frac{1}{2}, -2\frac{1}{4}\right)$.

When reflected, these values become 2 and $\left(\frac{1}{2}, 2\frac{1}{4}\right)$ respectively.

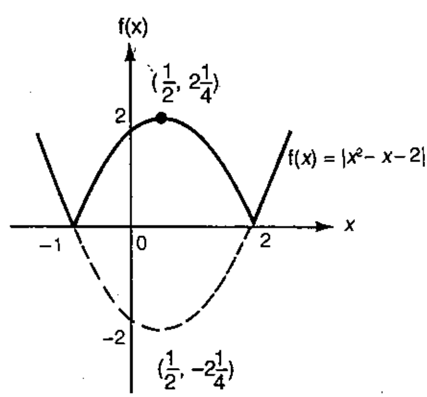


Fig. 4.9

RANGE OF A QUADRATIC FUNCTION

Example 15

Find the range of $f(x) = x^2 - 2x - 3$ *for the domain* $-2 \le x \le 5$.

At the end points, $f(-2) = (-2)^2 - 2(-2) - 3 = 5$ and $f(5) = 5^2 - 2(5) - 3 = 12$.

We might be tempted to say that the range is 5 to 12, but does the curve rise continuously from 5 to 12? It may go down to the minimum and then rise.
Verify that the minimum is -4 at $x = 1$ and sketch the curve (Fig.4.10).
The minimum lies within the domain.
So the actual range is $-4 \le f(x) \le 12$.
Hence for such questions it would be wise to make a sketch.

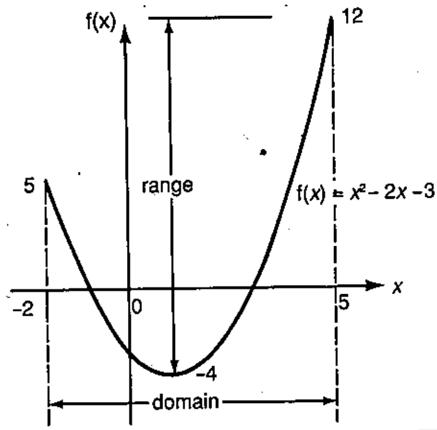


Fig. 4.10

Example 16

Sketch the graph of $3 - 5x - 2x^2$ *and state the range of the function* $f : x \longmapsto 3 - 5x - 2x^2$ *for real x.*

As $a < 0$, the curve has a maximum where $x = -\frac{b}{2a} = -\frac{(-5)}{2(-2)} = -\frac{5}{4}$.
The maximum value is $3 + \frac{25}{4} - \frac{50}{16} = \frac{49}{8} = 6\frac{1}{8}$.
Fig. 4.11 shows the sketch. As x can take any real value, the function takes values $\le 6\frac{1}{8}$ so the range is $f(x) \le 6\frac{1}{8}$.

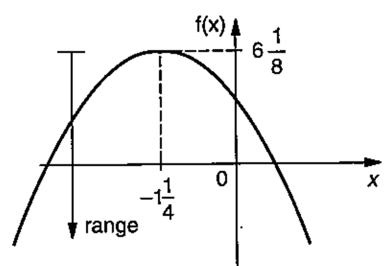


Fig. 4.11

Example 17

Find the range of the function $f(x) = |x(x-2)|$ *for the domain* $\frac{1}{2} \le x \le 2\frac{1}{2}$.

First sketch the graph (Fig.4.12).

The minimum of $x(x-2)$ is -1 at $x = 1$ which becomes a value of 1 when reflected.

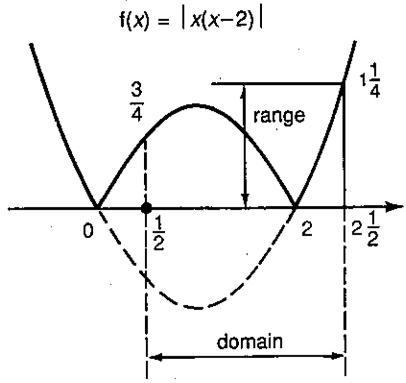


Fig.4.12

At the end points, $f(\frac{1}{2}) = \left|\frac{1}{2}(-\frac{3}{2})\right| = \frac{3}{4}$ and $f(2\frac{1}{2}) = \left|-\frac{5}{2}(\frac{1}{2})\right| = \frac{5}{4}$.

At $x = 2$ however, $f(2) = 0$.

Hence the range is $0 \le f(x) \le 1\frac{1}{4}$.

Exercise 4.2 *(Answers on page 616.)*

1 Find the maximum or minimum values of the following functions and the values of x where this occurs:

(a) $x^2 - 6x - 1$ (b) $x^2 + 2x - 3$

(c) $1 - 4x - 2x^2$ (d) $3 - x - 2x^2$

(e) $2x^2 - x - 4$ (f) $x^2 + 3$

(g) $4x^2 - 3x - 1$ (h) $5 - 2x - 4x^2$

(i) $(1 - x)(x + 2)$ (j) $x^2 + 2bx + c$

2 Sketch the graphs of the functions in Question 1 (except part (j)).

3 The graph of a quadratic function meets the x-axis where $x = 3$ and $x = k$. If the turning point of the function occurs where $x = \frac{1}{2}$, find the value of k.

4 Sketch the graph of $f(x) = |x^2 - 4x + 3|$ and find the range for the domain $0 \le x \le 2$.

5 Sketch the graph of $f(x) = |x(2 - x)|$. State the range if the domain is $-2 \le x \le 3$.

6 Sketch the graph of the function $f(x) = |x^2 + x - 2|$ and find its range for $0 \le x \le 2$.

7 Find the range of the function $y = |3 + 2x - x^2|$ for the domain $0 \le x \le 2$.

8 (a) A function V is given by $V(t) = 2t^2 - 8t + 30$. Find the minimum value of V and the value of t where this occurs.

(b) What is the range of this function for $0 \le t \le 3$?

9 Find the range of the function $f : x \longmapsto 2x^2 - 6x - 1$ for real values of x.

10 Find the range of the functions (a) $1 - 3x - x^2$ and (b) $2x^2 - x - 3$ for the domain $-1 \le x \le 2$.

11 Find the range of (a) $2x^2 - x - 3$ and of (b) $1 - 2x - x^2$ for the domain $-2 \le x \le 2$.

12 Convert $y = \frac{1}{2}[(x+4)^2 + (x-2)^2]$ to the form $y = (x+p)^2 + q$ and hence find the minimum value of y and the value of x where this occurs.

13 (a) Express $7 - x - 3x^2$ in the form $a - b(x+c)^2$, showing the values of a, b and c. Hence state the range of the function $f : x \longmapsto 7 - x - 3x^2$ for all real values of x.

(b) If the minimum value of $x^2 + 4x + k$ is -7 find the value of k.

14 The height (h m) of a ball above the ground is given by the function $h(t) = 15t - 5t^2$ where t is the time in seconds since the ball left ground-level. Find the range of the height for $1 \le t \le 3$.

15 A spot of light is made to travel across a computer screen in a straight line so that, at t seconds after starting, its distance from the left hand edge (d cm) is given by the function $d(t) = 7t - t^2 + 2$. Find the furthest distance the spot travels and how long it takes to travel this distance .

16 The function $f(x) = 1 + bx + ax^2$ has a maximum value of 4 where $x = -1$. Find the value of a and of b.

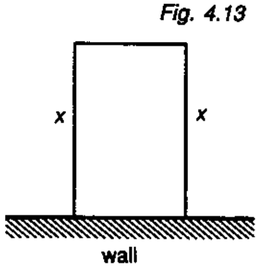
17 The function $f(x) = ax^2 + bx + c$ has a minimum value of $-5\frac{1}{4}$ where $x = \frac{1}{4}$ and $f(0) = -5$. Find the value of a, of b and of c.

18 A rectangular enclosure is made against a straight wall using three lengths of fencing, two of length x m (Fig.4.13). The total length of fencing available is 50 m.

(a) Show that the area enclosed is given by $50x - 2x^2$.

(b) Hence find the maximum possible area which can be enclosed and the value of x for this area.

Fig. 4.13



QUADRATIC INEQUALITIES

For $D > 0$ and $a > 0$, the equation $f(x) = ax^2 + bx + c = 0$ will have unequal roots. Call these α and β (where $\alpha < \beta$). Then we see from the graph of such a function (Fig.4.14) that

for $x < \alpha$, $f(x) > 0$
for $\alpha < x < \beta$, $f(x) < 0$
for $x > \beta$, $f(x) > 0$

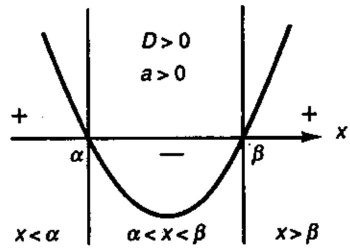


Fig. 4.14

For $D > 0$ and $a < 0$, the signs of f(x) will be reversed (Fig. 4.15),

for $x < \alpha$, $f(x) < 0$
for $\alpha < x < \beta$, $f(x) > 0$
for $x > \beta$, $f(x) < 0$

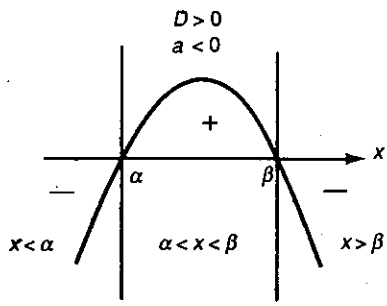


Fig.4.15

Keep the graphical illustrations in mind when dealing with such inequalities.

If, however, $D < 0$ (Fig.4.16) then

$D < 0$	f(x) is always positive if $a > 0$ f(x) is always negative if $a < 0$

(a) $D < 0, a > 0$

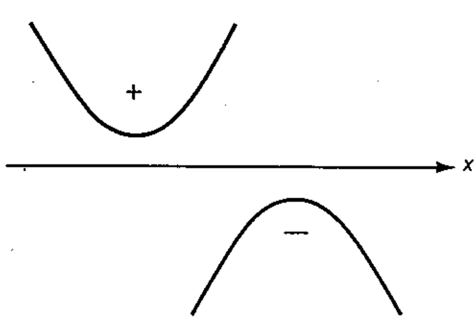


Fig. 4.16

(b) $D < 0, a < 0$

Example 18

Show that $3x^2 - 2x + 4$ is always greater than 1.

This means we have to show that $3x^2 - 2x + 3$ is always greater than 0.
Now for this function, $D = b^2 - 4ac = 4 - 36 < 0$. Therefore the function is always positive. (Similar to Fig.4.16 (a)).

Example 19

For what domain of values of x is $3x^2 - 2x \leq 1$?

This is equivalent to $3x^2 - 2x - 1 \leq 0$ i.e. $(3x + 1)(x - 1) \leq 0$.
The roots of the function are $\alpha = -\frac{1}{3}$ and $\beta = 1$.
Then $3x^2 - 2x - 1$ is 0 or negative if $-\frac{1}{3} \leq x \leq 1$ (Fig.4.17).

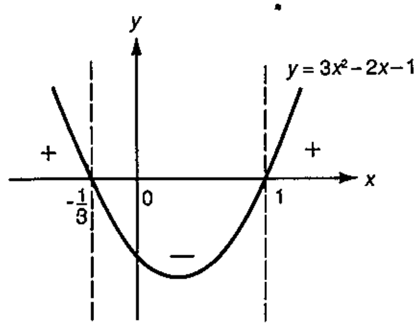
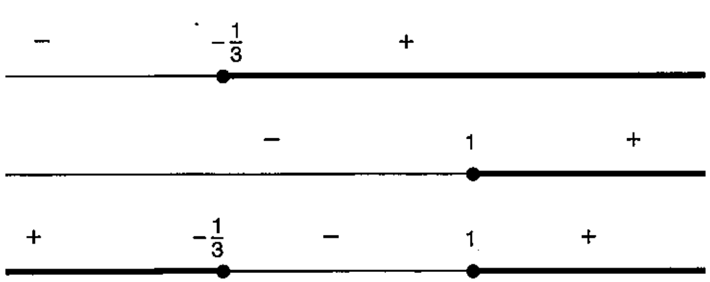


Fig. 4.17

Alternative method

This method uses the **signs** of the two factors $(3x + 1)$ and $(x - 1)$ to find the sign of the product $(3x + 1)(x - 1)$.



On the first number line, $3x + 1$ will be negative for $x < -\frac{1}{3}$, 0 at $x = -\frac{1}{3}$ and positive thereafter.
On the second number line, $(x - 1)$ will be negative if $x < 1$, 0 at $x = 1$ and positive thereafter.
The signs of the product are shown on the third number line. Hence we see that $(3x + 1)(x - 1) \leq 0$ for $-\frac{1}{3} \leq x \leq 1$ as before.

Example 20

For what range of values of p will the equation $x^2 - (p+2)x + p^2 + 3p = 3$ have real roots?

The equation is $x^2 - (p + 2)x + p^2 + 3p - 3 = 0$.

For real roots, $b^2 \geq 4ac$ where $a = 1$, $b = -(p + 2)$ and $c = p^2 + 3p - 3$.

Then $[-(p + 2)]^2 \geq 4(p^2 + 3p - 3)$
i.e. $p^2 + 4p + 4 \geq 4p^2 + 12p - 12$

which simplifies to $3p^2 + 8p - 16 \leq 0$
i.e. $(3p - 4)(p + 4) \leq 0$.

Hence, as in Fig.4.14, $-4 \leq p \leq \frac{4}{3}$.

Example 21

Find the domain of x for which $|x^2 - 3x - 7| \leq 3$.

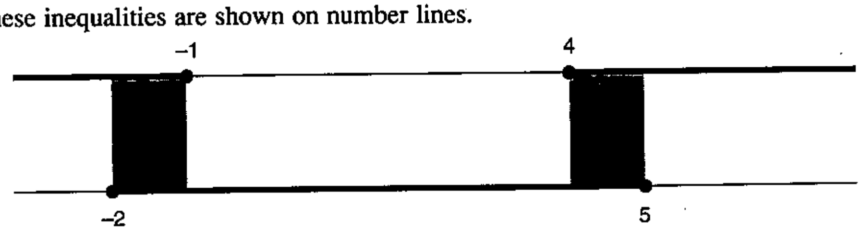
Extending the result for $|x| < k$ found in Chapter 3, if $|x^2 - 3x - 7| \leq 3$, then $-3 \leq x^2 - 3x - 7 \leq 3$.

Take these separately:

(1) $-3 \leq x^2 - 3x - 7$
gives $x^2 - 3x - 4 \geq 0$
i.e. $(x - 4)(x + 1) \geq 0$
This is true if $x \leq -1$ or $x \geq 4$.

(2) $x^2 - 3x - 7 \leq 3$
gives $x^2 - 3x - 10 \leq 0$
i.e. $(x - 5)(x + 2) \leq 0$
This is true if $-2 \leq x \leq 5$.

These inequalities are shown on number lines.



Now x must satisfy **both** sets of conditions. Hence x must lie in the shaded regions i.e. between –2 and –1 (inclusive) and between 4 and 5 (inclusive). Hence $-2 \le x \le -1$ and $4 \le x \le 5$.
This solution is shown on the graph of $y = |x^2 - 3x - 7|$ (Fig.4.18).

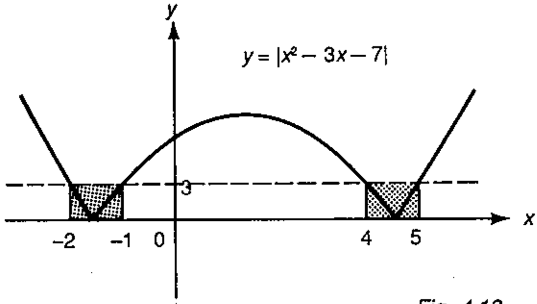


Fig. 4.18

Exercise 4.3 *(Answers on page 617.)*

1 Find the domain of x if

(a) $x^2 - x \ge 2$ (b) $x^2 + x \le 6$
(c) $x^2 + 5x > 6$ (d) $2x^2 \ge x + 1$
(e) $x(6x - 5) \le -1$ (f) $x^2 \ge 4x$
(g) $3x^2 \le x + 2$ (h) $(x + 3)(x + 1) > 24$
(i) $3x^2 < 4 - 11x$ (j) $2x^2 + 7x \ge 4$

2 If $8x^2 + 4x + k$ is never negative, find the least possible value of k.

3 Find the range of values of t if the equation $3x^2 - 3tx + (t^2 - t - 3) = 0$ has real roots.

4 If the roots of $p(x^2 + 2) = 1 - 2x$ are real, find the range of values of p.

5 If the equation $px(x - 1) + p + 3 = 0$ has real roots, find the range of values of p.

6 Find the domain of x if

(a) $|x^2 + x - 7| < 5$ (b) $|x^2 - 5x - 10| \ge 4$
(c) $|4 + x - x^2| \le 2$

7 Find the range of values of p if the roots of the equation $p^2x^2 - (p + 2)x + 1 = 0$ are real.

8 Show that the equation $(t + 3)x^2 + (2t + 5)x + (t + 2) = 0$ has real roots for all values of t.

9 Show that the equation $px^2 + (2p + 1)x + (p + 1) = 0$ has real roots for all values of p.

10 A rectangle has sides of length $(2x + 3)$ cm and $(x + 1)$ cm. What is the domain of x if the area of the rectangle lies between 10 cm² and 36 cm² inclusive?

11 If the equation $x^2 + 3 = t(x + 1)$ does not have real roots, find the range of values of t.

12 For what domain of x is $|2x^2 - x - 3| \leq 3$? Illustrate your result on a sketch of $y = |2x^2 - x - 3|$.

13 If $|(x + 3)(x - 2)| < 6$, find the domain of x and show your result on a sketch graph.

14 State the domain of x for which $3x - 2$ and $x + 3$ are
(a) both positive,
(b) both negative.
Hence find the domain of x for which $3x^2 + 7x \leq 6$.

15 The equation $px^2 + px + 2p = 3$ has real roots. Find the range of values of p.

16 The function $x^2 + 3x + k$ is never negative. Find the least whole number value of k. If $k = 4$, find the minimum value of the function.

SUMMARY

- The roots of $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Types of roots: if $b^2 > 4ac$, the roots are real and different
  if $b^2 = 4ac$, the roots are real and equal
  if $b^2 < 4ac$, the equation has no real roots.
- If $a > 0$, the function $ax^2 + bx + c$ has a minimum value;
  if $a < 0$, it has a maximum value.
- To find the maximum/minimum, write as $a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)$ and complete the square on $x^2 + \frac{bx}{a}$.
- If the roots of $f(x) = ax^2 + bx + c = 0$ are α and β (where $\alpha < \beta$), then

$$\left.\begin{array}{l} f(x) > 0 \text{ for } x < \alpha \\ f(x) < 0 \text{ for } \alpha < x < \beta \\ f(x) > 0 \text{ for } x > \beta \end{array}\right\} \text{ when } b^2 > 4ac \text{ and } a > 0 \text{ (Fig.4.19)}$$

and

$$\left.\begin{array}{l} f(x) < 0 \text{ for } x < \alpha \\ f(x) > 0 \text{ for } \alpha < x < \beta \\ f(x) < 0 \text{ for } x > \beta \end{array}\right\} \text{ when } b^2 > 4ac \text{ and } a < 0 \text{ (Fig.4.20)}$$

Fig. 4.19

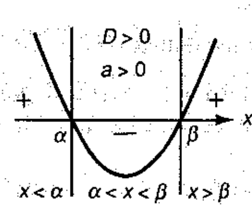
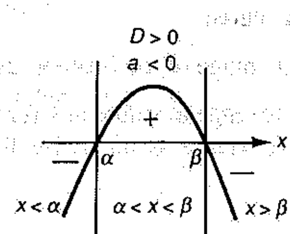


Fig. 4.20



If $b^2 < 4ac$ and $a > 0$, $f(x)$ is always > 0.
If $b^2 < 4ac$ and $a < 0$, $f(x)$ is always < 0.

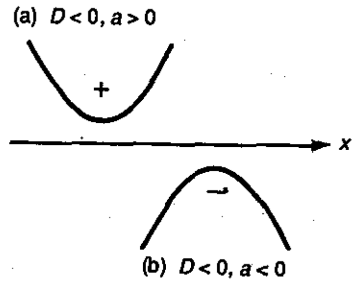


Fig.4.21

REVISION EXERCISE 4 *(Answers on page 617.)*

A

1 Without solving the following equations, state the nature of their roots:
(a) $3x^2 - x = 1$ (b) $(x + 1)(x - 2) = 5$
(c) $(1 - x) = \frac{4}{x+2}$ (d) $\frac{1}{x} + 3 = \frac{1}{x-1}$
(e) $(2x + 5)(2x + 3) = 2(6x + 7)$

2 Find the range of values of x for which $3x^2 < 10x - 3$.

3 Show that the equation $(t - 3)x^2 + (2t - 1)x + (t + 2) = 0$ has rational roots for all values of t.

4 Show that the equation $(p + 1)x^2 + (2p + 3)x + (p + 2) = 0$ has real roots for all values of p. (C)

5 The quadratic equation $x^2 + px + q = 0$ has roots -2 and 6. Find (i) the value of p and of q, (ii) the range of values of r for which the equation $x^2 + px + q = r$ has no real roots. (C)

6 Express $8 + 2x - x^2$ in the form $a - (x + b)^2$. Hence or otherwise find the range of $8 + 2x - x^2$ for $-1 \leq x \leq 5$.

7 (a) Find the range of values of x for which $6x^2 - 11x \geq 7$.
(b) Find the coordinates of the turning point of the curve $y = (2x - 3)^2 + 6$ and sketch the curve. (C)

8 Find the range of the function $2x^2 - 7x + 3$ for the domain $0 \leq x \leq 4$.

9 State the range of values of k for which $2k - 1$ and $k + 2$ are (i) both positive, (ii) both negative. Hence, or otherwise, find the range of values of k for which $2k^2 + 3k < 2$. (C)

10 (a) Find the value of p for which the line $y = 6$ is a tangent to the curve $y = x^2 + (1 - p)x + 2p$.

(b) Find the range of values of q for which the line $x + 2y = q$ meets the curve $x(x + y) + 8 = 0$. (C)

11 Find the domain of x for which $| 2x^2 - 4x - 3 | > 3$.

12 (a) The quadratic equation $kx^2 + 2(k + a)x + (k + b) = 0$ has equal roots. Express k in terms of a and b.

(b) The quadratic equation $(p + 1)x^2 + 2px + (p + 2) = 0$ has real roots. Find the range of values of p. (C)

13 The function $2ax^2 - 4x - a$ has a maximum value of 3. Find the values of a.

14 Sketch the graph of the function $| x^2 - x - 6 |$ and find its range for $0 \leq x \leq 3$.

15 The curve $y = ax^2 + bx + c$ has a maximum point at (2,18) and passes through the point (0,10). Evaluate a, b and c. (C)

16 The two shortest sides of a right-angled triangle have lengths $(x + 1)$ cm and $(x + 2)$ cm. If the area A cm^2 of the triangle is such that $15 \leq A \leq 28$, find the range of values of x.

17 The equation of a curve is $y = 4x^2 - 8x - 5$. Find

(i) the range of values of x for which $y \geq 0$,

(ii) the coordinates of the turning point of the curve.

State the coordinates of the maximum point of the curve $y = | 4x^2 - 8x - 5 |$ and sketch the curve $y = | 4x^2 - 8x - 5 |$. (C)

18 A square has side x cm and a rectangle has sides x cm and $2(x + 1)$ cm. For what range of values of x is the total area not less than 1 cm^2 and not more than 5 cm^2?

19 (a) Find the value of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has two equal roots.

(b) Show that the line $x + y = q$ will intersect the curve $x^2 - 2x + 2y^2 = 3$ in two distinct points if $q^2 < 2q + 5$. (C)

20 (a) The function $f : x \longmapsto x^2 + px + q$ is negative only between the values $x = 2$ and $x = 5$.

(i) Find the value of p and of q.

(ii) If $f(x) = -2$, find the value of x.

(b) The function $ax^2 + bx + c$ is positive only when $-\frac{3}{2} < x < 2$ and meets the y-axis where $y = 6$. Find the value of a, of b and of c.

21 Find the domain of x if $\sqrt{5x - 2 - 2x^2}$ is real.

22 $f(x) = 0.3x^2 - 0.2x$. If $0.1 \leq f(x) \leq 0.5$, find the domain of x.

B

23 Find the domain of x if $2 < \sqrt{(2x^2 + x + 3)} < 3$.

24 If the equation $x^2 + 3 = k(x + 1)$ has real roots, find the range of values of k. Hence find the two values of x for which the function $\frac{x^2 + 3}{x + 1}$ has (i) a maximum, (ii) a minimum value.

25 The function $f(x) = ax^2 + bx + c$ has a minimum value of 5 when $x = 1$ and $f(2) = 7$. Find the values of a, b and c.

26 The roots of the quadratic equation $x^2 + 2x + 3 = p(x^2 - 2x - 3)$ are real. Show that p cannot have a value between -1 and $\frac{1}{2}$.

27 The function $x^2 + px + q$ is negative for $2 < x < 4$. Find
(a) the values of p and q,
(b) the domain of x if $15 \leq x^2 + px + q \leq 48$.

28 (a) Solve the equation $x^2 + 2ax + 2 = 2a^2 + 5a$ to obtain x in terms of a.
(b) If these values of x are real, find the range of values of a.

29 If α and β are the roots of $ax^2 + bx + c = 0$, show that $\alpha + \beta = -\frac{b}{a}$ and that $\alpha\beta = \frac{c}{a}$. Hence show that $\alpha^2 + \beta^2 = \frac{b^2 - 2ac}{a^2}$ and that $\alpha - \beta = \frac{\sqrt{b^2 - 4ac}}{a}$.

Binomial Expansions

5

A **binomial** is an expression of two terms, such as $(x + y)$, $(a - b)$, etc. If the **binomial** $(a + b)$ is squared, the result is the **expansion** of $(a + b)^2$. Write down this expansion.

Now examine the pattern obtained if we expand $(a + b)^3$, $(a + b)^4$, etc.

$$(a + b)^3 = (a + b)^2(a + b)$$
$$= (a^2 + 2ab + b^2)(a + b)$$

To find this, multiply each term of $(a^2 + 2ab + b^2)$ by a, then by b and add the results.

Multiplying by a:	$a^3 + 2a^2b + ab^2$
Multiplying by b:	$a^2b + 2ab^2 + b^3$
Adding:	$a^3 + 3a^2b + 3ab^2 + b^3$

Note that the powers of a and b add up to 3 and that the coefficients are **1 3 3 1.**

Now find $(a + b)^4 = (a + b)^3(a + b)$ in the same way.
You should obtain $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
The powers add up to 4 and the coefficients are **1 4 6 4 1.**
Once more, find the expansion of $(a + b)^5$. Can you see the pattern?

1 2 1	coefficients of $(a + b)^2$
1 3 3 1	coefficients of $(a + b)^3$
1 4 6 4 1	coefficients of $(a + b)^4$
1 5 10 10 5 1	coefficients of $(a + b)^5$

Each line starts and ends with 1. Go along the $(a + b)^2$ line and add the coefficients in pairs. You will find that the sum of each pair gives the coefficient in the next line. Repeat for the other lines. Hence find the coefficients for $(a + b)^6$ and $(a + b)^7$.

Note that the coefficients are symmetrical and that the second coefficient is equal to the power of the expansion. For $(a + b)^n$ there are $(n + 1)$ terms, where n is an integer. Make a copy of the triangle up to a power of 8 to keep for reference.

This pattern is called **Pascal's Triangle** after the French mathematician Pascal (1623 – 1662) but it was known in China long before his time. By working through the triangle we can find the coefficients for any power n of $(a + b)$.

Later in this chapter, we will introduce the **Binomial Theorem** which gives a formula for the coefficients, but for most of our work the triangle will be sufficient.

Example 1

Expand $(a+b)^8$.

From the triangle the coefficients are 1 8 28 56 70 56 28 8 1.

Then $(a+b)^8 = 1a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1b^8$

Note that the powers of a decrease from 8 to 0 while the powers of b increase from 0 to 8. The sum of these powers is always 8.

$(a+b)$ is the model binomial but we can replace a or b by other expressions.

Example 2

Expand $(2x-1)^4$.

The initial coefficients are 1 4 6 4 1. Here $a = 2x$, $b = -1$.

$$\begin{aligned}\text{Then } (2x-1)^4 &= \mathbf{1}(2x)^4 + \mathbf{4}(2x)^3(-1) + \mathbf{6}(2x)^2(-1)^2 + \mathbf{4}(2x)(-1)^3 + \mathbf{1}(-1)^4 \\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

The coefficients are now quite different. The powers of x are in **descending** order.

Example 3

Find in ascending powers of x the expansion of $(2-\frac{x}{2})^6$.

The initial coefficients are 1 6 15 20 15 6 1. The expansion is

$$\begin{aligned}&2^6 + \mathbf{6}(2^5)\left(-\tfrac{x}{2}\right) + \mathbf{15}(2^4)\left(-\tfrac{x}{2}\right)^2 + \mathbf{20}(2^3)\left(-\tfrac{x}{2}\right)^3 + \mathbf{15}(2^2)\left(-\tfrac{x}{2}\right)^4 + \mathbf{6}(2)\left(-\tfrac{x}{2}\right)^5 + \left(-\tfrac{x}{2}\right)^6 \\ &= 64 - 6(2^4)x + 15(2^2)x^2 - 20x^3 + 15\left(\tfrac{x^4}{2^2}\right) - 6\left(\tfrac{x^5}{2^4}\right) + \tfrac{x^6}{2^6} \\ &= 64 - 96x + 60x^2 - 20x^3 + \tfrac{15x^4}{4} - \tfrac{3x^5}{8} + \tfrac{x^6}{64}\end{aligned}$$

Exercise 5.1 *(Answers on page 618.)*

1 Find, in descending powers of x, the expansions of:

(a) $(x-2)^4$ (b) $(2x-3)^3$ (c) $(2x+1)^5$

(d) $\left(x-\frac{1}{2}\right)^5$ (e) $\left(x+\frac{1}{x}\right)^6$ (f) $\left(\frac{x}{4}-2\right)^4$

2 Expand, in ascending powers of x:

(a) $(1-2x)^5$ (b) $(2-3x)^4$ (c) $\left(2-\frac{x}{2}\right)^6$ (d) $(1-x^2)^3$

3 Find, in ascending powers of x, the first four terms in the expansion of:

(a) $(2-x)^5$ (b) $(1-2x)^7$ (c) $\left(1-\frac{x}{2}\right)^8$ (d) $\left(4-\frac{x}{2}\right)^5$

4 Find the expansions of (a) $(3x - 2y)^4$ (b) $\left(x - \frac{1}{x}\right)^5$.

5 Expand $(a + b)^5$. If $a = \frac{3}{4}$ and $b = \frac{1}{4}$, find the value (as a fraction) of the fourth term of the expansion.

6 Write down the first four terms of the expansion of $(1 - x)^6$ in ascending powers of x. Using these terms, find an approximate value of $(0.99)^6$.

7 (a) Write down the expansions of $(1 + x)^3$ and $(1 - x)^3$.
(b) Hence simplify $(1 + x)^3 + (1 - x)^3$. Use your result to find the exact value of $(1 + \sqrt{2})^3 + (1 - \sqrt{2})^3$.

8 By using the expansions of $(2 + x)^4$ and $(2 - x)^4$, find the exact value of $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.

9 (a) Write down the expansions of $(1 + x)^4$ and $(1 - x)^4$.
(b) Hence simplify the expression $(1 + x)^4 - (1 - x)^4$. Use your result to find the value of $1.01^4 - 0.99^4$.

10 (a) Obtain the expansions of $\left(x + \frac{1}{x}\right)^5$ and $\left(x - \frac{1}{x}\right)^5$.
(b) Hence simplify $\left(x + \frac{1}{x}\right)^5 - \left(x - \frac{1}{x}\right)^5$.
(c) Choosing a suitable value of x, find the value of $2.5^5 - 1.5^5$.

Example 4

(a) Expand $(1 + y)^4$ in ascending powers of y.
(b) Hence find the expansion of $(1 + x - x^2)^4$ as far as the term in x^3.

(a) $(1 + y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4$

(b) Now substitute $(x - x^2)$ for y.

$$\begin{aligned}(1 + x - x^2)^4 &= 1 + 4(x - x^2) + 6(x - x^2)^2 + 4(x - x^2)^3 + (x - x^2)^4 \\ &= 1 + 4x - 4x^2 + 6(x^2 - 2x^3 ...) + 4(x^3 ...) + ... \\ &\qquad \text{(where we do not keep any terms higher than } x^3\text{)} \\ &= 1 + 4x - 4x^2 + 6x^2 - 12x^3 + 4x^3 \quad \text{(up to the term in } x^3\text{)} \\ &= 1 + 4x + 2x^2 - 8x^3 \quad \text{(up to the term in } x^3\text{)}\end{aligned}$$

Example 5

(a) Find, in ascending powers of x, the expansions of $(1 - 2x)^3$ and $(2 + x)^4$.
(b) Hence find the first four terms of the expansion of $(1 - 2x)^3(2 + x)^4$.

(a)
$$\begin{aligned}(1 - 2x)^3 &= 1 + 3(-2x) + 3(-2x)^2 + (-2x)^3 \\ &= 1 - 6x + 12x^2 - 8x^3\end{aligned}$$

$$\begin{aligned}(2 + x)^4 &= 2^4 + 4(2^3)(x) + 6(2^2)(x^2) + 4(2)(x^3) + x^4 \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

(b) $(1-2x)^3(2+x)^4 = (1-6x+12x^2-8x^3)(16+32x+24x^2+8x^3+x^4)$

The first four terms will go up to the power of x^3. So we multiply the terms in the first bracket by 16, $32x$, $24x^2$ and $8x^3$ and leave out any terms higher than x^3.

Multiplying by 16	$16-96x+192x^2-128x^3$
Multiplying by $32x$	$32x-192x^2+384x^3$
Multiplying by $24x^2$	$24x^2-144x^3$
Multiplying by $8x^3$	$8x^3$
Adding	$16-64x+24x^2+120x^3$

Example 6

(a) Find the terms in x^3 and x^4 in the expansion of $(3-\frac{x}{3})^6$ in ascending powers of x.

(b) Hence find the coefficient of x^4 in the expansion of $(1-\frac{x}{2})(3-\frac{x}{3})^6$.

(a) $(3-\frac{x}{3})^6 = 3^6 + 6(3^5)(-\frac{x}{3}) + 15(3^4)(-\frac{x}{3})^2 + 20(3^3)(-\frac{x}{3})^3 + 15(3^2)(-\frac{x}{3})^4 \ldots$

So the x^3 term is $-20x^3$ and the x^4 term is $+\frac{5x^4}{3}$.

(b) Then $(1-\frac{x}{2})(3-\frac{x}{3})^6 = (1-\frac{x}{2})(\ldots-20x^3+\frac{5x^4}{3}\ldots)$

The term in x^4 is found by multiplying the relevant terms as shown, and is $10x^4 + \frac{5x^4}{3}$ giving a coefficient of $\frac{35}{3}$.

Example 7

Write down and simplify the first three terms in the expansions (in ascending powers of x) of (a) $(1-\frac{3x}{2})^5$ and (b) $(2+x)^5$.

Hence find the coefficient of x^2 in the expansion of $(2-2x-\frac{3x^2}{2})^5$.

(a) $(1-\frac{3x}{2})^5 = 1 + 5(-\frac{3x}{2}) + 10(-\frac{3x}{2})^2 \ldots = 1 - \frac{15x}{2} + \frac{45x^2}{2}$

(b) $(2+x)^5 = 2^5 + 5(2^4)(x) + 10(2^3)(x^2) \ldots = 32 + 80x + 80x^2$

We notice that $(2-2x-\frac{3x^2}{2})^5$ is the product of (a) and (b)

$$= [(1-\frac{3x}{2})(2+x)]^5$$

$$= [1-\frac{15x}{2}+\frac{45x^2}{2}\ldots][32+80x+80x^2\ldots]$$

The term in x^2 will be the sum of the products linked together, so the coefficient of x^2 is $80 - (\frac{15}{2}\times 80) + (\frac{45}{2}\times 32) = 200$.

Example 8

Find, in ascending powers of x, the first three terms in the expansions of
(a) $(1+2x)^5$ *and* *(b)* $(1+px)^4$.
(c) If the coefficient of x^2 *in the expansion of* $(1+2x)^5(1+px)^4$ *is* -26*, find the value of* p.

(a) The first three terms of $(1 + 2x)^5$ are $1 + 5(2x) + 10(2x)^2 = 1 + 10x + 40x^2$.

(b) The first three terms of $(1 + px)^4$ are $1 + 4(px) + 6(px)^2 = 1 + 4px + 6p^2x^2$.

(c) $(1 + 2x)^5(1 + px)^4 = (1 + 10x + 40x^2)(1 + 4px + 6p^2x^2)$

We only require the term in x^2 so we pick out the terms (linked together above) whose products produce x^2:

$$1 \times 6p^2x^2 = 6p^2x^2$$
$$10x \times 4px = 40px^2$$
$$40x^2 \times 1 = 40x^2$$

giving $(6p^2 + 40p + 40)x^2$.
Hence $6p^2 + 40p + 40 = -26$
i.e. $3p^2 + 20p + 33 = 0$ or $(3p + 11)(p + 3) = 0$
and so $p = -\frac{11}{3}$ or -3.

Example 9

(a) Find the first three terms in the expansion of $(1-3x)^5$ *in ascending powers of* x.
(b) If the first three terms in the expansion of $(p+qx)(1-3x)^5$ *are* $3+rx+300x^2$*, state the value of* p *and find the values of* q *and* r.

(a) The first three terms of $(1 - 3x)^5$ are $1 + 5(-3x) + 10(-3x)^2 = 1 - 15x + 90x^2$.

(b) The first three terms of $(p + qx)(1 - 3x)^5$ will come from $(p + qx)(1 - 15x + 90x^2)$.
The first term is p so $p = 3$.
The term in x is $qx - 15px$ so $q - 15p = r$ (i)
The term in x^2 is $90px^2 - 15qx^2$ so $90p - 15q = 300$ (ii)

From (ii), $270 - 15q = 300$ so $q = -2$.
From (i), $r = -2 - 45 = -47$.

Example 10

(a) Expand $(1 - \frac{x}{2})^4$ in ascending powers of x.
(b) If this expansion is used to find the exact value of $(0.995)^4$, what value should be taken for x?
(c) Using this value, find $(0.995)^4$.

(a) The coefficients in the expansion of $(a + b)^4$ are 1 4 6 4 1 and, in this case, $a = 1, b = -\frac{x}{2}$.
All the powers of $a = 1$.
Then $(1 - \frac{x}{2})^4 = 1 + 4(-\frac{x}{2}) + 6(-\frac{x}{2})^2 + 4(-\frac{x}{2})^3 + (-\frac{x}{2})^4$
$= 1 - 2x + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{16}$

(b) If $1 - \frac{x}{2} = 0.995$, then $\frac{x}{2} = 0.005$ and $x = 0.01$.

(c) Substitute $x = 0.01$ in the expansion.
$(0.995)^4 = 1 - 2(0.01) + \frac{3(0.01)^2}{2} - \frac{(0.01)^3}{2} + \frac{(0.01)^4}{16}$
Writing the positive and negative terms separately:

positive	*negative*
1	$-2(0.01) = -0.02$
$\frac{3(0.01)^2}{2} = 0.000\ 15$	$-\frac{(0.01)^3}{2} = -0.000\ 000\ 5$
$\frac{(0.01)^4}{16} = 0.000\ 000\ 000\ 625$	
1.000 150 000 625	$-0.020\ 000\ 5$

which gives a sum of

$$\begin{array}{r} 1.000\ 150\ 000\ 625 \\ -0.020\ 000\ 5 \\ \hline 0.980\ 149\ 500\ 625 \end{array}$$

This is the exact value of $(0.995)^4$.
Compare this value with that obtained by using a calculator.

Exercise 5.2 *(Answers on page 619.)*

1 Write down the expansion of $(1 - x)^4$. Use your result to find the expansion of $(1 - x + \frac{x^2}{2})^4$ in ascending powers of x as far as the term in x^2.

2 Use the expansion of $(1 + x)^3$ to find the first three terms in the expansion of $(1 + \frac{x}{2} - x^2)^3$ in ascending powers of x.

3 Find the first three terms in the expansions in ascending powers of x of (a) $(2 - x)^4$ and (b) $(3 - \frac{x}{2})^4$. Hence find the coefficients of x and x^2 in the expansion of $(6 - 4x + \frac{x^2}{2})^4$.

4 (a) Write down the expansion of $(1 + x)^5$ in ascending powers of x as far as the term in x^3.
(b) Hence find the first four terms in the expansion of $(1 + x - x^2)^5$.

5 Expand in ascending powers of x, (a) $(1 + 2x)^4$ and (b) $(1 - x)^3$.
Hence find the first three terms in the expansion of $(1 + 2x)^4(1 - x)^3$.

6 Write down the expansions of $(1 + 2x)^3$ and $(2 - \frac{x}{2})^4$ in ascending powers of x. Hence find the coefficient of the term in x^2 in the expansion of $(1 + 2x)^3(2 - \frac{x}{2})^4$.

7 Find the coefficient of x^3 in the expansion of $(1 - 2x)^3(1 + \frac{3x}{2})^4$.

8 Expand each of the binomials $(1 + x)^5$ and $(2 - x)^5$ as far as the term in x^3. Hence find the coefficient of x^3 in the expansion of $(2 + x - x^2)^5$.

9 In the expansion of $(a + bx)^4$ in ascending powers of x, the first two terms are $16 - 96x$. Find the values of a and b.

10 The coefficient of the third term in the expansion of $(ax - \frac{1}{x})^5$ in descending powers of x is 80. Find the value of a.

11 (a) Expand $(1 + ax)^3$ and $(b + x)^4$ in ascending powers of x.
(b) If the first two terms in the expansion of $(1 + ax)^3(b + x)^4$ are $16 - 64x$, state the value of b, where $b > 0$, and find the value of a.

12 In the expansion of $(p + qx)^4$ in ascending powers of x, the first two terms are $16 - \frac{8x}{3}$. Find the values of p (> 0) and q. Hence find the third term in the expansion.

13 (a) Expand $(1 + px)^4$ and $(1 + qx)^3$ as far as the terms in x^2.
(b) Given that the coefficient of x^2 in the expansion of $(1 + px)^4(1 + qx)^3$ is -6 and that $p + q = 1$, find the values of p and q.

14 (a) State the expansions of (i) $(1 + ax)^3$ and (ii) $(1 + bx)^4$ in ascending powers of x.
(b) If the second and third terms in the expansion of $(1 + ax)^3(1 + bx)^4$ are $5x$ and $3x^2$ respectively, find the values of a and b.

15 (a) Find the coefficients of x^4 and x^5 in the expansion of $(2x - \frac{1}{2})^7$.
(b) Hence find the coefficient of x^5 in the expansion of $(\frac{x}{5} - 2)(2x - \frac{1}{2})^7$.

16 Find the coefficients of x^3 and x^4 in the expansion of $(\frac{2}{3} - x)^6$.
Hence find the coefficient of x^4 in the expansion of $(1 + 3x)(\frac{2}{3} - x)^6$.

17 Write down
(a) the first four terms in the expansion of $(1 - 2x)^4$, and
(b) the first three terms in the expansion of $(1 - x)^8$.
If the sum of the terms in (a) equals the sum of the terms in (b) where $x \neq 0$, find the value of x.

18 State the first three terms in the expansion of $(1 + x)^4$ and hence find the first three terms in the expansion of $(1 + ax + bx^2)^4$. If these are $1 + 8x + 12x^2$, find the values of a and b.

19 If the expansion of $(1 - x - x^2)^{10}$ is used to find the value of $(0.89)^{10}$, what value of x should be substituted?

20 Write down the first three terms in the expansion of $(1 - x)^8$ in ascending powers of x. Use this expansion to find the value of $(0.999)^8$ correct to 5 significant figures.

THE BINOMIAL THEOREM

The expansion of $(a + b)^n$ is given in full by a formula known as the **Binomial Theorem.** The formula is as follows:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where $\binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$, and

$r!$ is **factorial r** and $r! = 1 \times 2 \times 3 \times 4 \times \ldots \times r$.

For example, $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, $5! = 1 \times 2 \times 3 \times 4 \times 5$ etc.

Hence $\binom{n}{1} = \frac{n}{1}$, $\binom{n}{2} = \frac{n(n-1)}{1 \times 2}$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}$ and so on.

There are $(n + 1)$ terms in the expansion of $(a + b)^n$. The coefficients of the expansion are

	1	$\binom{n}{1}$	$\binom{n}{2}$	...	$\binom{n}{r}$	...	1
term	1st	2nd	3rd		$(r + 1)$th		$(n + 1)$th

The first and last coefficients are always 1 when n is a positive integer (which it always will be in our work). Note that the coefficient for the $(r + 1)$th term is $\binom{n}{r}$.

Some calculators give the numerical value of $\binom{n}{r}$ (shown as ${}^nC_r$) but the formula needs to be known for algebraic terms.

Example 11

Show that $\binom{10}{3} = \binom{10}{7}$.

$\binom{10}{3} = \dfrac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$

$\binom{10}{7} = \dfrac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 120$ after cancelling $4 \times 5 \times 6 \times 7$.

We could have expected this as the coefficients of $(a + b)^{10}$ are symmetrical.

This is an example of a general rule: $\binom{n}{r} = \binom{n}{n-r}$

Hence to find, say $\binom{12}{8}$ it is easier and quicker to find $\binom{12}{4}$.

Example 12

Using the theorem, find the coefficients in the expansion of $(a + b)^7$.

The coefficients are $1, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \binom{7}{4}, \binom{7}{5}, \binom{7}{6}$ and 1.

$\binom{7}{1} = \frac{7}{1} = 7$; $\binom{7}{2} = \frac{7 \times 6}{1 \times 2} = 21$; $\binom{7}{3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$;

$\binom{7}{4} = \binom{7}{3} = 35$; $\binom{7}{5} = \binom{7}{2} = 21$; $\binom{7}{6} = \binom{7}{1} = 7$

So the coefficients are 1, 7, 21, 35, 35, 21, 7 and 1 as we have found from Pascal's triangle.

Pascal's Triangle is the easier and quicker way of finding coefficients provided n is not too large. If n is large or is not known, then the Binomial Theorem must be used. The theorem is essential in more advanced work when n may not be a positive integer.

Example 13

Find the first four terms in the expansion of $(x-2)^{12}$.

Here $a = x$, $b = -2$ and $n = 12$.

The first four terms are

$$x^{12} + \binom{12}{1}x^{11}(-2) + \binom{12}{2}x^{10}(-2)^2 + \binom{12}{3}x^9(-2)^3$$

$$= x^{12} + \frac{12}{1}x^{11}(-2) + \frac{12\times 11}{1\times 2}x^{10}(4) + \frac{12\times 11\times 10}{1\times 2\times 3}x^9(-8)$$

$$= x^{12} - 24x^{11} + 264x^{10} - 1760x^9$$

Example 14

Find the 5th and 6th terms in the expansion of $(2x - \frac{1}{2})^{10}$.

Here $a = 2x$, $b = -\frac{1}{2}$ and $n = 10$.

The 5th term is given by $r = 4$ and the 6th term by $r = 5$.

The 5th term $= \binom{10}{4}(2x)^{10-4}(-\frac{1}{2})^4 = \frac{10\times 9\times 8\times 7}{1\times 2\times 3\times 4}\ (2x)^6(\frac{1}{2})^4\ = 840x^6$.

Verify that the 6th term $= \frac{10\times 9\times 8\times 7\times 6}{1\times 2\times 3\times 4\times 5}\ (2x)^5\ (-\frac{1}{2})^5 = -252x^5$.

Example 15

Write down (without simplifying) the first three terms in the expansion of $(x + b)^n$ where n is a positive integer. If the coefficients of the second and third terms are -8 and 30 respectively, find the values of n and b.

$(x+b)^n = x^n + \binom{n}{1}x^{n-1}b + \binom{n}{2}x^{n-2}b^2$

Hence the coefficients of the second and third terms are nb and $\frac{n(n-1)}{1\times 2}b^2$ respectively.

Then $\quad nb = -8 \quad$ (i)

and $\quad \frac{n(n-1)}{2}b^2 = 30$ i.e. $n(n-1)b^2 = 60 \quad$ (ii)

We solve these equations for n and b.

In (ii), substitute $b = \frac{-8}{n}$,

$$n(n-1)\frac{64}{n^2} = 60 \quad \text{or} \quad \frac{n-1}{n} = \frac{60}{64}$$

Then $64n - 64 = 60n$ from which we find $n = 16$.

From (i), $b = \frac{-8}{16} = -\frac{1}{2}$.

Example 16

Find the term independent of x in the expansion of $(2x - \frac{1}{x})^{10}$.

From the theorem, the **$(r + 1)$th** term is

$\binom{10}{r}(2x)^{10-r}(-\frac{1}{x})^r = \binom{10}{r}2^{10-r}x^{10-r}(-1)^r(\frac{1}{x})^r$.

If this term is to be independent of x, then the x's must cancel i.e. the powers of x in the numerator and denominator must be equal.

Then $10 - r = r$ or $r = 5$.

So the **6th** term is independent of x.

This term is therefore $\binom{10}{5}2^5(-1)^5 = \frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5} \times (-32) = -8064$.

Exercise 5.3 *(Answers on page 619.)*

1 Find the value of (a) $3!$, (b) $4!$, (c) $\frac{9!}{6!}$ (d) $\frac{12!}{4! \times 8!}$.

2 Find the value of (a) $\binom{6}{2}$, (b) $\binom{9}{2}$, (c) $\binom{12}{8}$, (d) $\binom{15}{12}$.

3 What is the value of x if $\binom{11}{3x} = \binom{11}{x^2 - 7}$?

4 Write down and simplify the first three terms of (a) $(1 + x)^{10}$, (b) $(x - \frac{1}{2})^{12}$, (c) $(x - \frac{1}{x})^9$.

5 For the following expansions, find
(a) the coefficient of the ninth term in $(2x - 1)^{12}$;
(b) the coefficient of the fourth term in $(1 - 3x)^{10}$;
(c) the coefficient of the fifth term in $(x - \frac{1}{x})^9$.

6 The coefficient of the second term in the expansion of $(1 + 2x)^n$ in ascending powers of x is 40. Find the value of n.

7 If the first three terms in the expansion of $(1 + ax)^n$ in ascending powers of x are $1 + 6x + 16x^2$, find the values of n and a.

8 In the expansion of $(1 + px)^n$ in ascending powers of x, the second term is $18x$ and the third term is $135x^2$. Find the values of n and p.

9 Find the term independent of x in the expansion of $(x - \frac{1}{2x^2})^9$.

10 If the ratio of the 5th to the 6th term in the expansion of $(a + \frac{1}{x})^{11}$ is $5x : 1$, find the value of a.

SUMMARY

- The coefficients in the expansion of $(a + b)^n$, where n is a positive integer can be found from Pascal's Triangle:

$$\begin{array}{ccccccccccc} & & & 1 & & 2 & & 1 & & & \qquad (a+b)^2 \\ & & 1 & & 3 & & 3 & & 1 & & \qquad (a+b)^3 \\ & 1 & & 4 & & 6 & & 4 & & 1 & \qquad (a+b)^4 \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \qquad (a+b)^5 \end{array}$$

etc.

- The powers of a decrease from n to 0, the powers of b increase from 0 to n. The sum of these powers is always n.

- Alternatively, the expansion of $(a + b)^n$ can be found using the Binomial Theorem, where n is a positive integer:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where $\binom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$, and $r! = 1 \times 2 \times 3 \times \ldots \times r$.

REVISION EXERCISE 5 *(Answers on page 619.)*

A

1. Find, in ascending powers of x, the first four terms in the expansion of (i) $(1 - 3x)^5$, (ii) $(1 + 5x)^7$. Hence find the coefficient of x^2 in the expansion of $(1 - 3x)^5(1 + 5x)^7$. (C)

2. Obtain the first three terms in the expansion of $(a + \frac{x}{b})^6$ in ascending powers of x. If the first and third terms are 64 and $\frac{80x^2}{3}$ respectively, find the values of a and b and the second term.

3. Find the first three terms in the expansion of $(1 - 2x)^5$ in ascending powers of x, simplifying the coefficients.
   Given that the first three terms in the expansion of $(a + bx)(1 - 2x)^5$ are $2 + cx + 10x^2$, state the value of a and hence find the value of b and of c. (C)

4. (a) Expand $(1 + 2x)^5$ and $(1 - 2x)^5$ in ascending powers of x.
   (b) Hence reduce $(1 + 2x)^5 - (1 - 2x)^5$ to its simplest form.
   (c) Using this result, evaluate $(1.002)^5 - (0.998)^5$.

5. Find, in ascending powers of t, the first three terms in the expansions of (i) $(1 + \alpha t)^5$, (ii) $(1 - \beta t)^8$. Hence find, in terms of α and β, the coefficient of t^2 in the expansion of $(1 + \alpha t)^5(1 - \beta t)^8$. (C)

6. The first three terms in the expansion of $(1 + \frac{x}{p})^n$ in ascending powers of x are $1 + x + \frac{9x^2}{20}$. Find the values of n and p.

7 Write down and simplify the expansion of $(1-p)^5$. Use this result to find the expansion of $(1-x-x^2)^5$ in ascending powers of x as far as the term in x^3. Find the value of x which would enable you to estimate $(0.9899)^5$ from this expansion. (C)

8 Find which term is independent of x in the expansion of $(x-\frac{1}{3x^2})^{15}$.

9 Obtain and simplify
(i) the first four terms in the expansion of $(2+x^2)^6$ in ascending powers of x,
(ii) the coefficient of x^4 in the expansion of $(1-x^2)(2+x^2)^6$. (C)

10 In the expansion of $(1-x)^{10}$, the sum of the first three terms is $\frac{4}{5}$ when a certain value of x is substituted. Find this value of x.

11 Evaluate the coefficients of x^5 and x^4 in the binomial expansion of $(\frac{x}{3}-3)^7$. Hence evaluate the coefficient of x^5 in the expansion of $(\frac{x}{3}-3)^7(x+6)$. (C)

12 If the first three terms in the expansion of $(1+kx)^n$ in ascending powers of x are $1-6x+\frac{33x^2}{2}$, find the values of k and n.

13 Find, in ascending powers of x, the first three terms in the expansion of $(1+ax)^6$. Given that the first two non-zero terms in the expansion of $(1+bx)(1+ax)^6$ are 1 and $\frac{-21x^2}{4}$, find the possible value of a and of b. (C)

14 Find the ratio of the 6th term to the 8th term in the expansion of $(2x+3)^{11}$ when $x=3$.

15 In the expansion of $(1+px)(1+qx)^4$ in ascending powers of x, the coefficient of the x term is -5 and there is no x^2 term. Find the value of p and of q.

16 If the fifth term in the expansion of $(x+\frac{1}{x})^n$ is independent of x find the value of n.

B

17 In the expansion of $(x^2+\frac{2}{x^3})^7$, find which term will have the form $\frac{A}{x}$ where A is an integer. Hence find the value of A.

18 The first three terms in the expansion of $(1+x+ax^2)^n$ are $1+7x+14x^2$. Find the values of n and a.

19 (a) Obtain the expansions of $(1+x)^5$ and $(1+x^2)^5$ in ascending powers of x.
(b) Show that $(1+x)(1+x^2)=1+x+x^2+x^3$.
(c) Hence find the first four terms in the expansion of $(1+x+x^2+x^3)^5$ in ascending powers of x.

20 For what value of x is the fifth term of $(1+2x)^{10}$ equal to the sixth term of $(2+x)^8$?

21 Show that (a) $(x-\frac{1}{x})^3 = x^3-\frac{1}{x^3}-3(x-\frac{1}{x})$ and
(b) $(x-\frac{1}{x})^5 = x^5-\frac{1}{x^5}-5(x^3-\frac{1}{x^3})+10(x-\frac{1}{x})$.
Hence show that $x^5-\frac{1}{x^5} = p^5+5p^3+5p$ where $p=x-\frac{1}{x}$.

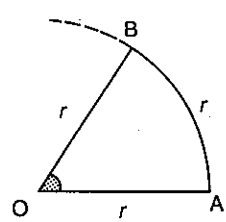
Radians, Arcs and Sectors

6

The practical unit of measurement for angles is the degree (°) which is $\frac{1}{360}$ th of a complete revolution. The number 360 comes from Babylonian times but it is an arbitrary choice. There is another system of measurement called **circular** or **radian** measure which is more suitable for further mathematics, particularly in Calculus. This system does not depend on the choice of any particular number.

RADIAN MEASURE

Fig. 6.1a



In a circle of radius r, centre O, we take an arc AB also of length r (Fig. 6.1a). Then the angle AOB is the *unit of radian measurement*, one **radian**.

In Fig. 6.1b, for example, arc PQ = $\frac{1}{2}r$ so $\angle POQ = \frac{1}{2}$ radian. If arc PR = $2r$, then $\angle POR$ is 2 radians, and so on. If the arc is kr then the angle subtended is k radians. Note that the size of 1 radian does not depend on the length of r or on any arbitrary number.

So we define 1 radian thus:

$1 - x - x^2 = 0.95$

$0.5 = x + x^2$

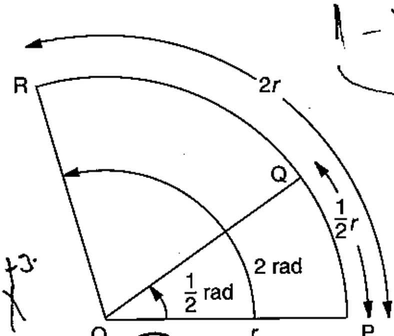


Fig. 6.1b

One radian is the angle made by an arc of length equal to the radius.

θ radians is sometimes written θ rad or θ^r or θ^c but normally just as θ. So we write $\sin\theta$ meaning $\sin(\theta$ radians). If degree measure is used, the degree symbol $\circ$ **must** be written.

Now the circumference of a circle of radius r has length $2\pi r$ (Fig.6.2).

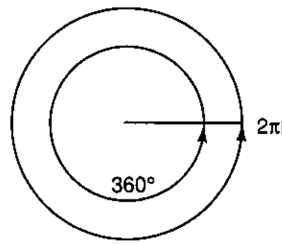


Fig. 6.2

So a complete revolution is 2π radians = 360°.

Therefore $\pi \text{ rad} = 180°$

Hence $\frac{\pi}{2}$ rad = 90°, $\frac{\pi}{4}$ rad = 45°, etc.

$x \text{ rad} = \left(\frac{180x}{\pi}\right)^\circ$ and $x^\circ = \frac{\pi x}{180}$ rad.

As π rad = 180°, then 1 rad = $\left(\frac{180}{\pi}\right)^\circ \approx 57.3°$.

This value cannot be found exactly as π is an irrational number. Usually radian measures are left as multiples of π, for example $\frac{3}{4}\pi$.

Tables or calculators may be used if necessary for conversion. When we use a calculator to work with trigonometrical ratios (sine, cosine and tangent) involving radians, it is convenient to put the calculator in the 'radian' mode. The input and output of angles will then be in radians.

Example 1

Convert (a) 36° to radian measure and (b) $\frac{5\pi}{6}$ to degree measure.

(a) $180° = \pi$ rad so $36° = \frac{\pi}{180} \times 36 = \frac{\pi}{5}$ rad.

(b) π rad = 180° so $\frac{5\pi}{6}$ rad = $\frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$.

Example 2

Find the value of (a) sin 0.4 (b) tan 1.5.

Put the calculator in 'radian' mode and key in the appropriate function.

(a) $\sin 0.4 = 0.389$

(b) $\tan 1.5 = 14.1$

Example 3

Find the value of θ (in radians) for $0 \le \theta \le \frac{\pi}{2}$ if
(a) $\cos\theta = 0.5$ (b) $\tan\theta = 0.5$

Again put the calculator in the 'radian' mode.

(a) $\cos\theta = 0.5$

$\theta = \cos^{-1} 0.5 = 1.05$ rad

(b) $\tan\theta = 0.5$

$\theta = \tan^{-1} 0.5 = 0.46$ rad

Exercise 6.1 *(Answers on page 619.)*

1 Convert the following radians to degree measure:

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{10}$ (c) $\frac{2\pi}{3}$ (d) 4π (e) $\frac{\pi}{6}$ (f) $\frac{\pi}{9}$

(g) $\frac{3\pi}{4}$ (h) $\frac{11\pi}{6}$ (i) $\frac{5\pi}{4}$ (j) $\frac{\pi}{8}$ (k) 2 (l) 1.5

2 Convert the following to radian measure as a multiple of π:

(a) 30° (b) 135° (c) 270° (d) 540° (e) 105°

(f) 40° (g) 200° (h) $22\frac{1}{2}$° (i) 400° (j) 75°

3 Find the value of

(a) $\sin\frac{\pi}{6}$ (b) $\cos\frac{\pi}{3}$ (c) $\tan\frac{\pi}{4}$ (d) $\cos\frac{3\pi}{4}$

(e) $\sin\frac{\pi}{2}$ (f) $\sin 2$ (g) $\cos 0.5$

4 Find the value of θ (in radians) for $0 \le \theta \le \frac{\pi}{2}$ if

(a) $\sin\theta = 0.5$ (b) $\cos\theta = 0.6$

(c) $\tan\theta = 1.5$ (d) $\cos\theta = 0.25$

5 Find the value of $\theta - \sin\theta$ if $\theta = 0.75$ rad.

6 Using a calculator, investigate the value of $\frac{\sin\theta}{\theta}$ when θ is small. (Take $\theta = 0.5, 0.3, 0.1, 0.05, 0.01$ for example).

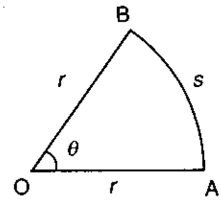
LENGTH OF AN ARC

In Fig.6.3, the arc AB is of length s in a circle with centre O and radius r. The arc subtends an angle of θ radians ($\angle$AOB) at the centre.

As we saw above, an arc of length kr subtends an angle of k radians. Here the arc length is s so $s = kr$ and $k = \frac{s}{r} = \theta$.

So $$s = r\theta; \quad \theta = \frac{s}{r}$$

Fig. 6.3



This formula is only valid if θ is in radians.

If $\angle AOB = \theta°$, then $s = \pi r \times \frac{\theta}{180}$. The formula is simpler in radian measure.

Example 4

In a circle of radius 8 cm, find
(a) the length of the arc which subtends an angle of $\frac{3\pi}{4}$ radians at the centre,
(b) the angle subtended by an arc of length 6 cm.

(a) $s = r\theta = 8 \times \frac{3\pi}{4} = 6\pi$ cm ($\approx$ 18.8 cm)

(b) From the formula, $\theta = \frac{s}{r} = \frac{6}{8} = 0.75$ rad ($= 0.75 \times \frac{180}{\pi} \approx 43.0°$)

AREA OF A SECTOR OF A CIRCLE

In Fig.6.4, AOB is a sector of angle θ rad in a circle with centre O and radius r. $\angle AOB = \theta$ rad.

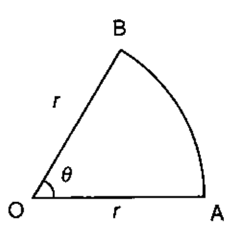
The area of the sector will be proportional to θ.

Hence $\frac{\text{area of sector AOB}}{\text{area of circle}} = \frac{\theta}{2\pi}$ and

area of sector $= \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$.

$$\text{Area of sector of angle } \theta = \tfrac{1}{2}r^2\theta$$

Fig. 6.4



Once again, this formula is only valid if θ is in radians.

Example 5

In Fig.6.5, O is the centre of a circle of radius r and $\angle AOB = \theta$. State
(a) the area of sector AOB
(b) the area of $\triangle AOB$.
Hence deduce the area of the segment which is shaded.
(c) Find the difference in length between the arc AB and the chord AB.

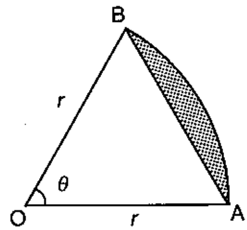


Fig. 6.5

(a) Area of sector $= \frac{1}{2}r^2\theta$

(b) Area of $\triangle$AOB $= \frac{1}{2}r^2 \sin\theta$ (using the formula: area of $\triangle = \frac{1}{2}bc \sin A$)

Hence the area of the segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$.

(c) Length of arc AB $= r\theta$

The length of the chord AB is found from the isosceles $\triangle$AOB (Fig.6.6), where AD is the perpendicular bisector of AB.

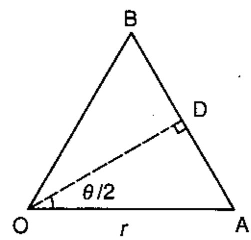
$\angle DOA = \frac{\theta}{2}$

Then AD $= r\sin\frac{\theta}{2}$ and AB $= 2r\sin\frac{\theta}{2}$.

Hence the difference in length between arc AB and chord AB

$= r\theta - 2r\sin\frac{\theta}{2} = r(\theta - 2\sin\frac{\theta}{2})$.

Fig. 6.6



Example 6

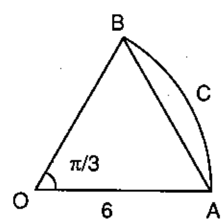
O is the centre of a circle of radius 6 cm. AOB is a sector of angle $\frac{\pi}{3}$ (Fig.6.7). Find

(a) the area of sector AOB,

(b) the area of segment ABC,

(c) the difference in length between arc AB and chord AB.

Fig. 6.7



(a) Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 36 \times \frac{\pi}{3}$
$= 6\pi \ (\approx 18.8 \text{ cm}^2)$

(b) Area of segment = area of sector – area of $\triangle$
$= 6\pi - \frac{1}{2} \times 6^2 \times \sin\frac{\pi}{3}$
$= 6\pi - 18 \times 0.866 = 3.26 \text{ cm}^2$

(c) Difference in lengths $= r\theta - 2r\sin\frac{\theta}{2}$
$= 6 \times \frac{\pi}{3} - 12\sin\frac{\pi}{6}$
$= 2\pi - 6$ (as $\sin\frac{\pi}{6} = 0.5$) ≈ 0.28 cm

Example 7

In Fig.6.8, OACB is a sector of a circle centre O and radius 5 cm. AB = 8 cm. Find
(a) θ in radians,
(b) the length of the arc ACB.

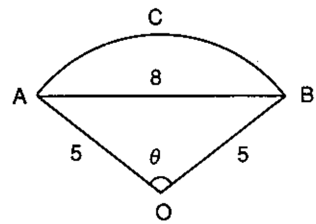
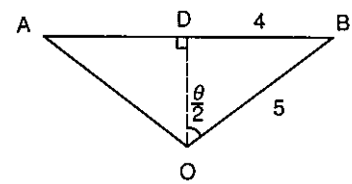


Fig. 6.8

(a) If OD bisects $\angle AOB$ (Fig.6.9)
then D is the midpoint of AB and BD = 4 cm.
Then $\sin \frac{\theta}{2} = \frac{4}{5}$ and $\frac{\theta}{2} \approx 0.927$ rad.
$\theta \approx 1.85$ rad

(b) Length of the arc = $5 \times 1.85 = 9.25$ cm

Fig. 6.9



Example 8

A circular disc, centre O and radius 30 cm, rests on two vertical supports AB, CD, each 20 cm tall and 45 cm apart (Fig. 6.10). Calculate, correct to 3 significant figures
(a) $\angle AOC$ in radians,
(b) the height of the lowest point of arc AC above BD,
(c) the fraction of the area of the disc that lies above the level of AC.

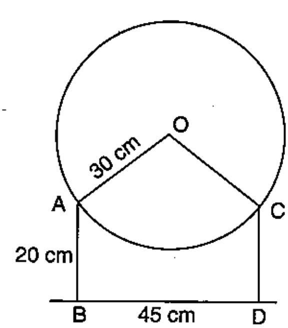


Fig. 6.10

Take $\angle AOC = \theta$.

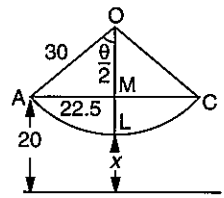


Fig. 6.11

(a) In Fig. 6.11, OM bisects $\angle AOC$, AM = 22.5, OA = 30 and $\angle AOM = \frac{\theta}{2}$.
$OM^2 = 30^2 - 22.5^2$ giving OM = 19.8 cm.
$\sin\frac{\theta}{2} = \frac{22.5}{30}$ so $\frac{\theta}{2} = 0.848$ rad and $\theta = 1.70$ rad.

(b) L is the lowest point of AC. OL = 30 and OM = 19.8 so ML = 10.2.
Then ML + x = 20 so x = 9.8 cm.

(c) Area of the disc = $\pi r^2 = 900\pi$ cm^2.
Area of segment ALC = $\frac{1}{2} \times 30^2 \times \theta - \frac{1}{2} \times 30^2 \times \sin\theta$
$= 450(1.70 - \sin 1.70) = 318.8$ cm^2
Then area above AC = $900\pi - 318.8$ and fraction of area above AC
$= \frac{900\pi - 318.8}{900\pi} = 0.887$.

Example 9

The area of the sector OAB (Fig.6.12) is 150 cm^2. Calculate
(a) θ (in radians),
(b) the length of arc AB.
(c) If this sector is folded up to form a cone, what is the radius of the cone?

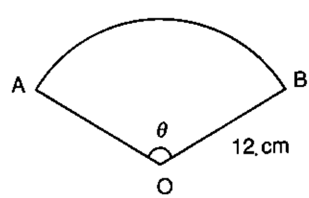


Fig.6.12

(a) Area of sector = $\frac{1}{2}r^2\theta = 150$
Then $\frac{1}{2} \times 144\theta = 150$ and $\theta = 2.083$ rad.

(b) Length of arc = $r\theta = 12 \times 2.083 = 25.0$ cm

(c) When folded up, the arc AB becomes a circle of radius, say R (Fig.6.13).
Then $2\pi R = 24.96$ and $R = 3.97$ cm.

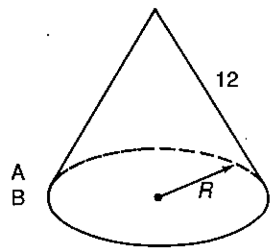


Fig.6.13

Example 10

A sector of a circle has radius r and angle θ. Find the value of θ correct to 3 significant figures if the perimeter of the sector equals half the circumference of the circle.

The perimeter of the sector $= r + r + r\theta = r(2 + \theta)$.

Then $r(2 + \theta) = \pi r$ and $2 + \theta = \pi$ giving $\theta = \pi - 2 \approx 1.14$ rad.

Example 11

A sector of angle θ in a circle of radius r cm has an area of 5 cm² and its perimeter is 9 cm. Find the values of r and θ.

Area $= \frac{1}{2}r^2\theta = 5$ so $r^2\theta = 10$ (i)

Perimeter $= r + r + r\theta = 2r + r\theta = 9$ (ii)

We solve these two equations.

From (i), $\theta = \frac{10}{r^2}$.

Substitute in (ii), $2r + r\frac{10}{r^2} = 9$ which gives $2r^2 + 10 = 9r$ or $2r^2 - 9r + 10 = 0$.

Hence $(2r - 5)(r - 2) = 0$ and $r = 2\frac{1}{2}$ or 2 cm.

From (i), the corresponding values of θ are 1.6 or 2.5 rad.

Exercise 6.2 *(Answers on page 619.)*

Where the answer is not exact, 3-figure accuracy is sufficient.

1 The length of an arc in a circle of radius 5 cm is 6 cm. Find the angle subtended at the centre.

2 An arc of length 5 cm is drawn with radius 3 cm. What angle does it subtend at the centre?

3 The area of a sector of a circle is 9 cm^2. If its radius is 6 cm find the angle of the sector and the length of its arc.

4 The area of a sector of a circle is 15 cm^2 and the length of its arc is 3 cm. Calculate (a) the radius of the sector and (b) its angle.

5 Find the missing values in the following table for sectors:

	Radius (cm)	$\angle$ of sector (radians)	Length of arc (cm)	Area of sector (cm^2)
(a)	4	1.3		
(b)		0.8	8	
(c)	8			16
(d)			3	18
(e)		0.4		7.2

6 A sector of a circle of radius 4 cm has an angle of 1.2 radians. Calculate
(a) the area of the sector,
(b) its perimeter.

7 In Fig. 6.14, DOBC is a semicircle, centre O and radius 6 cm. AC is perpendicular to DOB where AB is 2 cm.
Calculate
(a) the length of AC,
(b) $\angle$COA in radians and
(c) the perimeter of the shaded region.
(d) Express the area of the shaded region as a percentage of the area of the semicircle.

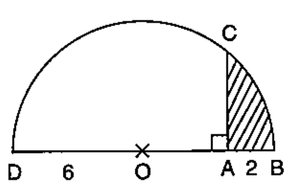


Fig. 6.14

8 If the area of a sector is 6.4 cm^2 and its angle is 0.8 radians, calculate the radius of the sector.

9 A chord AB is 8 cm long in a circle of radius 5 cm. Calculate
(a) the angle it subtends at the centre of the circle,
(b) the length of the shorter arc AB.

10 The perimeter of a sector is 128 cm and its area is 960 cm^2.
Find the possible values of the radius of the sector and its angle.

11 A wheel of radius 0.6 m rotates on its axis at a rate of 4.5 radians per second. Calculate the speed with which a point on its rim is moving.

12 A disc is rotating at $33\frac{1}{3}$ revolutions per minute.
(a) At what rate, in radians per second, is it rotating?
(b) At what speed, in metres per second, is a point on the rim moving, if the radius of the disc is 15 cm?

13 Fig.6.15 shows a cross–section through a tunnel, which is part of a circle of radius 5 m. The width AB of the floor is 8 m. Calculate
(a) the angle subtended at the centre of the circle by the chord AB,
(b) the length of the arc ACB.

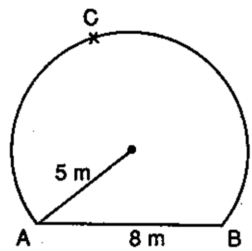


Fig. 6.15

14 Fig.6.16 represents the action of a windscreen wiper of a car. It rotates about O and travels from AB to A'B'and back. Calculate
(a) the area AA'B'B swept clear,
(b) the perimeter of this area.

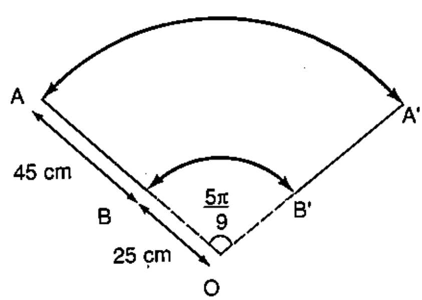


Fig. 6.16

15 A cylindrical barrel floats in water (Fig. 6.17). The diameter of the barrel is 120 cm and its highest point P is 80 cm above the water level AB.
(a) Calculate ∠AOB in radians, where O is the centre of the circular face.
(b) What fraction of the volume of the barrel is below the water line?

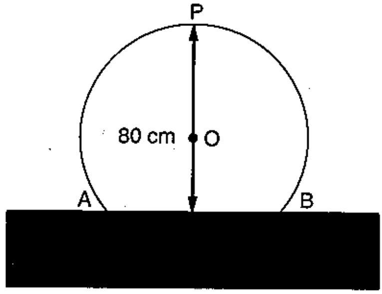


Fig. 6.17

16 In a circle centre O, AOB and COD are two concentric sectors as shown in Fig.6.18. The lengths of the arcs AB and DC are 2.8 cm and 2 cm respectively and AD = 2 cm. Calculate

(a) the length of OC,
(b) $\angle AOB$ in radians,
(c) the area of ABCD.

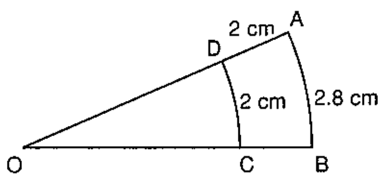
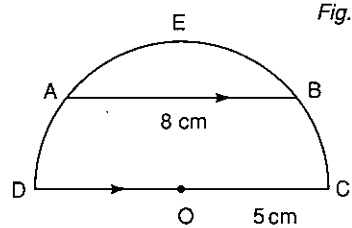


Fig. 6.18

17 In Fig.6.19, the chord AB, of length 8 cm, is parallel to the diameter DOC of the semicircle with centre O and radius 5 cm. Calculate

(a) $\angle AOB$ in radians,
(b) the area of the segment ABE,
(c) what fraction the area of the segment ABE is of the area of the semicircle.

Fig. 6.19



18 A wheel of radius 20 cm rolls without slipping on level ground. A point P on the rim is in the position P_1 at the start (Fig.6.20). When the centre of the wheel has moved through 50 cm, P is now in the position P_2. Calculate the angle (in radians) through which the wheel has turned.

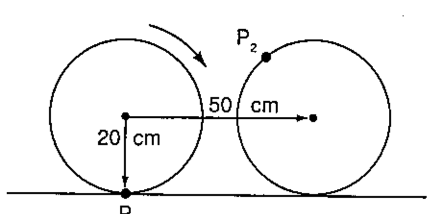


Fig. 6.20

19 A piece of wire, 10 cm long, is formed into the shape of a sector of a circle of radius r cm and angle θ radians.

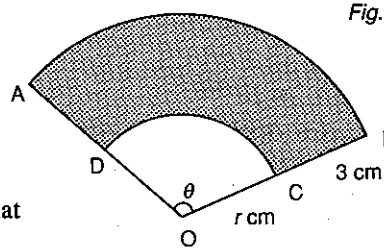
(a) Show that $\theta = \frac{10 - 2r}{r}$.

(b) Show also that the area A cm² of the sector is given by $A = 5r - r^2$.

(c) If $4 \leq A \leq 6$ and $\theta \leq 3$, find the limits within which r must lie.

20 In Fig.6.21, O is the centre of the circle containing the sector OAB. DC is a parallel arc and BC = 3 cm.

Fig. 6.21



If OC = r cm and $\angle$AOB = θ rad, show that

(a) the shaded area = $\frac{\theta}{2}(6r + 9)$ cm²,

(b) the perimeter of the shaded area equals $6 + \theta(2r + 3)$ cm.

(c) Given that the shaded area is three-quarters of the area of the sector OAB, calculate the value of r.

(d) If, however, the total perimeter of the shaded area equals the total perimeter of the sector OAB, find the value of θ.

SUMMARY

- 1 radian is the angle subtended by an arc of length equal to the radius.
- π rad = 180°
- Length s of an arc of radius r subtending an angle θ is $s = r\theta$ (θ in radians).
- Area A of a sector of radius r and angle θ is $A = \frac{1}{2}r^2\theta$ (θ in radians)

REVISION EXERCISE 6 *(Answers on page 620.)*

Where the answer is not exact, 3-figure accuracy is sufficient.

A

1 In Fig.6.22, ADB is a semicircle with centre O and radius 20 cm. DC is perpendicular to AB where C is the midpoint of OB. Calculate

(a) $\angle$DOC in radians,

(b) the area of the shaded region.

Fig. 6.22

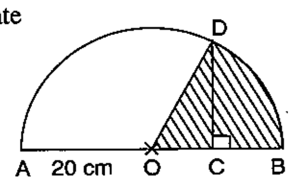


Fig. 6.22

2 Fig.6.23 shows a circle, centre O, radius 10 cm. The tangent to the circle at A meets OB produced at T. Given that the area of the triangle OAT is 60 cm² calculate the area of the sector OAB. (C)

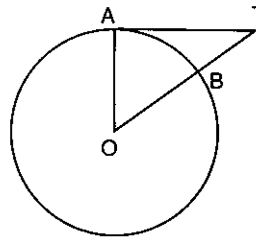


Fig. 6.23

3 In Fig.6.24, O is the centre of the sector OAB. CD is another arc, with centre O and radius r cm. DB = 2 cm. If the area of ABDC is one-third the area of the sector OCD, find the value of r.

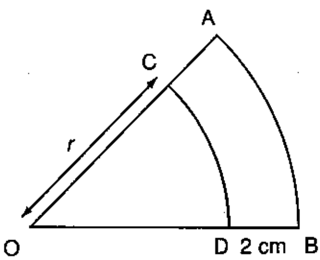


Fig. 6.24

4 In Fig.6.25, ADC is an arc of a circle, centre O, radius r and $\angle AOC = 2\theta$ radians. ABC is a semicircle on AC as diameter. Show that $AC = 2r \sin \theta$.

Find expressions, in terms of r and θ, for the areas of

(i) the sector OADC,

(ii) the segment ADC,

(iii) the shaded region. (C)

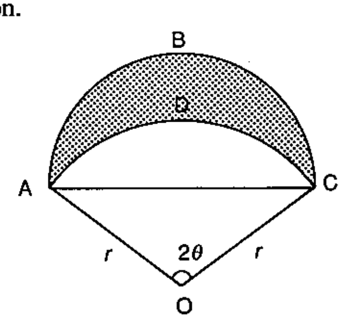


Fig. 6.25

5 OBD is a sector of a circle with centre O and radius 6 cm. $\angle BOD = \frac{2\pi}{5}$

A is a point on OB where OA = 2 cm and C is a point on OD such that OC = 4 cm. Find the area of the region bounded by BA, AC, CD and the arc BD.

6 Fig.6.26 shows a circle, centre O, radius 5 cm and two tangents TA and TB, each of length 8 cm. Calculate

(i) $\angle AOB$,

(ii) the length of the arc APB,

(iii) the area of the shaded region. (C)

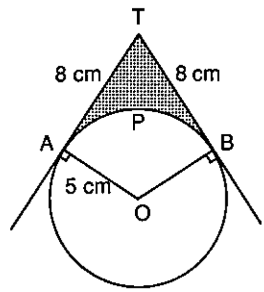


Fig. 6.26

7 Fig.6.27 shows part of a circle centre O of radius 6 cm.

(i) Calculate the area of sector BOC when $\theta = 0.8$ radians.

(ii) Find the value of θ in radians for which the arc length BC is equal to the sum of the arc length CA and the diameter AB. (C)

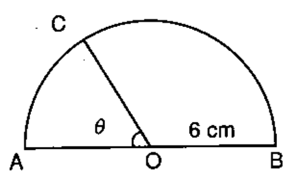


Fig. 6.27

8 In Fig.6.28, OAB is an equilateral triangle of side 10 cm. The arc ADB is drawn with centre O. A semicircle is drawn on AB as diameter. Find the area of the shaded region.

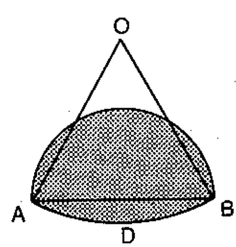


Fig.6.28

9 Fig.6.29 shows the circular cross-section of a uniform log of radius 40 cm floating in water. The points A and B are on the surface and the highest point X is 8 cm above the surface. Show that $\angle AOB$ is approximately 1.29 radians. Calculate

(i) the length of the arc AXB,

(ii) the area of the cross-section below the surface,

(iii) the percentage of the volume of the log below the surface. (C)

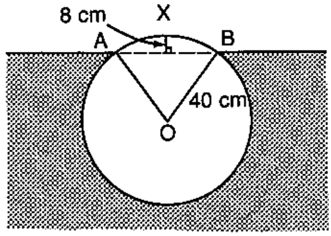


Fig.6.29

10 Fig.6.30 shows two arcs, AB and CD, of concentric circles, centre O. The radii OA and OC are 11 cm and 14 cm respectively and $\angle AOB = \theta$ radians.
Express in terms of θ the area of
(i) sector AOB,
(ii) the shaded region ABCD.

Given that the area of the shaded region ABCD is 30 cm^2, calculate
(iii) the value of θ,
(iv) the perimeter of the shaded region ABCD. (C)

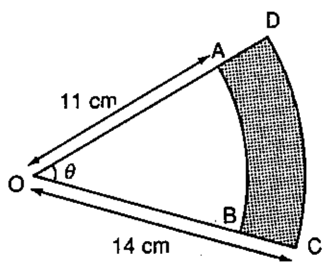


Fig. 6.30

11 Fig. 6.31 shows a semi-circle ABC, with centre O and radius 4 m, such that angle BOC = 90°.
Given that CD is an arc of a circle, centre B, calculate
(a) the length of the arc CD,
(b) the area of the shaded region. (C)

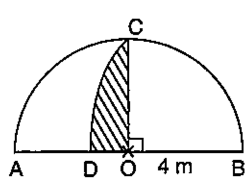


Fig. 6.31

B

12 In Fig.6.32, ABCD is a square of side 4 cm. Equal arcs AE and EB are drawn with radius 4 cm and centres B and A respectively. Calculate the area of the shaded region.

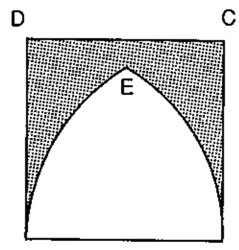


Fig. 6.32

13 The two circles in Fig.6.33 have centres A and B and radii 5 cm and 12 cm respectively. AB = 13 cm. Calculate the area of the shaded region.

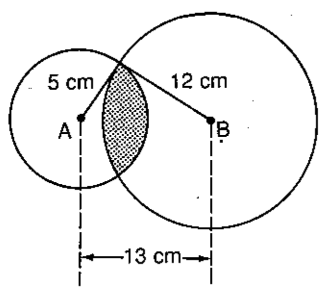


Fig. 6.33

14 A sector of a circle radius r has a total perimeter of 12 cm. If its area is A cm^2, show that $A = 6r - r^2$. Hence find the value of r for which A is a maximum and the corresponding value of the angle of the sector in radians.

15 In Fig.6.34, the sector OAB has centre O and radius 6 cm and $\angle AOB = \frac{\pi}{3}$ radians. OC is the bisector of $\angle AOB$ and P is the midpoint of OC. An arc DE of a circle is drawn with centre P to meet OA and OB at D and E respectively.

(a) Find the size of $\angle OPD$.

(b) Calculate the area of the shaded region.

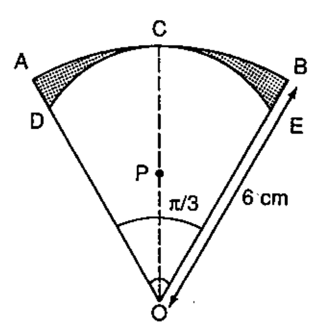


Fig. 6.34

16 In Fig.6.35, ABCD is a rhombus of side x and $\angle A = \theta$ radians. Arcs each of radius $\frac{x}{3}$ are drawn with centres A, B, C and D. If the shaded area is half the area of the rhombus, show that $\sin\theta = \frac{2\pi}{9}$ and find the two possible values of θ.

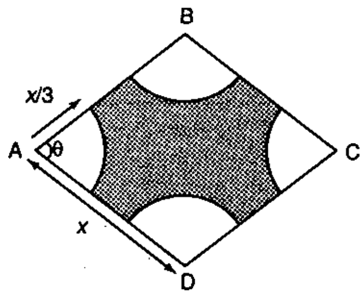


Fig. 6.35

17 A, B and C are three points in that order on the circumference of a circle radius 5 cm. The chords AB and BC have lengths 8 cm and 4 cm respectively. Find the ratio of the areas of the minor segments on AB and BC.

Trigonometry

7

TRIGONOMETRIC FUNCTIONS FOR A GENERAL ANGLE

The trigonometric functions **sine, cosine** and **tangent** of an angle θ were originally defined as ratios of the sides of a right-angled triangle, i.e. for a domain $0° \leq \theta \leq 90°$. We now extend the definition to deal with any angle (the **general** angle).

The actual *values* of $\sin \theta$, $\cos \theta$ and $\tan \theta$ for any given angle can be found directly using a calculator. To solve equations, however, we must know how to use these definitions inversely.

Suppose the arm OR (of unit length) in Fig.7.1 can rotate about O in an anticlockwise direction and makes an angle θ with the positive x–axis. We divide the complete revolution into 4 **quadrants** and take the positive y–axis at 90°. Let (x,y) be the coordinates of R. x and y will be positive or negative depending on which quadrant R lies in.

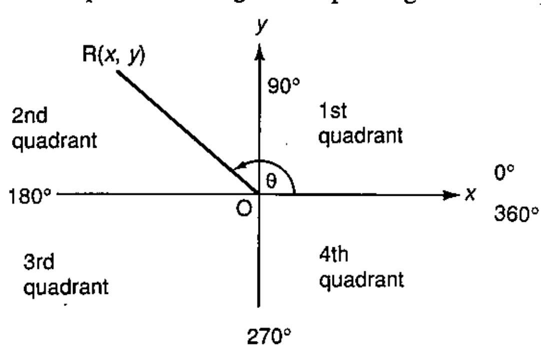


Fig.7.1

We define

$$\sin \theta = \frac{y\text{-coordinate of R}}{1}$$

$$\cos \theta = \frac{x\text{-coordinate of R}}{1}$$

$$\tan \theta = \frac{y\text{-coordinate of R}}{x\text{-coordinate of R}}$$

Note that both $|\sin\theta|$ and $|\cos\theta|$ are less than or equal to 1 as both $|x|$ and $|y|$ are less than or equal to 1, but that $\tan\theta$ can have any value. In the first quadrant, where $0° \leq \theta \leq 90°$, each of these functions will be positive (Fig.7.2).

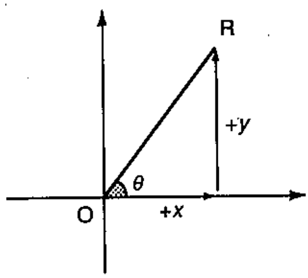


Fig. 7.2

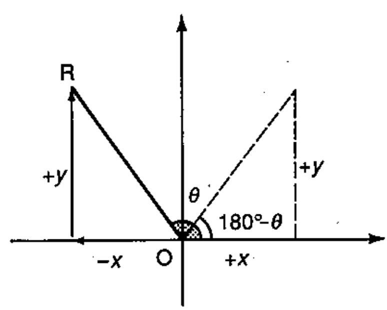
In the second quadrant (Fig.7.3), where $90° < \theta \leq 180°$, the angle θ is linked to the corresponding angle $180° - \theta$ in the first quadrant.

Fig.7.3

$$\sin\theta = +y = \sin(180° - \theta)$$

$$\cos\theta = -x = -\cos(180° - \theta)$$

$$\tan\theta = \tfrac{+y}{-x} = -\tan(180° - \theta)$$



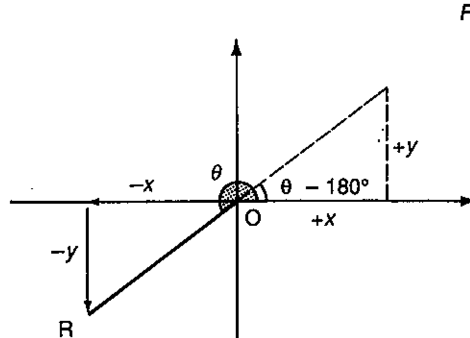
For the third quadrant (Fig.7.4), where $180° < \theta \leq 270°$, the corresponding angle in the first quadrant is $\theta - 180°$.

$$\sin\theta = -y = -\sin(\theta - 180°)$$

$$\cos\theta = -x = -\cos(\theta - 180°)$$

$$\tan\theta = \tfrac{-y}{-x} = \tan(\theta - 180°)$$

Fig.7.4



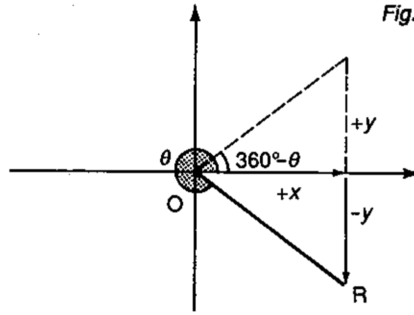
For the fourth quadrant (Fig.7.5), where $270° < \theta \leq 360°$, the corresponding angle in the first quadrant is $360° - \theta$.

Fig.7.5

$$\sin\theta = -y = -\sin(360° - \theta)$$

$$\cos\theta = +x = \cos(360° - \theta)$$

$$\tan\theta = \frac{-y}{x} = -\tan(360° - \theta)$$



Summarizing:

SIN +	All +
$\sin\theta = \sin(180° - \theta)$ $\cos\theta = -\cos(180° - \theta)$ $\tan\theta = -\tan(180° - \theta)$	$\sin\theta$ $\cos\theta$ $\tan\theta$
2nd	1st
3rd	4th
$\sin\theta = -\sin(\theta - 180°)$ $\cos\theta = -\cos(\theta - 180°)$ $\tan\theta = \tan(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$ $\cos\theta = \cos(360° - \theta)$ $\tan\theta = -\tan(360° - \theta)$
TAN +	COS +

Each function is positive (+) in the first quadrant and one other.
Each function is negative (–) in two quadrants.

Note on Special Angles 30°, 45°, 60°

As these angles are often used, it will be useful for future work to have their trigonometrical ratios in fractional form.

45°

In Fig.7.6(a), ABC is an isosceles right-angled triangle with AB = BC = 1. Hence AC = $\sqrt{2}$ and $\angle A = \angle C = 45°$.

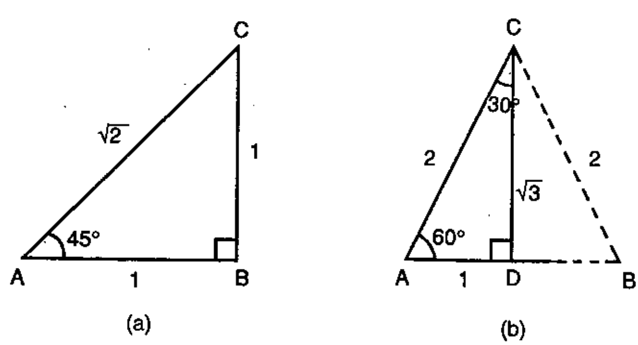


Fig.7.6

Then

$$\sin 45° = \frac{1}{\sqrt{2}} = \cos 45°$$
$$\tan 45° = 1$$

30°, 60°

In Fig.7.6(b), ABC is an equilateral triangle with side 2.
CD is the perpendicular bisector of AB so AD = 1 and CD = $\sqrt{3}$.
$\angle$A = 60° and $\angle$ACD = 30°.

Then

$$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$$
$$\sin 30° = \cos 60° = \frac{1}{2}$$
$$\tan 60° = \sqrt{3},\ \tan 30° = \frac{1}{\sqrt{3}}$$

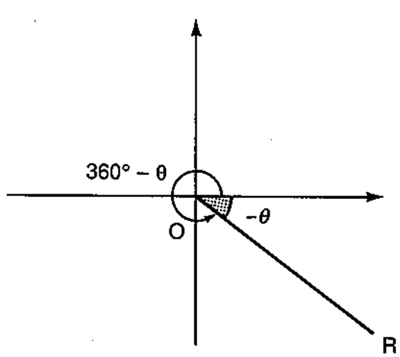
Using the special ratios above, the ratios for other angles related to 30°, 45° and 60° can be found in a similar form if required.
For example, $\cos 210° = -\cos(210° - 180°) = -\cos 30° = -\frac{\sqrt{3}}{2}$
Copy and complete this table:

θ	120°	135°	150°	210°	240°	300°	315°	330°
sin θ								
cos θ								
tan θ								

NEGATIVE ANGLES

If the arm OR rotates in a clockwise direction (Fig.7.7), it will describe a **negative** angle, $-\theta$. To find the value of a function of a negative angle, convert the angle to $360° - \theta$ or $2\pi - \theta$, if working in radians.

Fig.7.7



Thus $\sin(-30°) = \sin 330°$, $\tan(-\frac{\pi}{3}) = \tan(2\pi - \frac{\pi}{3}) = \tan(\frac{5\pi}{3})$ and so on.

BASIC TRIGONOMETRIC EQUATIONS

We apply the above trigonometric functions to the solution of basic trigonometric equations, i.e. equations in **one** function such as $\sin\theta = 0.44$, $\cos\theta = -0.78$ or $\tan\theta = 1.25$. As we shall see later, all other equations are reduced to one (or more) of these. **A basic equation will usually have two solutions for $0° \le \theta \le 360°$.**

To solve a basic equation, such as $\sin\theta = k$,

step 1 find the 1st quadrant angle α for which $\sin\alpha = |k|$;
step 2 find the quadrants in which θ will lie;
step 3 determine the corresponding angles for those quadrants.

Unless exact, angles in degrees are to be given to one decimal place.

Example 1

Solve (a) $\sin\theta = 0.57$, (b) $\sin\theta = -0.38$ for $0° \le \theta \le 360°$.

(a) If $\sin\alpha = 0.57$, then $\alpha = 34.75°$.
θ will lie in the 1st and 2nd quadrants (θ and $180° - \theta$)
Then $\theta = 34.75°$ or $180° - \theta = 34.75°$ i.e. $\theta = 145.25°$.
The solutions are $34.8°$ and $145.3°$.

(b) From $\sin\alpha = +0.38$, $\alpha = 22.33°$.
θ will lie in the 3rd and 4th quadrants.
Then $\theta - 180° = 22.3°$ or $360° - \theta = 22.3°$ giving $\theta = 202.3°$ and $337.7°$.

Solutions for the equations $\cos\theta = k$ and $\tan\theta = k$ are found in the same way.

Example 2

Solve (a) $\cos\theta = -0.3814$, (b) $\tan\theta = 1.25$ for $0° \le \theta \le 360°$.

(a) The 1st quadrant angle for $\cos\alpha = +0.3814$ is $67.58°$.
θ lies in the 2nd and 3rd quadrants.
Then $180° - \theta = 67.58°$ or $\theta - 180° = 67.58°$ giving $\theta = 112.4°$ and $247.6°$.

(b) The 1st quadrant angle for $\tan\theta = 1.25$ is $51.34°$.
θ lies in the 2nd and 3rd quadrants.
Then $\theta = 51.34°$ and $\theta - 180° = 51.34°$ i.e. $\theta = 231.34°$.
Hence the solutions are $\theta = 51.3°$ and $231.3°$.

Example 3

Solve the equation $3\cos^2\theta + 2\cos\theta = 0$ for $0° \le \theta \le 360°$.

The left hand side factorizes giving $\cos\theta(3\cos\theta + 2) = 0$ which separates into 2 basic equations:

$\cos\theta = 0$

and $3\cos\theta + 2 = 0$ which gives $\cos\theta = -\frac{2}{3} = -0.6667$.

Note: Do not divide through by the factor $\cos\theta$. This would lose the equation $\cos\theta = 0$. Never divide by a factor containing the unknown angle.

For $\cos\theta = 0$, $\theta = 90°$ or $270°$.

For $\cos\theta = -0.6667$, θ lies in the 2nd and 3rd quadrants.

The 1st quadrant angle is 48.19°.

Then $180° - \theta = 48.19°$ and $\theta - 180° = 48.19°$ giving $\theta = 131.8°$ and $\theta = 228.2°$.

Hence the solutions are 90°, 131.8°, 228.2° and 270°.

Example 4

For $0° \leq \theta \leq 360°$, solve $6\cos^2\theta + \cos\theta = 1$.

This is a quadratic equation in $\cos\theta$:

$$6\cos^2\theta + \cos\theta - 1 = 0$$

and so $$(3\cos\theta - 1)(2\cos\theta + 1) = 0$$

which separates into $\cos\theta = 0.3333$ and $\cos\theta = -0.5$.

Verify that the solutions are $\theta = 70.5°$, $120°$, $240°$ and $289.5°$.

Example 5

Solve the equation $\sin(\theta - 30°) = 0.4$ for $0° \leq \theta \leq 360°$.

Write $\phi = \theta - 30°$.

Then $\sin\phi = 0.4$.

Solve for ϕ.

Verify that $\phi = 23.6°$ and $156.4°$.

Then $\theta = 53.6°$ and $186.4°$.

OTHER TRIGONOMETRIC FUNCTIONS

There are three other functions which are the reciprocals of the sine, cosine and tangent. They are

cosecant: $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

secant: $\sec\theta = \dfrac{1}{\cos\theta}$

cotangent: $\cot\theta = \dfrac{1}{\tan\theta}$

Example 6

Solve (a) cosec $\theta = -1.58$, (b) $4 \cot \theta = \tan \theta$, for $0° \le \theta \le 360°$.

(a) Replace cosec θ by $\frac{1}{\sin \theta}$.

$\frac{1}{\sin \theta} = -1.58$ so $\sin \theta = -\frac{1}{1.58} = -0.6329$

Now verify that $\theta = 219.3°$ or $320.7°$.

(b) Replace cot θ by $\frac{1}{\tan \theta}$.

Then $\frac{4}{\tan \theta} = \tan \theta$ i.e. $\tan^2 \theta = 4$.

So $\tan \theta = \pm 2$ (NB: don't forget the negative root)

Verify that the solutions of these equations are 63.4°, 116.6°, 243.4° and 296.6°.

Exercise 7.1 *(Answers on page 620.)*

1 Solve the following equations for $0° \le \theta \le 360°$:

(a) $\sin \theta = \frac{1}{3}$ (b) $\cos \theta = 0.762$ (c) $\tan \theta = 1.15$
(d) $\cos \theta = -0.35$ (e) $\sin \theta = -0.25$ (f) $\tan \theta = -0.81$
(g) $\sin \theta = -0.1178$ (h) $\sin \theta = -0.65$ (i) $\cos \theta = 0.23$
(j) $\tan \theta = -1.5$ (k) $\operatorname{cosec} \theta = 1.75$ (l) $\cos \theta = -0.14$
(m) $\sec \theta = -1.15$ (n) $\cot \theta = 0.54$ (o) $\sec \theta = 2.07$

2 Solve the following equations for $0° \le \theta \le 360°$:

(a) $5 \sin^2 \theta = 2 \sin \theta$ (b) $9 \tan \theta = \cot \theta$
(c) $3 \tan^2 \theta + 5 \tan \theta = 2$ (d) $4 \cos^2 \theta + 3 \cos \theta = 0$
(e) $5 \sin^2 \theta = 2$ (f) $6 \sin^2 \theta + 7 \sin \theta + 2 = 0$
(g) $\cos(\theta + 20°) = -0.74$ (h) $\tan(\theta - 50°) = -1.7$
(i) $3 \sin^2 \theta = \sin \theta$ (j) $4 \sec^2 \theta = 5$
(k) $\cos^2 \theta = 0.\dot{6}$ (l) $6 \sin^2 \theta = 2 + \sin \theta$
(m) $2 \sec^2 \theta = 3 - 5 \sec \theta$ (n) $\sec(\theta - 50°) = 2.15$
(o) $\sin(\theta + 60°) = -0.75$

3 Find θ for $0° \le \theta \le 360°$ if $3 \cos^2 \theta - 2 = 0$.

4 If $5 \tan \theta + 2 = 0$, find θ in the range $0° \le \theta \le 360°$.

5 Solve the equation $5 \cos \theta - 3 \sec \theta = 0$ for $0° \le \theta \le 360°$.

6 Find all the angles between 0° and 180° which satisfy the equations

(a) $\sin x = 0.45$ (b) $\cos y = -0.63$ (c) $\tan \theta = 2.15$

7 Find the values of

(a) $\sin(-30°)$ (b) $\cos(-\frac{\pi}{4})$ (c) $\tan(-200°)$ (d) $\sec(-150°)$
(e) $\cot(-300°)$ (f) $\sin(-\frac{4\pi}{3})$ (g) $\operatorname{cosec}(-\frac{2\pi}{5})$

8 Show that

(a) $\sin(-\theta) = -\sin \theta$, (b) $\cos(-\theta) = \cos \theta$, (c) $\tan(-\theta) = -\tan \theta$.

9 Solve the equations
(a) $\sin(-\theta) = 0.35$, (b) $\sin(-\theta) = -0.27$
(c) $\cos(-\theta) = -0.64$ (d) $\tan(-\theta) = 1.34$,
for $0° \le \theta \le 360°$.

GRAPHS OF TRIGONOMETRIC FUNCTIONS

sin θ and cos θ

Complete the following table of values of sin θ and cos θ, taking a domain of 0° to 360° at 30° steps:

θ	0°	30°	60°	90°	120°	150°	180°	...	270°	...	360°
sin θ	0	0.5		1			0		−1		0
cos θ	1		0.5	0			−1				1

Plot these values on graph paper using scales of say 1 cm ≡ 30° on the θ-axis and 4 cm ≡ 1 unit on the function axis (Fig.7.8).

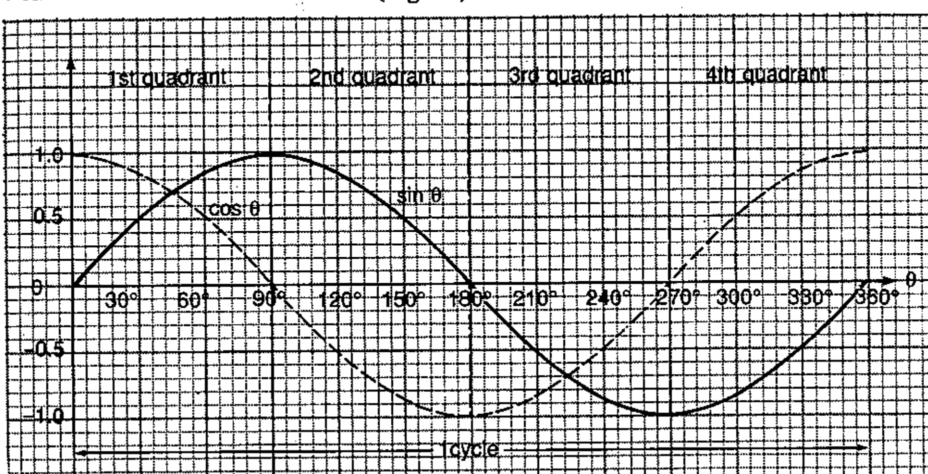


Fig.7.8

The graph shows **one cycle** of each function.

The sine curve has a maximum of 1 when θ = 90° and a minimum of −1 when θ = 270°. The cosine curve is identical to the sine curve but is shifted 90° to the left. This difference is called the *phase difference* between the two functions.

For angles greater than 360° or less than 0° the curves repeat themselves in successive cycles (Fig.7.9). Functions which repeat themselves like this are called **periodic** functions. The sine and cosine functions each have a **period** of 360° (or 2π). Hence

$$\sin(\theta + n360°) = \sin\theta \text{ or } \cos(\theta + 2n\pi) = \cos\theta$$

where n is any integer. This means that we can add or subtract 360° from any solution of $\sin\theta = k$ or $\cos\theta = k$ and obtain other solutions outside the domain $0° \le \theta \le 360°$.

For example, if the solutions of $\sin\theta = 0.5$ for $0° \le \theta \le 360°$ are 30° and 150°, then 30° + 360° = 390° and 150° − 360° = −210° are also solutions of the equation. These solutions are marked by dots on the graph of sin θ in Fig.7.9.

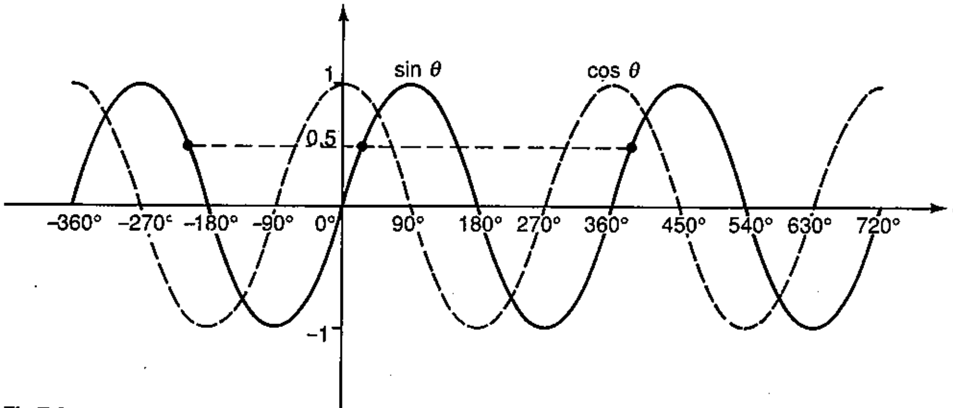


Fig.7.9

tan θ

Values of tan θ begin at 0 for θ = 0°, increase to 1 when θ = 45° and then increase rapidly as θ approaches 90°. tan 90° is undefined. Between 90° and 270° the function increases from large negative values through 0 to large positive values. The curve approaches the 90° and 270° axes but never reaches them. Hence the curve consists of 3 separate branches between 0° and 360° (Fig.7.10).

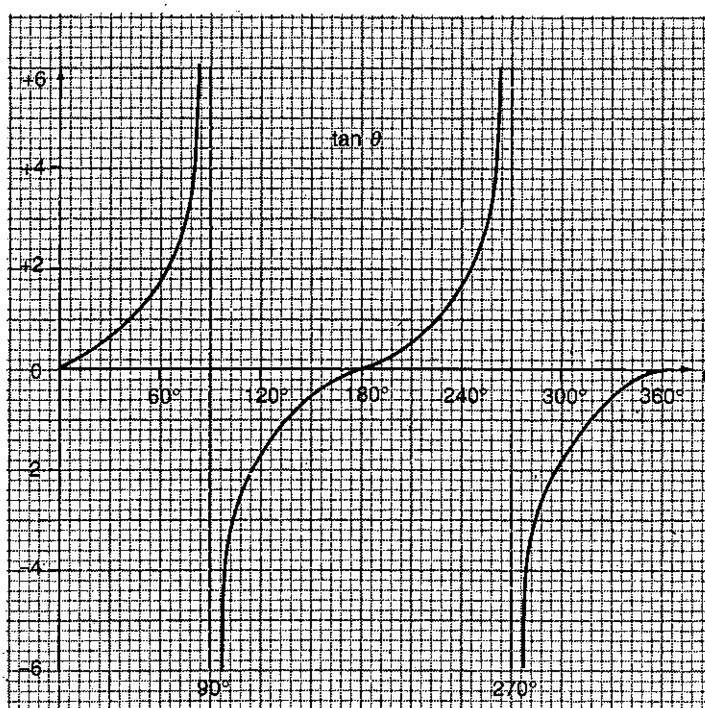


Fig.7.10

tan θ is also a periodic function but with a period of 180°. Hence tan(θ + $n\pi$) or tan(θ + n180°) = tan θ where n is an integer.

MULTIPLE ANGLE FUNCTIONS

Functions such as $\sin 2\theta$, $\cos \frac{\theta}{2}$, etc. are multiple angle functions as 2θ, $\frac{\theta}{2}$ are multiples of θ.

Example 7

(a) Sketch the graph of $y = \sin 2\theta$.
(b) Solve the equation $\sin 2\theta = 0.55$ for $0° \leq \theta \leq 360°$ and show the solutions on the graph.

(a) If the domain of θ is 0° to 360°, 2θ will take values from 0° to 720°. Hence the curve completes **two** cycles as θ increases from 0° to 360° (Fig.7.11).

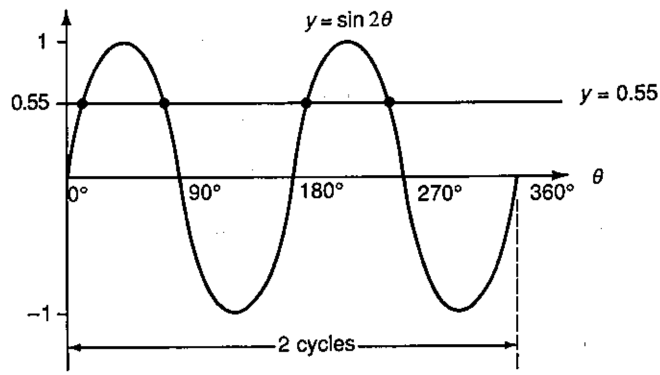


Fig.7.11

(b) For convenience, write $2\theta = \phi$ so $\sin \phi = 0.55$.
ϕ lies in the 1st and 2nd quadrants so $\phi = 33.37°$ or $180° - \phi = 33.37°$.
Hence $\phi = 33.37°$ or $146.63°$.

But ϕ takes values from 0° to 720°, so we add 360° to each of these to obtain further solutions.

Then $\phi = 2\theta = 33.37°$ or $146.63°$ or $393.37°$ or $506.63°$.
Hence $\theta = 16.7°$ or $73.3°$ or $196.7°$ or $253.3°$.

So we obtain 4 solutions, 2 for each cycle. These solutions are marked on the graph.

Note that all the solutions for 2θ must be obtained first before dividing by 2 to obtain the values of θ, which are then corrected to 1 decimal place.

Example 8

(a) Sketch the graph of $y = \cos \frac{\theta}{2}$ for $0° \leq \theta \leq 360°$.
(b) Solve the equation $\cos \frac{\theta}{2} = -0.16$ for this domain.

(a) If the domain of θ is 0° to 360°, then $\frac{\theta}{2}$ will take values from 0° to 180° only. So the graph will be a half-cycle of the cosine curve (Fig. 7.12).

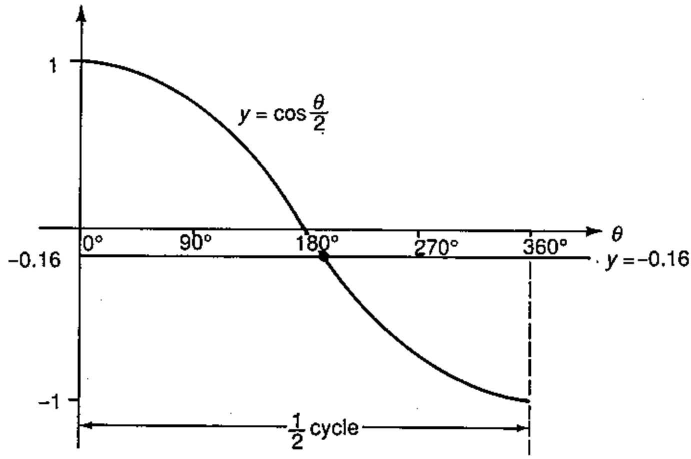


Fig.7.12

(b) Write $\frac{\theta}{2} = \phi$. Then $\cos \phi = -0.16$.
ϕ lies in the 2nd and 3rd quadrants.
Then $180° - \phi = 80.79°$ or $\phi - 180° = 80.79°$.
Hence $\phi = 99.21°$ or $260.79°$ and therefore $\theta = 198.4°$ or $521.6°$.

The second solution is outside the domain and is therefore discarded.
The only solution to the equation is $\theta = 198.4°$. This is to be expected as there is only a half cycle of the function.

Example 9

Solve the equation 5 sin $\frac{3\theta}{4} + 4 = 0$ for the domain $0° \leq \theta \leq 360°$.

Let $\frac{3\theta}{4} = \phi$. Then $\sin \phi = -\frac{4}{5} = -0.8$.
ϕ lies in the 3rd and 4th quadrants.
Then $\phi - 180° = 53.13°$ and $360° - \phi = 53.13°$.
Hence $\phi = 233.13°$ or $306.87°$.

If the domain of θ is 0° to 360°, then $\phi = \frac{3\theta}{4}$ takes values from 0° to 270°.

Hence the only solution is $\phi = 233.13°$ i.e. $\theta = \frac{4}{3} \times 233.13° = 310.8°$.
($\phi = 306.87°$ would give $\theta = 409.2°$).

Example 10

Solve $\cos(2\theta + 60°) = -0.15$ for $0° \leq \theta \leq 360°$.

Put $\phi = 2\theta + 60°$. Then $\cos \phi = -0.15$ giving $\phi = 98.63°$ and $261.37°$.
However if the domain of θ is 0° to 360°, then the domain of ϕ is 60° to 780°.
So we must add 360° to each of the above values.
Therefore $\phi = 2\theta + 60° = 98.63°$ or $261.37°$ or $458.63°$ or $621.37°$
and hence $\theta = 19.3°$ or $100.7°$ or $199.3°$ or $280.7°$.

Exercise 7.2 *(Answers on page 620.)*

1 Sketch the graphs of (a) $y = \sin 3\theta$, (b) $y = \cos 3\theta$ for $0° \le \theta \le 360°$. What is the period of each of these functions?

2 Sketch the graphs of (a) $y = \tan 2\theta$, (b) $y = \tan \frac{\theta}{2}$ for $0° \le \theta \le 360°$.

3 On the same diagram, sketch the graphs of $y = \sin 2\theta$ and $y = \cos\theta$ for $0° \le \theta \le 360°$. How many solutions of the equation $\sin 2\theta = \cos\theta$ are there in this domain?

4 Sketch on the same diagram, the graphs of $y = \sin \frac{\theta}{2}$ and $y = \cos\theta$ for $0° \le \theta \le 360°$. State the number of solutions which the equation $\sin\frac{\theta}{2} = \cos\theta$ will have in this domain.

5 On the same diagram, sketch the graphs of $y = \cos 3\theta$ and $y = \sin\frac{\theta}{2}$ for $0° \le \theta \le 360°$. State the number of solutions of the equation $\cos 3\theta = \sin\frac{\theta}{2}$ you would expect to obtain in this domain.

6 Solve, for $0° \le \theta \le 360°$, the following equations:

(a) $\sin 2\theta = 0.67$ (b) $\cos 3\theta = 0.58$
(c) $\tan\frac{\theta}{2} = 1.5$ (d) $\sin\frac{\theta}{3} = 0.17$
(e) $3\cos 2\theta = 2$ (f) $\sec\frac{\theta}{2} = -1.7$
(g) $\sin\frac{\theta}{3} = -0.28$ (h) $3\tan 2\theta + 1 = 0$
(i) $3\sin\frac{2\theta}{3} = 2$ (j) $4\cos\frac{3\theta}{2} + 3 = 0$
(k) $2\operatorname{cosec} 2\theta + 3 = 0$ (l) $\cot\frac{\theta}{2} = 1.35$
(m) $\cos\frac{3\theta}{4} = \frac{3}{4}$ (n) $\tan 2\theta = -1$
(o) $3\sin^2 2\theta + 2\sin 2\theta = 1$ (p) $2\cos^2\frac{\theta}{2} = \cos\frac{\theta}{2}$
(q) $\sin 2\theta = -0.76$ (r) $\sec\frac{\theta}{2} = 1.88$
(s) $\cos 2\theta = -0.65$ (t) $\tan\frac{2\theta}{3} + 2 = 0$
(u) $5\sin\frac{4\theta}{5} + 3 = 0$ (v) $2\operatorname{cosec}\frac{\theta}{2} = 3$

7 For $0° \le \theta \le 360°$, solve the following

(a) $\sin(\frac{\theta}{3} + 20°) = 0.47$ (b) $\tan(2\theta - 60°) = 1.55$
(c) $\cos(\frac{\theta}{2}) = 0.75$ (d) $\sin(2\theta + 80°) = -0.54$
(e) $\sec^2(\frac{\theta}{3} - 50°) = 1.2$

8 State the values of (a) $\sin(30° + n360°)$, (b) $\cos(n360° - 50°)$, (c) $\tan(45° + n180°)$ where n is an integer.

9 State the values of (a) $\sin(2n + 1)\pi$, (b) $\cos(6n - 1)\frac{\pi}{3}$, (c) $\tan(3n + 1)\frac{\pi}{3}$, where n is an integer.

10 Solve the equation $4\cos^2\frac{2\theta}{3} = 1$ for $0° \le \theta \le 360°$.

MODULUS OF TRIGONOMETRIC FUNCTIONS

$|\sin\theta|$ has the same meaning as $|x|$, i.e. it is the numerical value of $\sin\theta$. For example, $|\sin 300°| = |-0.866| = 0.866$, and so on.

Example 11

For $0° \le \theta \le 360°$, sketch separate graphs of (a) $y = 2 \sin \theta$, (b) $y = |2 \sin \theta|$, (c) $y = 1 + |\cos 2\theta|$, (d) $y = 1 - |\cos 2\theta|$.

(a) First sketch $y = \sin \theta$ (Fig.7.13)
For $y = 2 \sin \theta$, each value of $y = \sin \theta$ is doubled to give the graph of $y = 2 \sin \theta$.

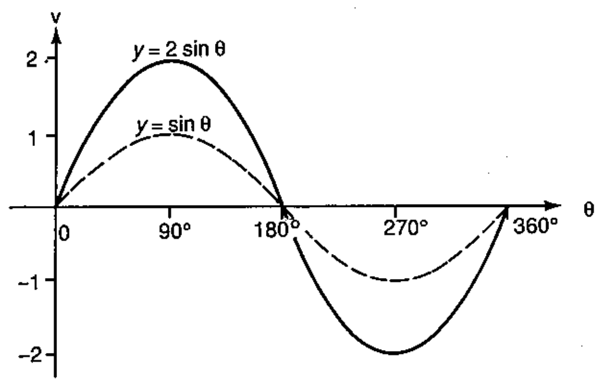


Fig. 7.13

(b) As we did earlier, we reflect the negative part of $y = 2 \sin \theta$ in the θ-axis to obtain $y = |2 \sin \theta|$ (Fig.7.14).

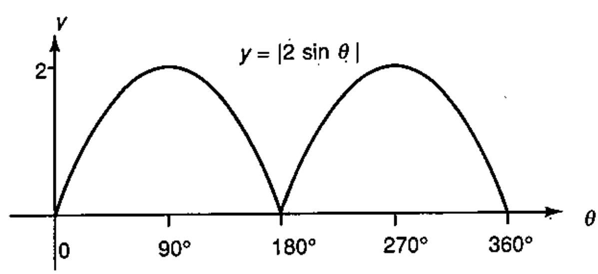


Fig. 7.14

(c) First sketch $y = \cos 2\theta$ (Fig.7.15) which has two cycles. Now reflect the negative part in the θ-axis to obtain $y = |\cos 2\theta|$.
This curve is now shifted up through 1 unit to obtain $y = 1 + |\cos 2\theta|$.

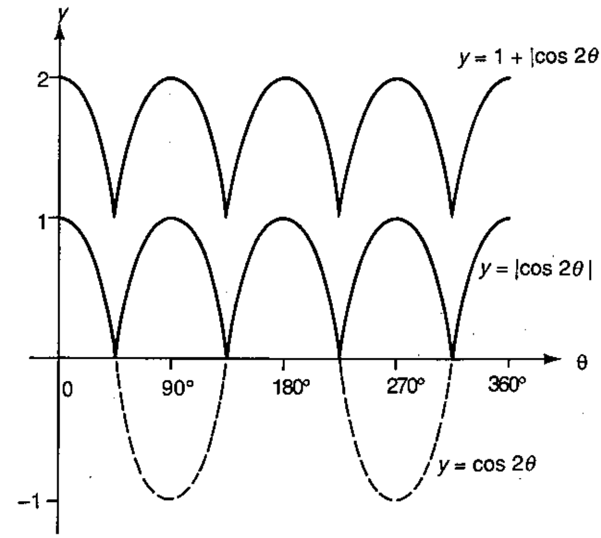


Fig.7.15

(d) Start by sketching $y = |\cos 2\theta|$ as in part (c) (Fig.7.16).
Then obtain $y = -|\cos 2\theta|$ by reflection of the whole curve in the θ-axis.
This is now shifted up through 1 unit to obtain $y = 1 - |\cos 2\theta|$ (Fig.7.17).

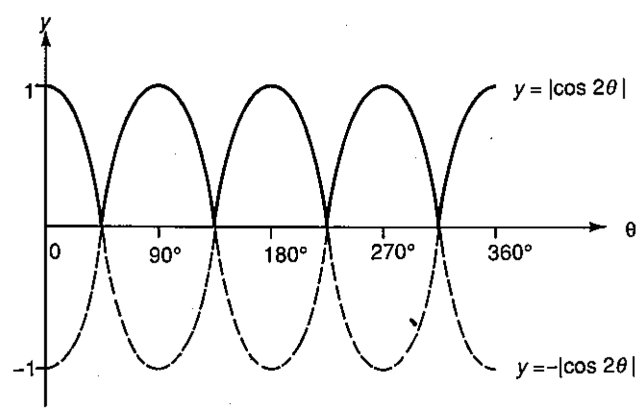


Fig.7.16

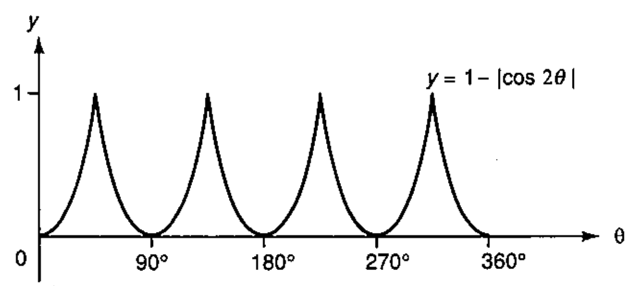


Fig.7.17

Example 12

Sketch on the same diagram, the graphs of $y = |2 \sin x|$ and $y = \frac{x}{\pi}$ for $0 \le x \le 2\pi$. Hence state the number of solutions of the equations $|2\pi \sin x| = x$ and $2\pi \sin x = x$ for $0 \le x \le 2\pi$.

We have to work in radians here as $y = \frac{x}{\pi}$ is a linear equation.
($y = \frac{x}{180°}$ is not meaningful.)

The graph of $y = 2 \sin x$ is drawn and then $y = |2 \sin x|$ (Fig.7.18).

To draw the line $y = \frac{x}{\pi}$ we take the points $x = 0$, $y = 0$ and $x = 2\pi$, $y = 2$.

The equation $|2\pi \sin x| = x$ is the same as $|2 \sin x| = \frac{x}{\pi}$ as π is positive. The solutions will occur at the intersections of the curve and the line, giving 4 solutions at the points marked O, A, B and C.

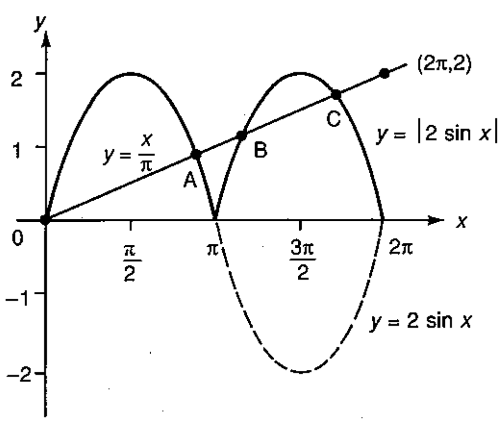


Fig.7.18

The equation $2\pi \sin x = x$ is the same as $2 \sin x = \frac{x}{\pi}$. So we look for the intersections of the original curve $y = 2 \sin x$ with the line, which reduces the number of solutions to 2 (points O and A).

Example 13

Sketch on the same diagram the graphs of $y = |2 \cos x|$ and $3y = x$ for the domain $0 \le x \le 2\pi$. Hence state the number of solutions in this domain of the equation $6|\cos x| = x$.

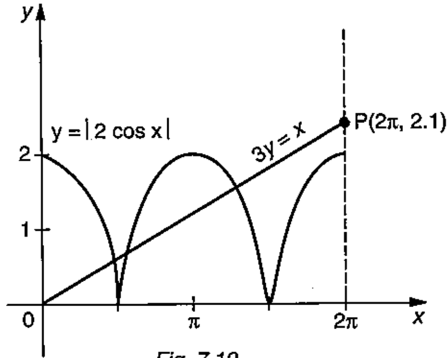


Fig. 7.19

Fig. 7.19 shows the graphs. The graph of $3y = x$ i.e. $y = \frac{x}{3}$ is the line OP, where O is the origin and P is the point $(2\pi, \frac{2\pi}{3} \approx 2.1)$. There are 3 solutions to the equation $|2\cos x| = \frac{x}{3}$ i.e. $6|\cos x| = x$.

Exercise 7.3 *(Answers on page 621.)*

1 State the values of (a) $|\sin 200°|$, (b) $|\cos \frac{2\pi}{3}|$, (c) $\sin|-200°|$, (d) $|\tan \frac{5\pi}{6}|$.

2 By sketching the graph of $y = \sin 2\theta$ for $0° \leq \theta \leq 360°$, find how many solutions the equation $\sin 2\theta = k$ will have in this interval, where $0 < k < 1$. How many solutions will the equation $|\sin 2\theta| = k$ have in the same interval?

3 Sketch the graphs of $y = |\cos\theta|$ and $y = |\cos\theta| - 1$ for $0° \leq \theta \leq 360°$.

4 On the same diagram, sketch the graphs of $y = |\sin\theta|$ and $y = |\cos\theta|$ for $0° \leq \theta \leq 360°$. How many solutions will the equation $|\sin\theta| = |\cos\theta|$ have in this interval?

5 Sketch the graphs of $y = 1 + 2\sin\theta$ and $y = |1 + 2\sin\theta|$ for $0° \leq \theta \leq 360°$. On another diagram, sketch the graph of $y = 1 + |2\sin\theta|$.

6 On the same diagram, for $0° \leq \theta \leq 360°$, sketch the graphs of $y = 2\cos\theta$ and $y = |2\cos\theta|$. Now add the graph of $y = 1 - |2\cos\theta|$.

7 On the same diagram, sketch the graphs of $y = |2\cos x|$ and $y = \frac{x}{2\pi}$ for $0 \leq x \leq 2\pi$. Hence state the number of solutions of the equations $|4\pi\cos x| = x$ and $4\pi\cos x = x$ for $0 \leq x \leq 2\pi$.

8 Sketch the graph of $y = |\tan\theta|$ for $0° \leq \theta \leq 360°$.

9 Sketch on the same diagram the graphs of $y = |\cos 2x|$ and $2y = x$ for the domain $0 \leq x \leq \pi$. Hence state the number of solutions in this domain of the equation $|2\cos 2x| = x$.

10 For $0 \leq x \leq 2\pi$, sketch the graphs of $y = |\cos x|$ and $y = \sin 2x$ on the same axes. State the number of solutions of the equation $\sin 2x = |\cos x|$ in this interval.

11 Sketch the graphs of $y = |\sin 3x|$ and $2\pi y = x$ for $0 < x \le 2\pi$. How many solutions do the equations $2\pi \sin 3x = x$ and $|2\pi \sin 3x| = x$ have in this interval?

12 On the same diagram, sketch the graphs of $y = |\sin x - 1|$ and $y = 2 \cos x$ for $0 \le x \le 2\pi$. Hence find the number of solutions of the equation $2 \cos x = |\sin x - 1|$ in this interval.

IDENTITIES

We have defined earlier, for an angle θ, $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \frac{y}{x}$ where (x,y) were the coordinates of R and OR = 1 unit (*Fig. 7.19*).

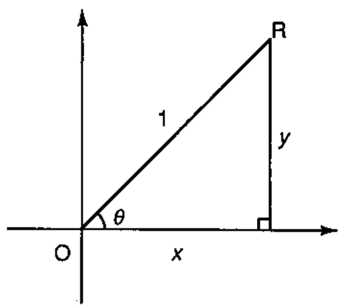


Fig. 7.20

Then $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (i)

This is an **identity** which is true for all values of θ. So we use the symbol $\equiv$ meaning 'identical to' or 'equivalent to'. In any expression, $\tan\theta$ could be replaced by $\dfrac{\sin\theta}{\cos\theta}$ or vice-versa.

As $\cot\theta = \dfrac{1}{\tan\theta}$, then $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$ (ii)

From Fig.7.19, $x^2 + y^2 = 1$ for all values of x and y.

Hence $\mathbf{\sin^2\theta + \cos^2\theta \equiv 1}$ (iii)

[*Note*: $\sin^2\theta$ means $(\sin\theta)^2$]

and $\mathbf{\sin^2\theta \equiv 1 - \cos^2\theta}$ (iv)

and $\mathbf{\cos^2\theta \equiv 1 - \sin^2\theta}$ (v)

Taking identity (iii), divide both sides by $\cos^2\theta$:

then $\dfrac{\sin^2\theta}{\cos^2\theta} + 1 \equiv \dfrac{1}{\cos^2\theta}$

i.e. $\mathbf{\tan^2\theta + 1 \equiv \sec^2\theta}$ (vi)

Dividing both sides of identity (iii) by $\sin^2\theta$:

then $1 + \dfrac{\cos^2\theta}{\sin^2\theta} \equiv \dfrac{1}{\sin^2\theta}$

i.e. $\mathbf{1 + \cot^2\theta \equiv \mathrm{cosec}^2\,\theta}$ (vii)

Summarizing:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \cot\theta \equiv \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\sin^2\theta \equiv 1 - \cos^2\theta \qquad \cos^2\theta \equiv 1 - \sin^2\theta$$

$$\tan^2\theta + 1 \equiv \sec^2\theta \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

These identities are used to transform trigonometric expressions into another form.

Example 14

Prove that $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta \sec\theta$.

We take one side and convert it to the expression on the other side. It is usually easier to start with the side which is more complicated or which involves sums of functions. This gives more scope for manipulation.

Taking the left hand side (LHS):

$$\begin{aligned}\cot\theta + \tan\theta &\equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\\ &\equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta} \times \frac{1}{\cos\theta}\\ &\equiv \operatorname{cosec}\theta\sec\theta\end{aligned}$$

If we start with the RHS, then

$$\frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \equiv \frac{1}{\sin\theta\cos\theta}$$

but it is not obvious that we should now replace 1 by $\sin^2\theta + \cos^2\theta$. Do this and then divide the numerator by $\sin\theta\cos\theta$ to complete the proof.

Example 15

Show that $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} \equiv 2\sec^2\theta$

We take the more complicated LHS.

Then
$$\begin{aligned}\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} &\equiv \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}\\ &\equiv \frac{2}{1-\sin^2\theta}\\ &\equiv \frac{2}{\cos^2\theta} \equiv 2\sec^2\theta\end{aligned}$$

Example 16

Prove that $\tan^2\theta \equiv \sin^2\theta(1 + \tan^2\theta)$

$$\text{RHS} \equiv \sin^2\theta\left(1 + \frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$\equiv \sin^2\theta\left(\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\right)$$

$$\equiv \sin^2\theta\left(\frac{1}{\cos^2\theta}\right) \equiv \tan^2\theta$$

Exercise 7.4

Prove the following identities:

1 $\sin\theta \cot\theta \equiv \cos\theta$

2 $(1 + \tan^2\theta)\cos^2\theta \equiv 1$

3 $(1 + \tan^2\theta)(1 - \sin^2\theta) \equiv 1$

4 $\cos^2\theta - \sin^2\theta \equiv 1 - 2\sin^2\theta$

5 $\sec\theta - \cos\theta \equiv \sin\theta \tan\theta$

6 $\cot^2\theta\,(1 - \cos^2\theta) \equiv \cos^2\theta$

7 $\dfrac{1}{\cos^2\theta} - \dfrac{1}{\cot^2\theta} \equiv 1$

8 $\dfrac{\cot\theta}{\tan\theta} + 1 \equiv \text{cosec}^2\theta$

9 $\tan^2\theta - \sin^2\theta \equiv \sin^4\theta \sec^2\theta$

10 $(\sin\theta + \cos\theta)(\tan\theta + \cot\theta) \equiv \sec\theta + \text{cosec}\,\theta$

11 $\sin^4\theta - \cos^4\theta \equiv 1 - 2\cos^2\theta$

12 $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 \equiv 2$

13 $\dfrac{1 - \tan^2\theta}{\cot^2\theta - 1} \equiv \tan^2\theta$

14 $\sec\theta + \tan\theta \equiv \dfrac{1}{\sec\theta - \tan\theta}$

15 $\sec^4\theta - \sec^2\theta \equiv \tan^2\theta + \tan^4\theta$

16 $(\text{cosec}\,\theta - \cot\theta)^2 \equiv \dfrac{1 - \cos\theta}{1 + \cos\theta}$

EQUATIONS WITH MORE THAN ONE FUNCTION

Further types of trigonometrical equations can be solved using the identities we have just learnt. Some methods of solution are now shown. The object is to reduce the equation to one function.

Example 17

Solve the equation $3\cos\theta + 2\sin\theta = 0$ for $0° \le \theta \le 360°$.

The equation contains two functions but if we divide throughout by $\cos\theta$, this will be reduced to one function.

Then $3 + 2\dfrac{\sin\theta}{\cos\theta} = 0$ or $\tan\theta = -1.5$.

Now solve this basic equation.
Verify that the solutions are 123.7° and 303.7°.

Example 18

Solve the equation $2\sin\theta = \tan\theta$ for $0° \le \theta \le 360°$.
Illustrate the solutions graphically.

Rewrite the equation as $2\sin\theta = \dfrac{\sin\theta}{\cos\theta}$

i.e. $2\sin\theta\cos\theta - \sin\theta = 0$
or $\sin\theta(2\cos\theta - 1) = 0$.

This can be separated into two basic equations $\sin\theta = 0$ and $2\cos\theta - 1 = 0$
i.e. $\cos\theta = 0.5$.

The solutions of $\sin\theta = 0$ are 0°, 180° and 360°.
The solutions of $\cos\theta = 0.5$ are 60° and 300°.
Hence the solutions are 0°, 60°, 180°, 300° and 360°.
The graphs of $y = 2\sin\theta$ and $y = \tan\theta$ are shown in Fig. 7.21, with the positions of the solutions marked.

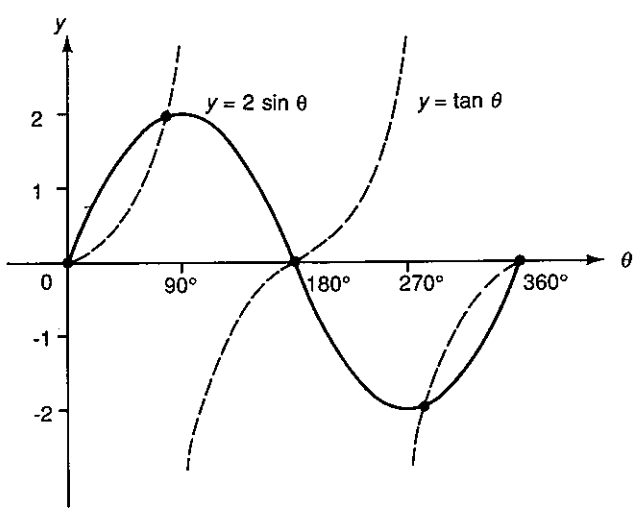


Fig. 7.21

Example 19

Solve $3 \sin \theta + 5 \cot \theta = \operatorname{cosec} \theta$ for $0° \leq \theta \leq 360°$.

This involves three functions. Reduce this to two by replacing $\cot \theta$ and $\operatorname{cosec} \theta$.

Then $3 \sin \theta + 5\dfrac{\cos \theta}{\sin \theta} = \dfrac{1}{\sin \theta}$.

Now remove the fractions: $3 \sin^2 \theta + 5 \cos \theta = 1$

We can now reduce to one function by replacing $\sin^2 \theta$ by $1 - \cos^2 \theta$.

Then $3(1 - \cos^2 \theta) + 5 \cos \theta = 1$ or $3 \cos^2 \theta - 5 \cos \theta - 2 = 0$.

This is a quadratic in $\cos \theta$ and gives $(3 \cos \theta + 1)(\cos \theta - 2) = 0$.

We now have two basic equations:

$\cos \theta = -\frac{1}{3}$ which gives $\theta = 109.47°$ or $250.53°$,

and $\cos \theta = 2$ which has no solution.

Hence, the solutions are $\theta = 109.5°$ and $250.5°$.

Example 20

Solve the equation $4 \operatorname{cosec}^2 \theta - 7 = 4 \cot \theta$ for $0° \leq \theta \leq 180°$.

If we replace $\operatorname{cosec}^2 \theta$ by $1 + \cot^2 \theta$, we shall have an equation in $\cot \theta$ only.

Then $4(1 + \cot^2 \theta) - 7 = 4 \cot \theta$ i.e. $4 \cot^2 \theta - 4 \cot \theta - 3 = 0$.

This is a quadratic in $\cot \theta$ and gives $(2 \cot \theta - 3)(2 \cot \theta + 1) = 0$ leading to the basic equations $\cot \theta = 1.5$ and $\cot \theta = -0.5$.

Hence $\tan \theta = 0.6667$ and $\tan \theta = -2$.

Now solve these but note that the domain is $0°$ to $180°$.

The only solutions are therefore $\theta = 33.7°$ and $116.6°$.

Exercise 7.5 *(Answers on page 622.)*

Solve the following equations for $0° \leq \theta \leq 360°$:

1 $8 \cot \theta = 3 \sin \theta$

2 $\sin \theta + 4 \cos^2 \theta = 1$

3 $8 \sin \theta = 3 \cos^2 \theta$

4 $2 \sec^2 \theta = 3 - \tan^2 \theta$

5 $\cot \theta + \tan \theta = 2 \sec \theta$

6 $\tan \theta + 3 \cot \theta = 4$

7 $\cot^2 \theta + 3 \operatorname{cosec}^2 \theta = 5$

8 $3(\sec \theta - \tan \theta) = 2 \cos \theta$

9 $2 \cot^2 \theta + 11 = 9 \operatorname{cosec} \theta$

10 $3 \sin^2 \theta = 1 + \cos \theta$

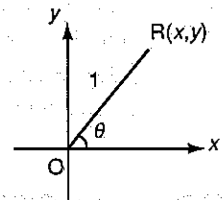
11 $5 \cos \theta - \sec \theta = 4$

12 $3 \cot 2\theta + 2 \sin 2\theta = 0$

SUMMARY

- If θ is any angle, $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \frac{y}{x}$ where (x,y) are the coordinates of R and OR = 1 (Fig. 7.22).

Fig. 7.22



SIN +	All +
$\sin\theta = \sin(180° - \theta)$	$\sin\theta$
$\cos\theta = -\cos(180° - \theta)$	$\cos\theta$
$\tan\theta = -\tan(180° - \theta)$	$\tan\theta$
2nd	1st
3rd	4th
$\sin\theta = -\sin(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$
$\cos\theta = -\cos(\theta - 180°)$	$\cos\theta = \cos(360° - \theta)$
$\tan\theta = \tan(\theta - 180°)$	$\tan\theta = -\tan(360° - \theta)$
TAN +	COS +

- To solve a basic equation such as $\sin\theta = k$:
  (1) find the angle α in the 1st quadrant such that $\sin\alpha = |k|$;
  (2) find the quadrants in which θ will lie;
  (3) determine the corresponding angles in these quadrants and solve for θ. A basic equation will usually have 2 solutions in the interval 0° to 360°.

- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\qquad \sec\theta = \dfrac{1}{\cos\theta}$ $\qquad \cot\theta = \dfrac{1}{\tan\theta}$

- Graphs of sin, cos, tan (Fig. 7.23).

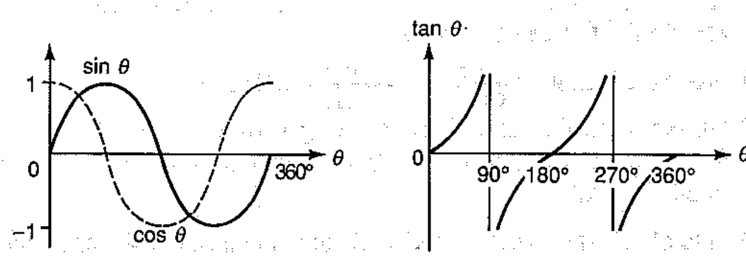


Fig. 7.23

sin and cos have a period of 360°:
$\sin(n360° + \theta) = \sin\theta$, $\cos(n360° + \theta) = \cos\theta$, where n is an integer.

tan has a period of 180°: $\tan(n180° + \theta) = \tan\theta$.

- For equations with a multiple angle $k\theta$, solve for $k\theta$ first and then derive the values of θ.

- Identities

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad \cot\theta \equiv \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\sin^2\theta \equiv 1 - \cos^2\theta \qquad \cos^2\theta \equiv 1 - \sin^2\theta$$

$$\tan^2\theta + 1 \equiv \sec^2\theta \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

To solve equations with more than one function, use the above identities to reduce to one function.

REVISION EXERCISE 7 *(Answers on page 623.)*

A

1 Find all the angles between 0° and 360° which satisfy the equations
(a) $\cot 2x = -\frac{1}{2}$, (b) $2\sin y = 3\cos y$.

2 Sketch on the same diagram, for $0 \le x \le 2\pi$, the graph of $y = 2\cos x - 1$ and the graph of $y = \sin 2x$. Hence state the number of solutions in this interval of the equation $2\cos x - 1 = \sin 2x$. (C)

3 Sketch the graph of (a) $y = |\cos x|$, (b) $y = |\cos x| - 1$ and (c) $y = 1 - |\cos x|$ for values of x between 0 and 2π.

4 Prove the identity $\sec x - \cos x \equiv \sin x \tan x$.

5 Find all the angles between 0° and 180° which satisfy the equations
(a) $\cos \frac{2}{3}x = \frac{2}{3}$, (b) $3\cot y - 4\cos y = 0$,
(c) $3\sec^2 z = 7 + 4\tan z$. (C)

6 Solve for $0° \le \theta \le 360°$, the equations
(a) $\operatorname{cosec} 2\theta = 3$ (b) $4\cot\theta = 5\cos\theta$
(c) $10\sin^2\theta + 31\cos\theta = 13$.

7 Prove the identity $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} \equiv 2\sec x$. (C)

8 On the same diagram, sketch the graphs of $y = 1 + \cos x$ and $y = |\sin x|$ for $0 \le x \le 2\pi$. Hence state the number of solutions of the equation $1 + \cos x = |\sin x|$ in this interval.

9 Find all the angles between 0° and 180° which satisfy the equations
(a) $\tan(x + 70°) = 1$,
(b) $8\sin y + 3\cos y = 0$,
(c) $3\sin^2\theta + 5\sin\theta\cos\theta - 2\cos^2\theta = 0$.

10 Sketch on the same diagram, the graphs of $y = |2\cos x|$ and $y = \frac{4x}{3\pi}$ for $0 \le x \le 2\pi$. State, for the range $0 \le x \le 2\pi$, the number of solutions of (i) $|3\pi\cos x| = 2x$, (ii) $3\pi\cos x = 2x$. (C)

11 State the range of $y = 2 - |\cos x|$ for the domain $0 \le x \le \frac{3\pi}{2}$.

12 On the same diagram, sketch the graphs of $y = \sin 2x$ and $y = \sin \frac{x}{2}$. for $0 \le x \le 2\pi$. Hence state the number of solutions of the equation $\sin 2x = \sin \frac{x}{2}$ in that interval. What would be the number of solutions of $|\sin 2x| = \sin \frac{x}{2}$?

13 For the domain $0° \le \theta \le 360°$, solve
(a) $\sin \theta + \cos \theta \cot \theta = 2$,
(b) $6 \cot^2 \theta = 1 + 4 \operatorname{cosec}^2 \theta$.

B

14 Solve the equation $\sin \theta = 4 \sin^3 \theta$ for $0° \le \theta \le 360°$.

15 Solve, for $0° \le \theta \le 360°$, the equations
(a) $8 \sin^2 \theta = \operatorname{cosec} \theta$,
(b) $4 \cos^2 \theta = 9 - 2 \sec^2 \theta$.

16 Sketch the graphs of $y = |2 \sin x|$ and $y = |\frac{x}{\pi} - 1|$ for $0 \le x \le 2\pi$. How many solutions are there of the equation $|2\pi \sin x| = |x - \pi|$ in this interval?

17 A segment ACB in a circle is cut off by the chord AB where $\angle AOB = \theta$ radians (O is the centre). If the area of this segment is $\frac{1}{4}$ of the area of the circle, show that $\theta - \sin \theta = \frac{\pi}{2}$.
Draw the graphs of $y = \sin \theta$ and $y = \theta - \frac{\pi}{2}$ for $0 \le \theta \le \pi$, taking scales of 4 cm for $\frac{\pi}{2}$ on the x-axis and 4 cm per unit on the y-axis. (Take $\pi = 3.14$). Hence find an approximate solution of the equation $\theta - \sin \theta = \frac{\pi}{2}$.

18 In Fig. 7.24, ACB is a semicircle of radius r, centre O and $\angle ABC = \theta°$.
(a) Using the identity $2 \sin \theta \cos \theta = \sin 2\theta$, show that the area of the shaded region is $r^2 (\frac{\pi}{2} - \sin 2\theta)$.
(b) State in terms of r, the maximum and minimum possible values of this area and the corresponding values of θ.
(c) Find the values of θ for which the area of the shaded region equals $\frac{1}{2}$ the area of the semicircle.

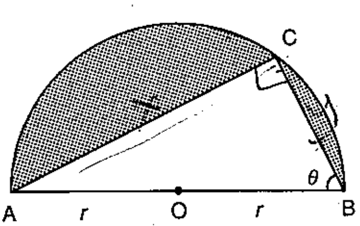


Fig. 7.24

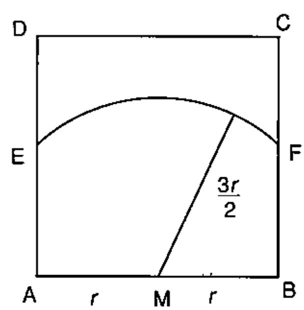


Fig. 7.25

19 A goat is tied to one end of a rope of length $\frac{3r}{2}$, the other end being fixed to the midpoint M of the side AB of a square field ABCD of side $2r$ (Fig. 7.25).
(a) Find, in radians, $\angle EMF$.
(b) Find in terms of r the area ABFE.
(c) Calculate what percentage of the area of the field the goat can cover.

20 In Fig. 7.26, OA and OB are two radii of a circle centre O where angle BOA = θ radians. The tangent to the circle at A meets OB produced at C. If the area of the sector OAB is twice the area of the shaded region, show that $2 \tan \theta = 3\theta$. By drawing the graphs of $y = \tan \theta$ and $y = \frac{3\theta}{2}$ for a suitable domain, or otherwise, find the approximate value of θ. (Otherwise, a solution could be found by trial and error using a calculator in radian mode. Test values of θ to make $\tan \theta - 1.5\theta$ reasonably small.)

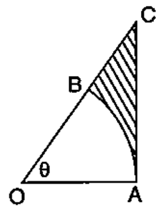


Fig. 7.26

Vectors

8

SCALARS AND VECTORS

A **scalar** is a purely numerical quantity with a unit, such as \$20 or a mass of 2 kg. No idea of *direction* is involved. A **vector** quantity, however, has a direction which must be stated, such as a velocity of 20 m s^{-1} northeast (NE). A velocity of 20 m s^{-1} southeast (SE) would be quite different.

To specify a vector, its **magnitude** (e.g. 20 m s^{-1}) *and* its direction (e.g. NE) must **both** be given.

Scalars are added and subtracted by the usual rules of arithmetic but to 'add' or 'subtract' vectors, we use a special rule – the **parallelogram law.**

REPRESENTATION OF VECTORS

A simple example of a vector is a **displacement.** Suppose a piece of board is moved, without rotation, across a flat surface (Fig.8.1).

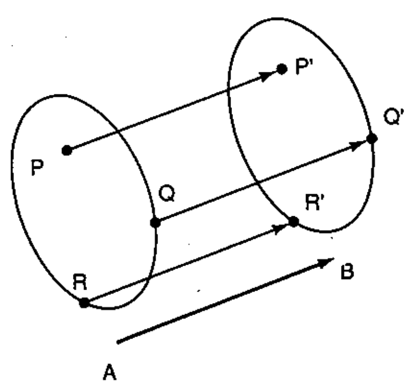


Fig. 8.1

Points on the board such as P, Q, R are displaced through the same distance and in the same direction to points P', Q', R'. So we can represent this vector by *any* line segment

AB where AB = PP' = QQ' = RR' and AB // PP' // QQ' // RR'. The arrow head shows the sense of the direction. AB is drawn to scale to give the correct magnitude of the displacement. We write such a vector as $\overrightarrow{AB}$.

EQUALITY OF VECTORS

In Fig.8.2, the line segments AB, CD and EF are parallel (in the same direction) and equal in length. Then these lines can each represent the same vector and $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$.

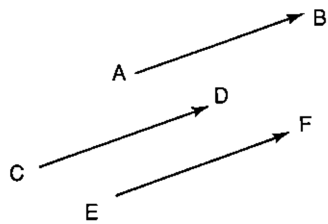


Fig. 8.2

Conversely, if $\overrightarrow{AB} = \overrightarrow{CD}$ (Fig.8.3), then
(a) the line segments AB and CD are equal in length **and**
(b) AB // CD.

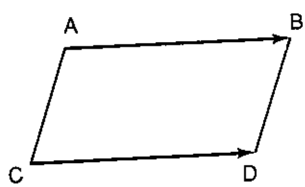


Fig. 8.3

It is important to remember that both parts are implied by the statement $\overrightarrow{AB} = \overrightarrow{CD}$. The figure ABDC is therefore a parallelogram.

NOTATION FOR VECTORS

We state the endpoints of a vector by writing it as $\overrightarrow{AB}$, as above or we can use a single letter (Fig.8.4). A vector could then be given as **a** (printed in **bold**). We write this as $\vec{a}$ or $\underset{\sim}{a}$. Always distinguish a vector **a** in this way from an algebraic quantity *a*.

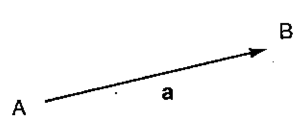


Fig. 8.4

MAGNITUDE OF A VECTOR

The magnitude or **modulus** of a vector $\overrightarrow{AB}$ is the length of the line segment representing the vector to the scale used. We denote this as $|\overrightarrow{AB}|$.
If $\overrightarrow{AB}$ in Fig.8.4 is drawn to a scale of 1 cm ≡ 10 m s^{-1} for example, then $|\overrightarrow{AB}| = 30$ m s^{-1}.
The magnitude of the vector **a** is written as | **a** | or as $\overline{a}$.
Note this carefully: **a** is the vector but | **a** | or a is its magnitude.

Zero Vector

The vector which has no magnitude (and of course no direction) is the **zero** vector, written $\underset{\sim}{0}$ or $\vec{0}$.

Scalar Multiple of a Vector

Given a vector **a** (Fig.8.5), we can make multiples of this vector.
For example, $\overrightarrow{PQ} = 2\mathbf{a}$. $\overrightarrow{PQ}$ has the same direction as **a** but twice its magnitude.
$|\overrightarrow{PQ}| = 2|\mathbf{a}| = 2a$.

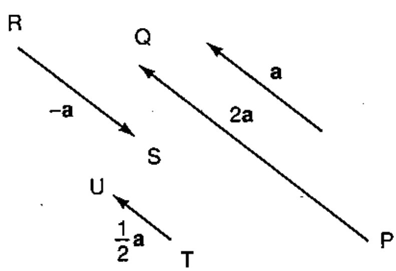


Fig. 8.5

$\overrightarrow{RS} = -\mathbf{a}$, i.e. it has the same magnitude as **a** but is in the reverse direction.
Note that $\overrightarrow{RS} = -\overrightarrow{SR}$.
$\overrightarrow{TU} = \frac{1}{2}\mathbf{a}$.

> If $\mathbf{a} = k\mathbf{b}$, where k is a scalar (a number) $\neq 0$, then the vectors **a** and **b** are parallel and in the same direction if $k > 0$ but in opposite directions if $k < 0$.
> $|\mathbf{a}| = |k| \times |\mathbf{b}|$

Conversely, if **a** and **b** are parallel, then $\mathbf{a} = k\mathbf{b}$. (k will be positive if **a** and **b** are in the same direction, negative if they are in opposite directions.)
Scalar multiples of a vector can be combined arithmetically. For example $2\mathbf{a} + 3\mathbf{a} = 5\mathbf{a}$ and $4(2\mathbf{a}) = 8\mathbf{a}$.
So $m\mathbf{a} + n\mathbf{a} = (m + n)\mathbf{a}$ and $m(n\mathbf{a}) = mn\mathbf{a}$ for all values of m and n.

Example 1

Given the vector **a** *(Fig.8.6(a)), draw the vectors (i) 3***a***, (ii)* $-\frac{1}{3}$**a**.

The vectors are shown in Fig.8.6(b). They are all parallel but (ii) is in the opposite direction to **a**.

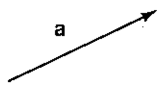


Fig. 8.6(a)

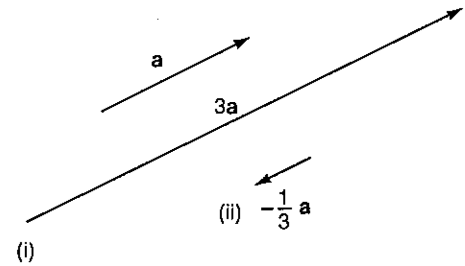


Fig. 8.6(b)

Example 2

What type of quadrilateral is ABCD if (a) $\overrightarrow{AB} = \overrightarrow{DC}$, *(b)* $\overrightarrow{AB} = 3\overrightarrow{DC}$?

(a) AB = DC and AB // DC. Then ABCD is a parallelogram (Fig.8.7(a)). It follows therefore that $\overrightarrow{AD} = \overrightarrow{BC}$.

(b) AB = 3DC and AB // DC. Then ABCD is a trapezium (Fig. 8.7(b)).

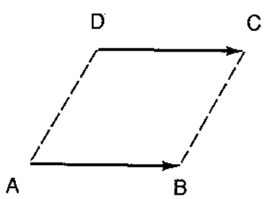


Fig. 8.7(a)

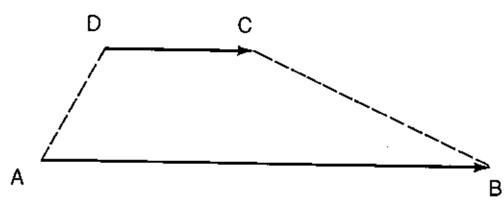
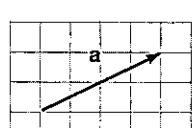


Fig. 8.7(b)

Exercise 8.1 *(Answers on page 624.)*

1 Copy Fig.8.8 and draw the vectors (a) 2**a**, (b) –**a**, (c) $\frac{3}{4}$**a**.

Fig. 8.8



2 In Fig.8.9, state each of the vectors **p**, **q** and **r** in the form $k\mathbf{a}$.

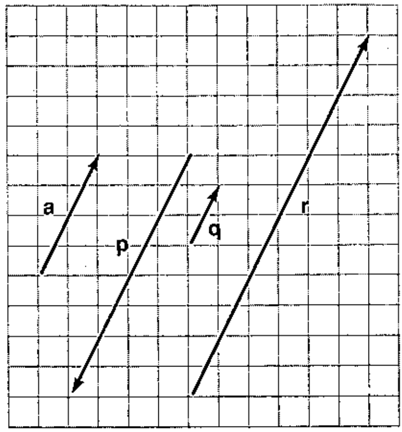


Fig. 8.9

3 The line AB is divided into three equal parts at C and D. If $\overrightarrow{AD} = \mathbf{a}$, state as scalar multiples of **a**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{CB}$, (c) $\overrightarrow{BD}$.

4 In Fig.8.10, ABCDEF is a regular hexagon. Given that $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{CD} = \mathbf{c}$, state the following vectors as scalar multiples of **a**, **b** or **c**:
(a) $\overrightarrow{DE}$, (b) $\overrightarrow{EF}$, (c) $\overrightarrow{FA}$, (d) $\overrightarrow{BE}$, (e) $\overrightarrow{AD}$.

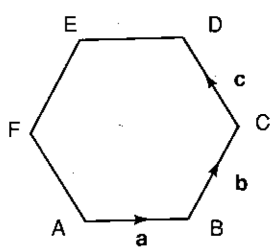


Fig. 8.10

5 If $\overrightarrow{AB} = k\overrightarrow{BC}$ ($k \neq 0$), what can be said about the points A, B and C?

6 A is the point (4,0) and B the point (0,3). State the value of $|\overrightarrow{AB}|$.

7 If P is (–2,–5) and Q is (3,7), find $|\overrightarrow{PQ}|$.

8 O is the origin, $|\overrightarrow{OR}| = 3$ and the line OR makes an angle θ with the x-axis where $\sin\theta = \frac{2}{3}$. Find the possible coordinates of R.

ADDITION OF VECTORS

To 'add' two vectors **a** and **b**, i.e. to combine them into one vector, we place them so as to start from the same point O (Fig.8.11).

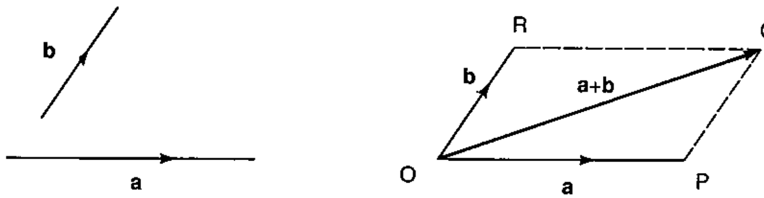


Fig. 8.11

Now complete the parallelogram OPQR.

We define **a** + **b** = $\overrightarrow{OQ}$ i.e. **the diagonal starting from O.**

$\overrightarrow{OQ}$ is called the **resultant** of **a** and **b**.

This is the **parallelogram law** for the addition of vectors. Note that we use the symbol '+' though here it means 'combined with' and not arithmetical addition.

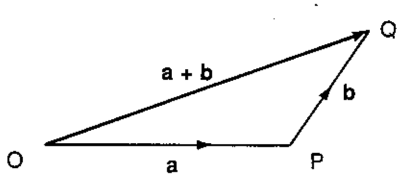
As RQ is parallel and equal to OP, $\overrightarrow{RQ}$ = **a**.

Then $\overrightarrow{OR} + \overrightarrow{RQ}$ = **b** + **a** = $\overrightarrow{OQ}$ = **a** + **b**.

Hence **a** + **b** = **b** + **a**.

In practice, it is not necessary to draw the parallelogram. The vectors can be placed 'end-on'. PQ is equal and parallel to OR so $\overrightarrow{PQ}$ = **b**. We draw **a** and then **b** starting from the end of **a** (Fig.8.12).

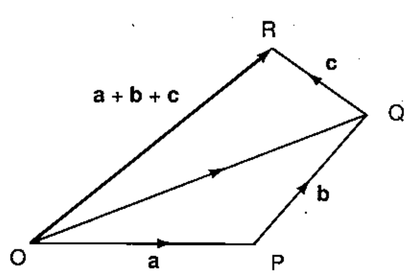
Fig.8.12



The third side OQ of the triangle gives **a** + **b**.

More than 2 vectors can be combined in this way. For example, in Fig.8.13,
a + **b** = $\overrightarrow{OQ}$ and $\overrightarrow{OQ}$ + **c** = **a** + **b** + **c** = $\overrightarrow{OR}$.

Fig. 8.13



DIAGONALS OF A PARALLELOGRAM

In Fig.8.14, $\overrightarrow{OP} = \mathbf{a}$, $\overrightarrow{OR} = \mathbf{b}$.
Then $\overrightarrow{OQ} = \mathbf{a} + \mathbf{b}$.

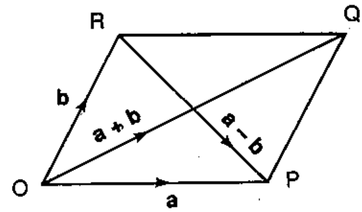


Fig.8.14

$\overrightarrow{RP} = \overrightarrow{RQ} + \overrightarrow{QP} = \mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$.
Also $\overrightarrow{PR} = -\overrightarrow{RP} = -(\mathbf{a} - \mathbf{b}) = \mathbf{b} - \mathbf{a}$.

These last two results are important and can be remembered as follows:

$\mathbf{a} - \mathbf{b}$ is the vector from the endpoint of **b** to the endpoint of **a**;
$\mathbf{b} - \mathbf{a}$ is the vector from the endpoint of **a** to the endpoint of **b**

where **a** and **b** start from the same point (Fig.8.15).

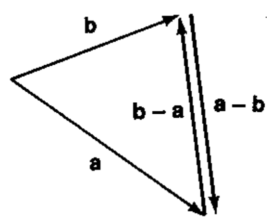


Fig.8.15

Example 3

The vectors **a** *and* **b** *are given (Fig.8.16(a)). Draw the vectors*
(i) $2\mathbf{a} + \mathbf{b}$, *(ii)* $\mathbf{a} - \mathbf{b}$, *(iii)* $\mathbf{a} - 2\mathbf{b}$.

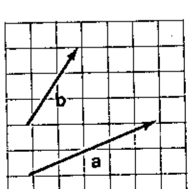


Fig. 8.16(a)

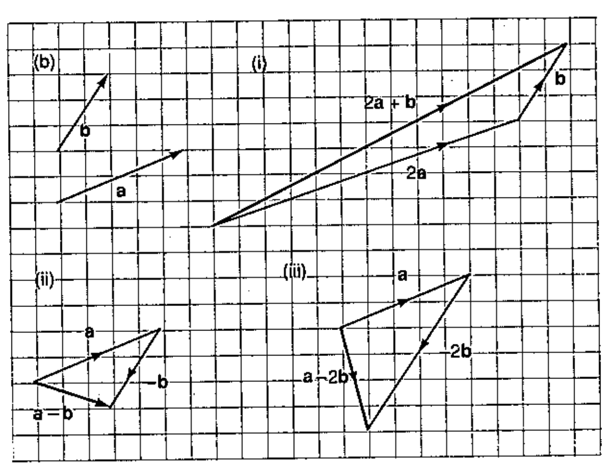


Fig. 8.16(b)

(The vectors are shown in Fig. 8.16(b).)

(i) Draw 2**a** followed by **b**.

(ii) Draw **a** followed by –**b**. Alternatively draw **a** and **b** from the same point and use the rule above.

(iii) Draw **a** followed by –2**b**.

Example 4

In △ABC, $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$ *and M is the midpoint of AB (Fig.8.17). State in terms of* **a** *and* **b**, *(a)* $\overrightarrow{AM}$, *(b)* $\overrightarrow{MC}$, *(c)* $\overrightarrow{CM}$.

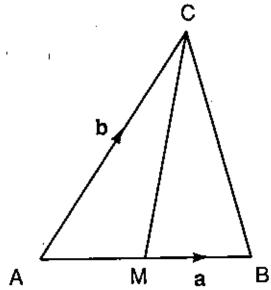
(a) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\mathbf{a}$

(b) $\overrightarrow{MC} = \overrightarrow{AC} - \overrightarrow{AM}$ (from the end of AM to end of AC)

$= \mathbf{b} - \frac{1}{2}\mathbf{a}$

(c) $\overrightarrow{CM} = -\overrightarrow{MC} = \frac{1}{2}\mathbf{a} - \mathbf{b}$.

Fig. 8.17



Example 5

$\overrightarrow{OA}$, $\overrightarrow{OB}$ *and* $\overrightarrow{OC}$ *are the vectors* **a**, **b** *and* **c** *respectively. D is the midpoint of AB and E lies on BC where BE = 2BC (Fig.8.18). State in terms of* **a**, **b** *and* **c**,

(a) $\overrightarrow{AB}$, *(b)* $\overrightarrow{AD}$, *(c)* $\overrightarrow{OD}$, *(d)* $\overrightarrow{BC}$, *(e)* $\overrightarrow{BE}$, *(f)* $\overrightarrow{OE}$, *(g)* $\overrightarrow{DE}$.

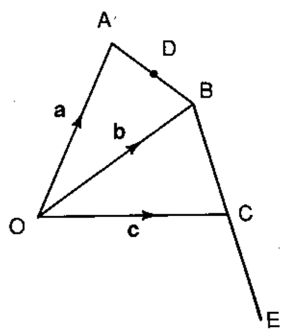


Fig. 8.18

(a) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

(b) $\overrightarrow{AD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

(c) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

(d) $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

(e) $\overrightarrow{BE} = 2(\mathbf{c} - \mathbf{b})$

(f) $\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE} = \mathbf{b} + 2(\mathbf{c} - \mathbf{b}) = 2\mathbf{c} - \mathbf{b}$

(g) $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = 2\mathbf{c} - \mathbf{b} - \frac{1}{2}(\mathbf{a} + \mathbf{b}) = 2\mathbf{c} - \frac{\mathbf{a}}{2} - \frac{3\mathbf{b}}{2}$

Exercise 8.2 *(Answers on page 624.)*

1 Given the vectors **a** and **b** in Fig.8.19, draw the vectors
(a) $\mathbf{a} + 2\mathbf{b}$, (b) $2\mathbf{a} - \mathbf{b}$, (c) $3\mathbf{a} - 2\mathbf{b}$.

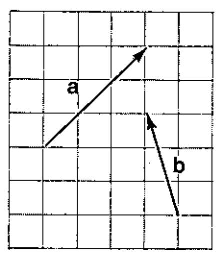


Fig. 8.19

2 In $\triangle ABC$, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. State in terms of **a** and **b**, (a) $\overrightarrow{AC}$ and (b) $\overrightarrow{CA}$.

3 Given the vectors **a**, **b** and **c** in Fig.8.20, draw
(a) $\mathbf{a} + 2\mathbf{b}$, (b) $\mathbf{a} + 2\mathbf{b} + \mathbf{c}$, (c) $\mathbf{a} - \mathbf{b} + \mathbf{c}$, (d) $\frac{1}{2}\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.

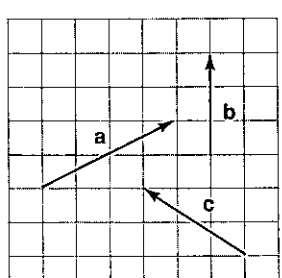


Fig.8.20

4 If $|\mathbf{a}| = |\mathbf{b}|$ but $\mathbf{a} \neq \mathbf{b}$, explain why $\mathbf{a} + \mathbf{b}$ bisects the angle between **a** and **b** and is perpendicular to $\mathbf{a} - \mathbf{b}$.

5 In $\triangle OAB$, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the midpoint of AB. State in terms of **a** and **b**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{AM}$, (c) $\overrightarrow{OM}$.

6 In $\triangle$OPQ, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$. R is the midpoint of OP and S lies on OQ such that OS = 3SQ. State in terms of **p** and **q**, (a) $\overrightarrow{OR}$, (b) $\overrightarrow{PQ}$, (c) $\overrightarrow{OS}$, (d) $\overrightarrow{RS}$.

7 In $\triangle$OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. BC is drawn parallel to OA (in the same direction) and BC = 2OA. State in terms of **a** and **b**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BC}$, (c) $\overrightarrow{OC}$, (d) $\overrightarrow{AC}$

8 OACB is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. AC is extended to D where AC = 2CD. Find, in terms of **a** and **b**, (a) $\overrightarrow{AD}$, (b) $\overrightarrow{OD}$, (c) $\overrightarrow{BD}$.

9 OAB is a triangle with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of OA and G lies on MB such that MG = $\frac{1}{2}$GB. State in terms of **a** and **b** (a) $\overrightarrow{OM}$, (b) $\overrightarrow{MB}$, (c) $\overrightarrow{MG}$, (d) $\overrightarrow{OG}$.

10 $\overrightarrow{OA} = \mathbf{p} + \mathbf{q}$, $\overrightarrow{OB} = 2\mathbf{p} - \mathbf{q}$, where **p** and **q** are two vectors and M is the midpoint of AB. Find in terms of **p** and **q**, (a) $\overrightarrow{AB}$, (b) $\overrightarrow{AM}$, (c) $\overrightarrow{OM}$.

POSITION VECTORS

If O is the origin, then the vector $\overrightarrow{OA}$ is called the **position vector** of A. For example, if the position vector of A is $2\mathbf{a} - 3\mathbf{b}$, then $\overrightarrow{OA}$ is $2\mathbf{a} - 3\mathbf{b}$.

Using Vectors

The following principles should be carefully noted:

(1) If $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$ then $m = p$ and $n = q$. (See Examples 6 and 9).

(2) If the points P, Q and R are collinear, then $\overrightarrow{PQ} = k\overrightarrow{QR}$ (and conversely) because $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel but meet at Q. (See Examples 7 and 8). We could also use $\overrightarrow{PQ} = k\overrightarrow{PR}$.

(3) If the vectors $m\mathbf{a} + n\mathbf{b}$ and $p\mathbf{a} + q\mathbf{b}$ are parallel, then $\frac{m}{p} = \frac{n}{q}$. (See Example 10.)

Example 6

If $\mathbf{p} = 2\mathbf{a} - 3\mathbf{b}$ *and* $\mathbf{q} = \mathbf{a} + 2\mathbf{b}$, *find numbers x and y such that* $x\mathbf{p} + y\mathbf{q} = \mathbf{a} - 12\mathbf{b}$.

$x\mathbf{p} + y\mathbf{q} = x(2\mathbf{a} - 3\mathbf{b}) + y(\mathbf{a} + 2\mathbf{b}) = (2x + y)\mathbf{a} + (-3x + 2y)\mathbf{b}$

By (1) above, if this vector is to equal $\mathbf{a} - 12\mathbf{b}$, then the multiples of **a** and the multiples of **b** on each side must be separately equal.
Hence $2x + y = 1$ and $-3x + 2y = -12$.
Solving these equations, $x = 2$, $y = -3$.
Checking this, $2(2\mathbf{a} - 3\mathbf{b}) - 3(\mathbf{a} + 2\mathbf{b}) = \mathbf{a} - 12\mathbf{b}$ as required.

Example 7

The position vectors of P, Q and R are $\mathbf{a}-2\mathbf{b}$, $2\mathbf{a}-3\mathbf{b}$ *and* $\mu\mathbf{a}-6\mathbf{b}$, *where* μ *is a scalar constant. If the points P, Q and R are collinear, find*
(a) the value of μ *and (b) the ratio PQ:QR.*

First we find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

(a) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\mathbf{a} - 3\mathbf{b} - (\mathbf{a} - 2\mathbf{b}) = \mathbf{a} - \mathbf{b}$
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \mu\mathbf{a} - 6\mathbf{b} - (2\mathbf{a} - 3\mathbf{b}) = (\mu - 2)\mathbf{a} - 3\mathbf{b}$

Now if P, Q and R are to be collinear, $\overrightarrow{PQ} = k\overrightarrow{QR}$.
$\overrightarrow{PQ} = \mathbf{a} - \mathbf{b}$ and $\overrightarrow{QR} = (\mu - 2)\mathbf{a} - 3\mathbf{b}$
Comparing these, the multiple of $-\mathbf{b}$ in $\overrightarrow{QR}$ is 3 so the multiple of $\mathbf{a}$ must also be 3. Hence $\mu - 2 = 3$ or $\mu = 5$.

(b) When $\mu = 5$, $|\overrightarrow{PQ}| = |\mathbf{a} - \mathbf{b}|$ and $|\overrightarrow{QR}| = |3(\mathbf{a} - \mathbf{b})|$ which gives the ratio PQ:QR as 1:3.

Example 8

$\overrightarrow{OP} = 3\mathbf{a} + \mathbf{b}$, $\overrightarrow{OQ} = \mu(\mathbf{a} - \mathbf{b})$ *and* $\overrightarrow{OR} = 4\mathbf{a} + 4\mathbf{b}$.
Given that P, Q and R are collinear, find the value of μ *and the ratio PQ:QR.*

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mu\mathbf{a} - \mu\mathbf{b} - 3\mathbf{a} - \mathbf{b} = (\mu - 3)\mathbf{a} - (\mu + 1)\mathbf{b}$
$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = 4\mathbf{a} + 4\mathbf{b} - \mu\mathbf{a} + \mu\mathbf{b} = (4 - \mu)\mathbf{a} + (\mu + 4)\mathbf{b}$

The relation between these vectors is not as straightforward as it was in Example 7. We shall have to find an equation for μ. If P, Q and R are collinear, $\overrightarrow{PQ} = k\overrightarrow{QR}$ so the multiples of $\mathbf{a}$ and of $\mathbf{b}$ in the two vectors must be in the same ratio.

Then $\frac{\mu - 3}{4 - \mu} = \frac{-\mu - 1}{\mu + 4}$ which leads to $\mu^2 + \mu - 12 = \mu^2 - 3\mu - 4$ giving $\mu = 2$.

Hence $\overrightarrow{PQ} = -\mathbf{a} - 3\mathbf{b}$ and $\overrightarrow{QR} = 2\mathbf{a} + 6\mathbf{b} = -2(-\mathbf{a} - 3\mathbf{b})$.

The ratio PQ:QR = 1:–2 which means that QR is twice as long as PQ but in the *opposite direction* as shown in Fig.8.21.

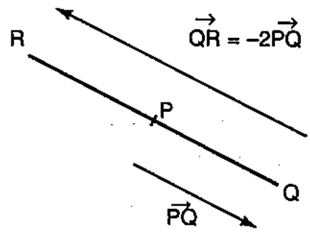


Fig.8.21

Example 9

In Fig.8.22, $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$. C lies on OA where $OC = \frac{2}{3}$ OA, D is the midpoint of AB and BC and OD intersect at M.

(a) By taking $\vec{OM} = p\vec{OD}$ and $\vec{BM} = q\vec{BC}$, where p and q are numbers, find two vector expressions for $\vec{OM}$.

Hence find (b) the values of p and q, (c) the ratios OM:MD and BM:MC.

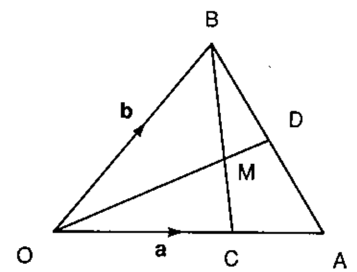


Fig.8.22

(a) First find $\vec{OM}$ as part of $\vec{OD}$. To obtain $\vec{OD}$ we find $\vec{AD}$.

$\vec{AB} = \mathbf{b} - \mathbf{a}$ so $\vec{AD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

Then $\vec{OD} = \vec{OA} + \vec{AD} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

So $\vec{OM} = p\vec{OD} = \frac{p}{2}(\mathbf{a} + \mathbf{b}) = \frac{p}{2}\mathbf{a} + \frac{p}{2}\mathbf{b}$. (i)

Now to find another expression for $\vec{OM}$, we use $\vec{OB} + \vec{BM}$.

First find $\vec{BM}$ as part of $\vec{BC}$.

$\vec{BC} = \vec{OC} - \vec{OB} = \frac{2}{3}\mathbf{a} - \mathbf{b}$ so $\vec{BM} = q(\frac{2}{3}\mathbf{a} - \mathbf{b})$

Then $\vec{OM} = \vec{OB} + \vec{BM} = \mathbf{b} + q(\frac{2}{3}\mathbf{a} - \mathbf{b}) = \frac{2q}{3}\mathbf{a} + (1 - q)\mathbf{b}$ (ii)

(b) As (i) and (ii) are expressions for the *same* vector, then by (1) above, the multiples of **a** and **b** are equal.

So $\frac{p}{2} = \frac{2q}{3}$ and $\frac{p}{2} = 1 - q$. We solve these equations.

$\frac{2q}{3} = 1 - q$ giving $q = \frac{3}{5}$ and hence $p = \frac{4}{5}$.

(c) OM = $\frac{4}{5}$OD so OM:MD = 4:1 and BM = $\frac{3}{5}$BC so BM:MC = 3:2.

Ratio Theorem (Optional)

This theorem is not necessary for this syllabus but may be found useful. It gives a direct way of finding the position vector of a point dividing a line in a given ratio. In Fig. 8.23, $\vec{OA} = \mathbf{a}$, $\vec{OB} = \mathbf{b}$ and P divides AB in the ratio $p:q$. We wish to find $\vec{OP}$.

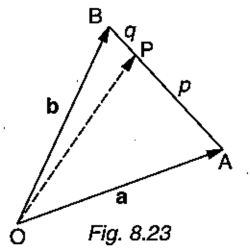


Fig. 8.23

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{p}{p+q}\ \overrightarrow{AB} = \mathbf{a} + \frac{p}{p+q}(\mathbf{b} - \mathbf{a}) = \frac{q\mathbf{a} + p\mathbf{b}}{q + p}$

This is known as the **ratio theorem** for vectors. Note carefully that q multiplies $\mathbf{a}$ (on the other side of P) and p multiplies $\mathbf{b}$.

For example, if P was the midpoint of AB, then $p = q = 1$.

So $\overrightarrow{OP} = \frac{\mathbf{a}+\mathbf{b}}{2}$. If $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$ then $p = 1$, $q = 2$ and $\overrightarrow{OP} = \frac{2\mathbf{a}+\mathbf{b}}{3}$. Similarly, if $\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AB}$ where the position vectors of A and B are $3\mathbf{a} - 2\mathbf{b}$ and $-\mathbf{a} + 5\mathbf{b}$, then $p = 3$, $q = 2$ and the position vector of P will be $\frac{2(3\mathbf{a} - 2\mathbf{b}) + 3(-\mathbf{a} + 5\mathbf{b})}{5} = \frac{3\mathbf{a} + 11\mathbf{b}}{5}$.

Note: Care must be taken when P divides $\overrightarrow{AB}$ **externally** i.e. P lies outside AB. One of p or q must then be taken as negative (Fig. 8.24).

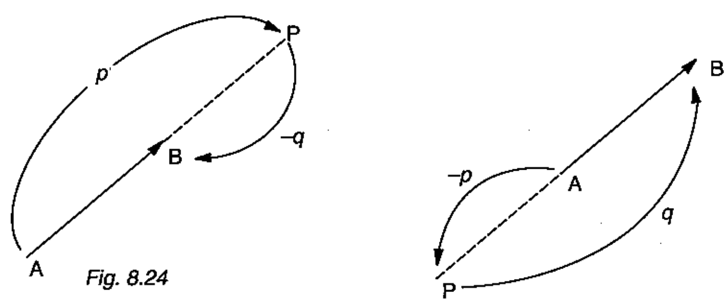


Fig. 8.24

$\overrightarrow{OP} = \frac{-q\mathbf{a} + p\mathbf{b}}{-q + p}$ $\qquad$ $\overrightarrow{OP} = \frac{q\mathbf{a} - p\mathbf{b}}{q - p}$

Example 10

In Fig. 8.25, the position vectors of the points A, B and C are $2\mathbf{a}-\mathbf{b}$, $4\mathbf{a}+5\mathbf{b}$ and $-\mathbf{a}+4\mathbf{b}$ respectively. L and N are the midpoints of AB and AC respectively. M is a point such that $\overrightarrow{LM} = \frac{1}{3}\overrightarrow{LC}$.

(a) Find the position vectors of L, M and N and (b) show that B, M and N are collinear and state the ratio BM:MN. (c) P is a point on BN produced such that BP = pBN. If PC is parallel to AM, find the value of p.

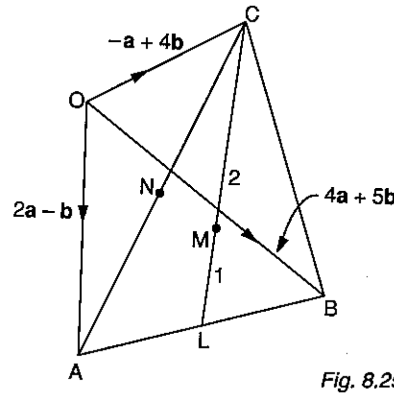


Fig. 8.25

(a) The position vector of L is $\frac{(2\mathbf{a}-\mathbf{b})+(4\mathbf{a}+5\mathbf{b})}{2} = 3\mathbf{a}+2\mathbf{b}$.

As $LM = \frac{1}{3}LC$, LM:MC = 1:2.

The position vector of M is $\frac{2(3\mathbf{a}+2\mathbf{b})+1(-\mathbf{a}+4\mathbf{b})}{2+1} = \frac{5\mathbf{a}+8\mathbf{b}}{3}$

The position vector of N is $\frac{(2\mathbf{a}-\mathbf{b})+(-\mathbf{a}+4\mathbf{b})}{2} = \frac{\mathbf{a}+3\mathbf{b}}{2}$

(b) We find $\overrightarrow{BM}$ and $\overrightarrow{MN}$.

$$\overrightarrow{BM} = \overrightarrow{OM} - \overrightarrow{OB} = \frac{5\mathbf{a}+8\mathbf{b}}{3} - (4\mathbf{a}+5\mathbf{b}) = \frac{-7\mathbf{a}-7\mathbf{b}}{3} = -\frac{7}{3}(\mathbf{a}+\mathbf{b})$$

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{\mathbf{a}+3\mathbf{b}}{2} - \frac{5\mathbf{a}+8\mathbf{b}}{3} = \frac{-7\mathbf{a}-7\mathbf{b}}{6} = -\frac{7}{6}(\mathbf{a}+\mathbf{b})$$

Then B, M and N are collinear as $\overrightarrow{BM}$ is a multiple of $\overrightarrow{MN}$.

$BM:MN = -\frac{7}{3}:-\frac{7}{6} = 2:1$.

(c) $\overrightarrow{BP} + \overrightarrow{PC} = \overrightarrow{BC}$

so $p\overrightarrow{BN} + \overrightarrow{PC} = (-\mathbf{a}+4\mathbf{b}) - (4\mathbf{a}+5\mathbf{b})$

i.e. $p[\frac{\mathbf{a}+3\mathbf{b}}{2} - (4\mathbf{a}+5\mathbf{b})] + \overrightarrow{PC} = -5\mathbf{a}-\mathbf{b}$

so $p(\frac{-7\mathbf{a}-7\mathbf{b}}{2}) + \overrightarrow{PC} = -5\mathbf{a}-\mathbf{b}$

giving $\overrightarrow{PC} = (\frac{7p-10}{2})\mathbf{a} + (\frac{7p-2}{2})\mathbf{b}$

$\overrightarrow{AM} = \frac{5\mathbf{a}+8\mathbf{b}}{3} - (2\mathbf{a}-\mathbf{b}) = \frac{-\mathbf{a}+11\mathbf{b}}{3} = -\frac{1}{3}\mathbf{a} + \frac{11}{3}\mathbf{b}$

If these are parallel, the multiples of **a** and of **b** must be in the same ratio.

Hence $\dfrac{\frac{7p-10}{2}}{-\frac{1}{3}} = \dfrac{\frac{7p-2}{2}}{\frac{11}{3}}$ i.e. $\frac{7p-10}{-1} = \frac{7p-2}{11}$

which simplifies to $77p - 110 = -7p + 2$ or $p = \frac{4}{3}$.

Exercise 8.3 *(Answers on page 625.)*

1 Given that $\mathbf{p} = 3\mathbf{a} - \mathbf{b}$ and $\mathbf{q} = 2\mathbf{a} - 3\mathbf{b}$, find numbers x and y such that $x\mathbf{p} + y\mathbf{q} = \mathbf{a} + 9\mathbf{b}$.

2 If $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{c} = 12\mathbf{i} - 5\mathbf{j}$, find numbers p and q such that $p\mathbf{a} + q\mathbf{b} = \mathbf{c}$.

3 Given $\mathbf{p} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{q} = \mathbf{i} - \mathbf{j}$, find numbers x and y such that $x\mathbf{p} + y\mathbf{q} = -4\mathbf{i} - 11\mathbf{j}$.

4 If $\mathbf{p} = 2\mathbf{a} - 5\mathbf{b}$, $\mathbf{q} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{r} = \mathbf{a} - 16\mathbf{b}$, find numbers x and y such that $x\mathbf{p} + y\mathbf{q} = \mathbf{r}$.

5 If $\overrightarrow{OP} = 2\mathbf{a} - 5\mathbf{b}$, $\overrightarrow{OQ} = 5\mathbf{a} - \mathbf{b}$ and $\overrightarrow{OR} = 11\mathbf{a} + 7\mathbf{b}$, show that P, Q and R are collinear and state the ratio PQ:QR.

6 The position vectors of P, Q and R are $\mathbf{a} - 2\mathbf{b}$, $2\mathbf{b}$ and $-4\mathbf{a} + k\mathbf{b}$ respectively. If P, Q and R are collinear, find the value of k. What is the ratio PQ:QR?

7 Given that $\overrightarrow{OP} = \mathbf{a} + \mathbf{b}$, $\overrightarrow{OQ} = k\mathbf{a}$ and $\overrightarrow{OR} = 7\mathbf{a} - 2\mathbf{b}$, find the value of k if Q lies on PR.

8 The position vectors of P, Q and R are $2\mathbf{a} - \mathbf{b}$, $\mu(\mathbf{a} - \mathbf{b})$ and $\mathbf{a} + \mathbf{b}$ respectively. Find the value of μ if PQR is a straight line. State the ratio PQ:QR.

9 (a) The position vectors of L, M and N are $\mathbf{p} + 2\mathbf{q}$, $m(\mathbf{p} + \mathbf{q})$ and $\mathbf{p} - \mathbf{q}$ respectively. Find the value of m for which LMN is a straight line, and state the ratio LM:MN.
(b) The position vectors of A, B and C are $\mathbf{a} + 2\mu\mathbf{b}$, $\mu\mathbf{a} - \mathbf{b}$ and $2\mathbf{a} - 3\mathbf{b}$ respectively. If AB is parallel to OC, where O is the origin, find the value of μ.

10 The position vectors of A and B are $\mathbf{a} - 2\mathbf{b}$ and $3\mathbf{a} + 4\mathbf{b}$ respectively. Using the ratio theorem or otherwise, find the position vector of P where (a) $\overrightarrow{AP} = 2\overrightarrow{PB}$, (b) $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$, (c) $4\overrightarrow{AP} = 3\overrightarrow{AB}$, (d) P lies on AB extended and AP = 3BP, (e) P lies on BA extended and AP = 2BA.

11 $\overrightarrow{OA} = 2\mathbf{a} - 4\mathbf{b}$ and $\overrightarrow{OB} = 4\mathbf{a} + 6\mathbf{b}$, where O is the origin. P and Q are the midpoints of OA and AB respectively. (a) State the position vectors of P and Q. (b) G lies on BP such that BG = 2GP. Find the position vector of G. (c) Show that O, G and Q are collinear and state the ratio OG:GQ. (d) R lies on OA where $\overrightarrow{OR} = p\overrightarrow{OA}$. If BR is parallel to GA find the value of p.

12 P and Q divide the sides BC and AC respectively of $\triangle$ABC in the ratio 2:1. If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$, find (a) $\overrightarrow{QP}$ and (b) show that QP is parallel to AB and one-third its length.

13 OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. D lies on OB where OD:DB = 1:4. AD meets OC at E. By taking $\overrightarrow{OE} = p\overrightarrow{OC}$ and $\overrightarrow{AD} = q\overrightarrow{AE}$, show that
$\frac{1}{5}(\mathbf{a} + \mathbf{c}) = (1 - q)\mathbf{a} + pq\mathbf{c}$.
Hence find the values of p and q and the ratios OE:EC and AD:DE.

14 OABC is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$. M is the midpoint of AB and MC meets OB at X.

(a) By taking $\overrightarrow{MX} = p\overrightarrow{MC}$ and $\overrightarrow{OX} = q\overrightarrow{OB}$, express $\overrightarrow{OX}$ in terms of (i) p, $\mathbf{a}$ and $\mathbf{b}$, (ii) q, $\mathbf{a}$ and $\mathbf{b}$.

(b) Hence evaluate p and q and state the ratios OX:XB and CX:XM.

15 C lies on the side OA of $\triangle$OAB where OC:CA = 2:1. D lies on the side OB where OD:DB = 1:2. AD meets BC at T.

(a) Taking $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AT} = p\overrightarrow{AD}$ and $\overrightarrow{CT} = q\overrightarrow{CB}$, find two expressions for $\overrightarrow{OT}$. Hence find (b) the values of p and q and (c) the ratios CT:TB and AT:TD.

16 C and D divide OA and OB respectively in the ratio 1:3. E divides CB in the ratio 1:4. Taking $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, use vector methods to prove that DEA is a straight line and find the ratio DE:EA.

17 In $\triangle$OAB, C divides OA in the ratio 2:3 and D divides AB in the ratio 1:2. OD meets CB at E.

(a) Taking $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OE} = p\overrightarrow{OD}$ and $\overrightarrow{CE} = q\overrightarrow{CB}$, obtain two expressions for $\overrightarrow{OE}$.

(b) Hence find the values of p and q.

(c) State the ratios OE:ED and CE:EB.

18 The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O. C is the midpoint of AB and D divides OB in the ratio 2:1. AD and OC meet at P.

(a) Taking $\overrightarrow{OP} = p\overrightarrow{OC}$ and $\overrightarrow{AP} = q\overrightarrow{AD}$, express $\overrightarrow{OP}$ in two different forms.

Hence find (b) the values of p and q and (c) the ratio OP:PC. (d) Q lies on BA produced where $\overrightarrow{AQ} = k\overrightarrow{BA}$. State the position vector of Q. If OQ is parallel to DC, find the value of k.

19 OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. $\overrightarrow{OB}$ is extended to D where $\overrightarrow{OB} = \overrightarrow{BD}$ and $\overrightarrow{OA}$ is extended to E where $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{OA}$. CE and AD meet at X.

(a) Taking $\overrightarrow{AX} = p\overrightarrow{AD}$ and $\overrightarrow{CX} = q\overrightarrow{CE}$, find two expressions for $\overrightarrow{OX}$.

(b) Hence find the values of p and q and the ratios AX:XD and CX:XE.

(c) F lies on AD and BF is parallel to CE. Taking $\overrightarrow{AF} = r\overrightarrow{AD}$, find the value of r.

(d) Hence state the ratio BF:CE.

20 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. OB is produced to C where $\overrightarrow{OB} = 2\overrightarrow{BC}$. D is the midpoint of AB. OD produced meets AC at E. Taking $\overrightarrow{OD} = p\overrightarrow{OE}$ and $\overrightarrow{AE} = q\overrightarrow{AC}$, derive two expressions for $\overrightarrow{OD}$ and hence find the values of p and q and the ratios OD:DE and AE:EC.

21 OABC is a parallelogram with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. D lies on OC where OD:DC = 1:2 and E is the midpoint of CB. DB meets AE at T. Taking $\overrightarrow{DT} = p\overrightarrow{DB}$ and $\overrightarrow{AT} = q\overrightarrow{AE}$, form two vector expressions for $\overrightarrow{OT}$ and hence find the values of p and q.

22 The position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to O. C lies on OB where OC:CB = 1:3. AC is produced to D where $\overrightarrow{AD} = p\overrightarrow{AC}$. If DB is parallel to OA, find the value of p.

23 The position vectors of the points A, B and C are $7\mathbf{a} - 2\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 2\mathbf{b}$ respectively. L is the point where $\overrightarrow{AL} = \frac{1}{3}\overrightarrow{AB}$. M is the midpoint of BC and N is the point such that $\overrightarrow{CN} = 2\overrightarrow{CA}$. Find the position vectors of L, M and N and show that these points are collinear. State the ratio ML:LN.

24 A, B and C have position vectors $\mathbf{a} - \mathbf{b}$, $3\mathbf{a} + 2\mathbf{b}$ and $4\mathbf{a} - 3\mathbf{b}$ respectively. P lies on AB where AP:AB = 2:3, Q lies on BC where BQ:BC = 3:4 and R lies on AC extended so that AC = CR. Find the position vectors of P, Q and R and show that P, Q and R are collinear. State the ratio PQ:QR.

COMPONENTS OF A VECTOR: UNIT COORDINATE VECTORS

Suppose $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$ (Fig.8.24).

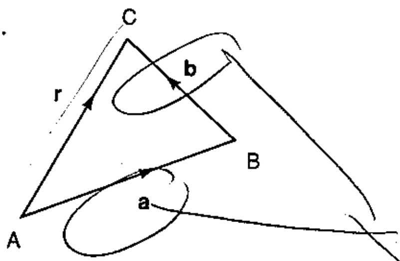


Fig. 8.26

The resultant of **a** and **b** is $\overrightarrow{AC} = \mathbf{r} = \mathbf{a} + \mathbf{b}$. The vectors **a** and **b** are called the **components** of **r**. The components of a vector **r** are *any* two vectors whose resultant is **r**. A vector can therefore be resolved into two components in an infinite number of ways. However if we take the components parallel to the x- and y-axes (Fig. 8.27), they will be unique and perpendicular.

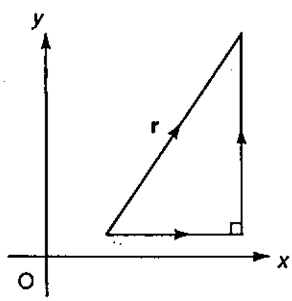


Fig. 8.27

We now define two standard unit vectors **i** and **j** called the **unit coordinate** (or **base**) **vectors** (Fig. 8.28).

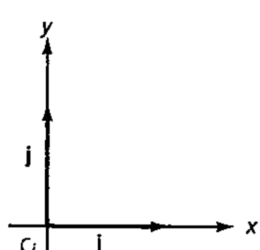


Fig. 8.28

i is a vector in the direction of the positive x-axis and $|\mathbf{i}| = 1$;
j is a vector in the direction of the positive y-axis and $|\mathbf{j}| = 1$.

The perpendicular components of any vector can now be expressed in terms of **i** and **j** in a standard form. For example, suppose the vector $\overrightarrow{AC} = \mathbf{r}$ has components of magnitude 3 and 4 parallel to the axes (Fig. 8.28).The horizontal component $\overrightarrow{AD} = 3\mathbf{i}$ and the vertical component $\overrightarrow{DC} = 4\mathbf{j}$.
Hence the vector $\mathbf{r} = \overrightarrow{AD} + \overrightarrow{DC} = 3\mathbf{i} + 4\mathbf{j}$.
r is now expressed in terms of the base vectors **i** and **j**.

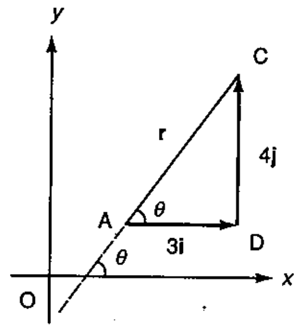


Fig. 8.29

r can also be written as $\binom{3}{4}$, i.e. in **column vector** form. [Do not confuse with coordinates (3,4)]. For example, $2\mathbf{i} - 3\mathbf{j}$ can be written as $\binom{2}{-3}$. $\mathbf{i} = \binom{1}{0}$ and $\mathbf{j} = \binom{0}{1}$.
Given **r** in terms of **i** and **j**, we can find $|\mathbf{r}|$ and the angle θ it makes with the positive x-axis.
From $\triangle$ADC (Fig. 8.28), $AC^2 = AD^2 + DC^2$, so $|\mathbf{r}^2| = 3^2 + 4^2 = 25$ and $|\mathbf{r}| = \sqrt{25} = 5$.
$\tan\theta = \frac{4}{3}$ giving $\theta = 53.13°$.

Note: To find θ for a given vector, draw a diagram to locate the correct quadrant as $\tan\theta = \frac{b}{a}$ will give two values for $0° \leq \theta \leq 360°$.

In general for a vector **r** with perpendicular components of magnitude a and b (Fig. 8.30):

$\mathbf{r} = a\mathbf{i} + b\mathbf{j} = \binom{a}{b}$
$|\mathbf{r}| = |a\mathbf{i} + b\mathbf{j}| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$

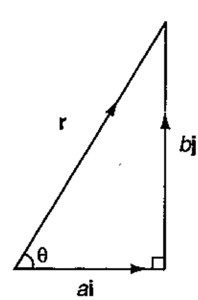


Fig. 8.30

Example 11

The position vector of A is $-2\mathbf{i} + 3\mathbf{j}$.
(a) State the coordinates of A.
(b) Find $|\overrightarrow{OA}|$ and the angle the vector OA makes with the x-axis.

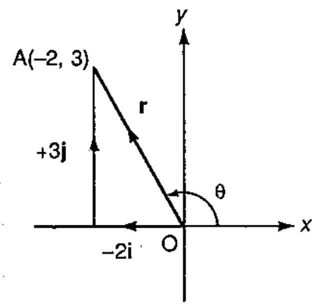


Fig. 8.31

(a) The coordinates of A are (–2,3) (Fig.8.29).
(b) $|\overrightarrow{OA}| = |-2\mathbf{i} + 3\mathbf{j}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13} \approx 3.6$
$\tan\theta = \frac{3}{-2}$ giving $\theta = 123.7°$ (2nd quadrant, Fig.8.29).

Example 12

(a) $\overrightarrow{OC}$ has the same direction as $\binom{-3}{4}$ and $|\overrightarrow{OC}| = 30$. Express $\overrightarrow{OC}$ as a column vector.

(b) The position vectors of A and B are $\binom{-2}{3}$ and $\binom{1}{-2}$ respectively. Find
(i) $\overrightarrow{AB}$, (ii) the equation of AB.

(a) $\overrightarrow{OC}$ must be a scalar multiple of $\binom{-3}{4}$ so $\overrightarrow{OC} = \binom{-3k}{4k}$ where $k > 0$. $|\overrightarrow{OC}|^2 = 9k^2 + 16k^2 = 25k^2$ so $|\overrightarrow{OC}| = 5k = 30$ and $k = 6$. Hence $\overrightarrow{OC} = \binom{-18}{24}$.

(b) (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} - (-2\mathbf{i} + 3\mathbf{j}) = 3\mathbf{i} - 5\mathbf{j}$.

(ii) The coordinates of A and B are (–2,3) and (1,–2).
Hence the equation of AB is $\frac{y-3}{-2-3} = \frac{x+2}{1+2}$ i.e. $5x + 3y = -1$.

Example 13

The position vectors of A, B and C are $2\mathbf{i} - \mathbf{j}$, $3\mathbf{i} + 2\mathbf{j}$ *and* $-3\mathbf{i} + 4\mathbf{j}$ *respectively (Fig. 8.32).*

(a) Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$ *and* $|\overrightarrow{AC}|$ *and show that* $\triangle ABC$ *is right angled.*

(b) If the position vector of D is $-4\mathbf{i} - \mathbf{j}$, *find the angle BD makes with the x-axis.*

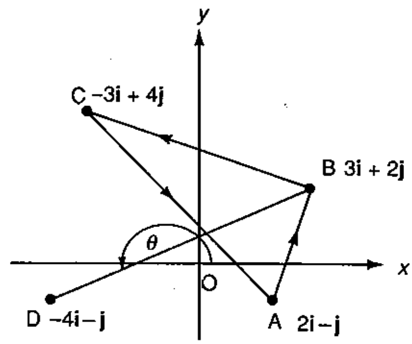


Fig. 8.32

(a) $\overrightarrow{AB} = 3\mathbf{i} + 2\mathbf{j} - (2\mathbf{i} - \mathbf{j}) = \mathbf{i} + 3\mathbf{j}$; $|\overrightarrow{AB}| = \sqrt{10}$

$\overrightarrow{BC} = -3\mathbf{i} + 4\mathbf{j} - (3\mathbf{i} + 2\mathbf{j}) = -6\mathbf{i} + 2\mathbf{j}$; $|\overrightarrow{BC}| = \sqrt{40}$

$\overrightarrow{CA} = 2\mathbf{i} - \mathbf{j} - (-3\mathbf{i} + 4\mathbf{j}) = 5\mathbf{i} - 5\mathbf{j}$; $|\overrightarrow{CA}| = \sqrt{50}$

As $|\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2$, the triangle is right angled.

(b) $\overrightarrow{BD} = -4\mathbf{i} - \mathbf{j} - (3\mathbf{i} + 2\mathbf{j}) = -7\mathbf{i} - 3\mathbf{j}$

$\tan\theta = \frac{-3}{-7}$ giving $\theta = 203.2°$ (3rd quadrant).

Unit Vectors

The magnitude of the vector $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ is $|3\mathbf{i} + 4\mathbf{j}| = 5$ so the vector is 5 units long. Hence the vector $\frac{3\mathbf{i} + 4\mathbf{j}}{5}$ is one unit long and is in the **same** direction as **a**. This is the **unit vector** in the direction of **a** (Fig. 8.33). It is written as $\hat{\mathbf{a}}$ (read 'vector **a** cap').

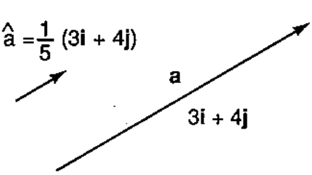


Fig. 8.33

If $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, then $\hat{\mathbf{r}} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}}$

For example, if $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ then $\hat{\mathbf{a}} = \frac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$

Again, if $\overrightarrow{OA} = \binom{2}{3}$, the unit vector parallel to $\overrightarrow{AO}$ would be $\frac{-2\mathbf{i} - 3\mathbf{i}}{\sqrt{13}}$.

Exercise 8.4 *(Answers on page 625.)*

1 On graph paper, mark the positions of the points with position vectors $\mathbf{i} + \mathbf{j}$, $-2\mathbf{i} - \mathbf{j}$, $3\mathbf{i} + 2\mathbf{j}$, $-3\mathbf{j}$.

2 A, B and C are points with position vectors $2\mathbf{i} - 3\mathbf{j}$, $\mathbf{i} + 2\mathbf{j}$ and $4\mathbf{i} - \mathbf{j}$ respectively. Find in terms of $\mathbf{i}$ and $\mathbf{j}$, the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

3 The position vectors of A and B are $3\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} + 3\mathbf{j}$ respectively. Points C and D have position vectors given by $\overrightarrow{OC} = \overrightarrow{AO}$ and $\overrightarrow{CD} = \overrightarrow{AB}$.
- (a) Find the position vectors of C and D in terms of $\mathbf{i}$ and $\mathbf{j}$ and show the positions of the four points on a diagram.
- (b) Express $\overrightarrow{DB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.
- (c) Find $|\overrightarrow{DB}|$ and the angle $\overrightarrow{DB}$ makes with the x-axis.

4 Find the magnitude and the angle made with the x-axis of the vectors
(a) $\binom{-1}{2}$ (b) $\binom{3}{-3}$ (c) $2\mathbf{i} + 3\mathbf{j}$ (d) $-4\mathbf{i} - 2\mathbf{j}$

5 (a) The coordinates of A are $(-3,2)$ and the position vector of B is $2\mathbf{i} + 4\mathbf{j}$. Find the vector $\overrightarrow{BA}$.
- (b) The vector $\overrightarrow{OA}$ has magnitude 25 units and is in the same direction as $\binom{3}{-4}$. The vector $\overrightarrow{OB}$ has magnitude 6.5 units and is opposite in direction to $\binom{5}{12}$. State the vectors $\overrightarrow{AO}$ and $\overrightarrow{AB}$ in column vector form.

6 The point with position vector $3\mathbf{i} - 2\mathbf{j}$ is displaced by a vector $\binom{4}{2}$. Find its new position vector.

7 If the coordinates of A are $(2, 4)$ and $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j}$, find the position vector of B.

8 (a) If $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$, find $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.
- (b) $\overrightarrow{OA} = \binom{3}{-4}$ and $\overrightarrow{OB} = \binom{7}{-1}$. Find the unit vectors parallel to $\overrightarrow{AO}$ and $\overrightarrow{BA}$.

9 In this question, take $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = 3\mathbf{i}$. Find (i) $\hat{\mathbf{a}}$ and (ii) $\hat{\mathbf{b}}$.
Express in terms of $\mathbf{i}$ and $\mathbf{j}$, (iii) $\mathbf{a} + 2\mathbf{b}$, (iv) $2\mathbf{c} - 3\mathbf{b}$, (v) $\mathbf{a} + \mathbf{b} - 2\mathbf{c}$.
Find (vi) $|\mathbf{a} + 2\mathbf{b}|$, (vii) $|2\mathbf{c} - 3\mathbf{b}|$, (viii) $|\mathbf{a} + \mathbf{b} - 2\mathbf{c}|$.

10 The position vectors of A, B, C and D are $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} - \mathbf{j}$, $-\mathbf{i} - 4\mathbf{j}$ and $3\mathbf{i} + 2\mathbf{j}$ respectively. Find in terms of $\mathbf{i}$ and $\mathbf{j}$ the vectors (a) $\overrightarrow{AB}$, (b) $\overrightarrow{BD}$, (c) $\overrightarrow{CA}$, (d) $\overrightarrow{AD}$.

11 The position vectors of A and B are $2\mathbf{i} + 3\mathbf{j}$ and $3\mathbf{i} - 8\mathbf{j}$ respectively. D is the midpoint of AB and E divides OD in the ratio 2:3. Find the coordinates of E.

12 P and Q have position vectors $5\mathbf{i} + 2\mathbf{j}$ and $\mathbf{i} - 4\mathbf{j}$ respectively. If $\overrightarrow{OP} = 3\overrightarrow{OQ} + 2\overrightarrow{OR}$, find the position vector of R.

13 A, B and C have coordinates $(1,2)$, $(2,5)$ and $(0,-4)$ respectively. If $\overrightarrow{AB} = \overrightarrow{CD}$, find the position vector of D.

14 The position vectors of A and B are $3\mathbf{i} + \mathbf{j}$ and $-4\mathbf{i} + 2\mathbf{j}$ respectively. Find the position vector of C if $\overrightarrow{AB} = \overrightarrow{BC}$.

15 Points A and B have position vectors $2\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$ respectively.

(a) Given that $\overrightarrow{OC} = \overrightarrow{AB}$ and $\overrightarrow{AD} = \overrightarrow{CB}$, find the position vectors of C and D.

(b) Show the positions of the four points on a diagram.

(c) Find $|\overrightarrow{CD}|$ and the angle $\overrightarrow{CD}$ makes with the x-axis.

16 The position vectors of A and B are $4\mathbf{i} + 5\mathbf{j}$ and $\mathbf{i} - 2\mathbf{j}$ respectively. Find the position vector of C if $3\overrightarrow{OA} = 2\overrightarrow{OB} + \overrightarrow{OC}$.

17 The coordinates of A and B are (2,3) and (–2,5) respectively. Find the position vector of C if $2\overrightarrow{OA} = 2\overrightarrow{OB} + \overrightarrow{BC}$.

18 Show that the points with position vectors $4\mathbf{i} + 5\mathbf{j}$, $3\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{j}$ are collinear.

19 What is the gradient of the line joining the points with position vectors $2\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 3\mathbf{j}$?

20 Show that the triangle whose vertices have position vectors $2\mathbf{i} + 4\mathbf{j}$, $5\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ is isosceles.

21 (a) The velocity $\mathbf{v}$ m s^{-1} of a body is given by the vector $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$.
Find the speed of the body and the angle its path makes with the x-axis.

(b) If its position vector at the start was $\mathbf{i} + \mathbf{j}$, what is its position vector (i) after 1 sec, (ii) after 3 secs, (iii) after t secs?

(c) After what time will it reach the position given by $7\mathbf{i} + 19\mathbf{j}$?

22 A body is moving with velocity $\mathbf{v}$ m s^{-1} where $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$. If it started from the position $\mathbf{i} + 4\mathbf{j}$, what is its position after 3 seconds? How long will it take to reach the position $11(\mathbf{i} - \mathbf{j})$?

23 The position vector $\mathbf{r}$ of a point on a straight line is given by $\mathbf{r} = \mathbf{i} + \mathbf{j} + t(2\mathbf{i} - \mathbf{j})$ where t is a number.

(a) What is its position vector when $t = 2$?

(b) Find the position vector of another point on the line by taking any other value of t.

(c) Hence find the gradient and the equation of the line .

24 Find the gradient and equation of the line given by $\mathbf{r} = \mathbf{i} - \mathbf{j} + k(\mathbf{i} - \mathbf{j})$ where k is a number.

25 The position vector $\mathbf{r}$ of a point is given by $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + t(\mathbf{i} + 2\mathbf{j})$, where t is a number. What is its position vector when (a) $t = -1$, (b) $t = 3$?

(c) What is the value of t when its position vector is $7\mathbf{i} + 9\mathbf{j}$?

26 If the vectors $m\mathbf{i} - 2\mathbf{j}$ and $4\mathbf{i} - 6\mathbf{j}$ are parallel, state the value of m.

27 The position vectors of A and B are $3\mathbf{i} - 2\mathbf{j}$ and $t\mathbf{i} + \mathbf{j}$ respectively. Find the value of t if OAB is a straight line.

28 OABC is a parallelogram where O is the origin. The position vectors of A and B are $4\mathbf{i} + 6\mathbf{j}$ and $6\mathbf{i} + 8\mathbf{j}$ respectively. D is the midpoint of CB and E is the midpoint of AB. OD meets CE at F.

(a) State the position vectors of C, D and E.

(b) By taking $\overrightarrow{OF} = m\overrightarrow{OD}$ and $\overrightarrow{CF} = n\overrightarrow{CE}$, find the values of m and n and the ratio OF:FD.

29 (a) State the condition for the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ to be perpendicular.

(b) The points A, B, C and D have position vectors $\mathbf{i} + \mathbf{j}$, $3\mathbf{i} - 2\mathbf{j}$, $-3\mathbf{i} - 3\mathbf{j}$ and $-\mathbf{j}$ respectively. Find the gradients of AB and CD and show that these lines are perpendicular.

30 The points A, B, C and D have position vectors $\mathbf{i}$, $2\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + \mathbf{j}$ and $5\mathbf{i}$ respectively. Show that AB and CD are perpendicular.

31 P, Q, R and S have position vectors $\mathbf{i} + 2\mathbf{j}$, $3\mathbf{i} - \mathbf{j}$, $-\mathbf{i} - \mathbf{j}$ and $k\mathbf{i} + \mathbf{j}$ respectively, where k is a number.

(a) Find the gradients of PQ and RS.

(b) For what value of k will the lines PQ and RS be perpendicular?

32 If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{b}| = 2$, what are the greatest and smallest values of $|\mathbf{a} + \mathbf{b}|$?

SCALAR PRODUCT OF TWO VECTORS

Vectors can be 'multiplied' in two ways. In one, the result is another vector, called the vector product but we shall not use this method. In the other, the result is a scalar so it is called the **scalar** (or **dot**) product.

We write the scalar product of **a** and **b** as **a.b** and define it as

$$\mathbf{a.b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta = ab\cos\theta$$

where θ is the angle between the vectors (Fig. 8.34).

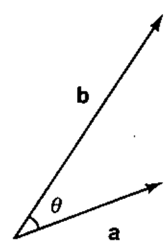


Fig. 8.34

For example, if $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and the angle between **a** and **b** = 60°, then $\mathbf{a.b} = 2 \times 3 \times 0.5 = 3$.

The scalar product is always a number. This will be negative if $90° < \theta < 180°$ as $\cos\theta$ is then negative. $|\mathbf{a}|$ and $|\mathbf{b}|$ are always positive. From the definition we can derive the following important facts:

I The scalar product is commutative

$\mathbf{a.b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta$ and $\mathbf{b.a} = |\mathbf{b}| \times |\mathbf{a}| \times \cos\theta$

Hence $\boxed{\mathbf{a.b} = \mathbf{b.a}}$

II Parallel vectors

If **a** and **b** are parallel but in the same direction, then $\theta = 0°$ (Fig. 8.35(a)).
a.b $= ab \cos 0° = ab = |\mathbf{a}| \times |\mathbf{b}|$.

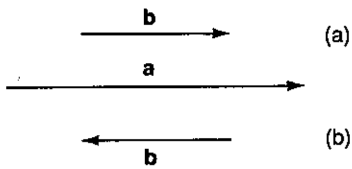


Fig. 8.35

If **a** and **b** are parallel but in opposite directions, then $\theta = 180°$ (Fig. 8.35(b)).
a.b $= ab \cos 180° = -ab$.

Hence **i.i** (written as $\mathbf{i}^2$) = 1 and $\mathbf{j}^2 = 1$.
Also $(3\mathbf{i} + 4\mathbf{j}).(3\mathbf{i} + 4\mathbf{j}) = |3\mathbf{i} + 4\mathbf{j}|^2 = 25$.

III Perpendicular vectors

If **a** and **b** are perpendicular, then $\theta = 90°$ and $\cos 90° = 0$.
Hence **a.b** $= ab \cos 90° = 0$.

a.b = 0 if a is perpendicular to b

Hence **i.j** = **j.i** = 0.

Conversely, if **a.b** = 0, then **a** and **b** are at right angles (unless either **a** or **b** or both are **0**).

IV Distributive law for a scalar product

In ordinary algebra, $a(b + c) = a \times (b + c) = ab + ac$, i.e. we can 'remove the brackets'. The '×' is distributed over the '$b + c$'. This is known as the **distributive law** for products. The same law is true for scalar products: **a.(b + c) = a.b + a.c.**

Example 14

If $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$, *find* **a.b**.

$$\begin{aligned}
\mathbf{a.b} &= (2\mathbf{i} + 3\mathbf{j}).(\mathbf{i} - 2\mathbf{j}). \\
&= 2\mathbf{i}.(\mathbf{i} - 2\mathbf{j}) + 3\mathbf{j}(\mathbf{i} - 2\mathbf{j}) \text{ by the distributive law} \\
&= 2\mathbf{i}.\mathbf{i} - 2\mathbf{i}.2\mathbf{j} + 3\mathbf{j}.\mathbf{i} - 3\mathbf{j}.2\mathbf{j} \text{ using the law again} \\
&= \mathbf{2} - 0 + 0 - \mathbf{6} \text{ as } \mathbf{i.i} = \mathbf{j.j} = 1 \text{ and } \mathbf{i.j} = \mathbf{j.i} = 0 \\
&= -4
\end{aligned}$$

From this we see that we only need to multiply the two **i** and the two **j** terms and add the results. The **i.j** terms are ignored.

So $\overbrace{(2\mathbf{i} + 3\mathbf{j}).(\mathbf{i}}^{2} \underbrace{- 2\mathbf{j})}_{-6}$ = 2 – 6 = –4.

By the same method we can show that in general:

$$(a_1\mathbf{i} + b_1\mathbf{j}).(a_2\mathbf{i} + b_2\mathbf{j}) = \binom{a_1}{b_1}.\binom{a_2}{b_2} = a_1a_2 + b_1b_2$$

Example 15

The position vectors of P and Q are $2\mathbf{i} + \mathbf{j}$ *and* $-3\mathbf{i} + 2\mathbf{j}$ *respectively. Find ∠POQ (Fig. 8.36).*

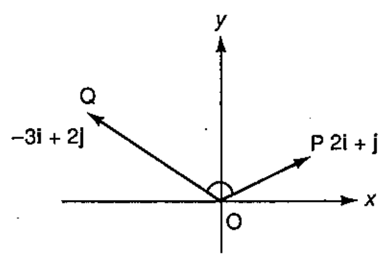


Fig. 8.36

From the definition, **a.b** = $ab\cos\theta$

Then $\cos\theta = \dfrac{\mathbf{a.b}}{ab}$

i.e. **the cosine of the angle between two vectors is the scalar product divided by the product of their moduli.**

Hence $\cos \angle POQ = \dfrac{(2\mathbf{i} + \mathbf{j}).(-3\mathbf{i} + 2\mathbf{j})}{|2\mathbf{i} + \mathbf{j}| \times |-3\mathbf{i} + 2\mathbf{j}|} = \dfrac{-6 + 2}{\sqrt{5}\,\sqrt{13}} = -0.4961$

giving ∠POQ = 119.74°.

Example 16

The position vectors of A and B are $2\mathbf{i} - 3\mathbf{j}$ *and* $t\mathbf{i} + 2\mathbf{j}$ *respectively.*

(a) Find the value of t for which $\overrightarrow{OA}$ *and* $\overrightarrow{OB}$ *are perpendicular.*

(b) If t = 4, find ∠AOB to the nearest degree.

(a) If $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$, then their scalar product = 0.
So $(2\mathbf{i} - 3\mathbf{j}).(t\mathbf{i} + 2\mathbf{j}) = 2t - 6 = 0$ and $t = 3$.

(b) $\cos \angle AOB = \dfrac{(2\mathbf{i} - 3\mathbf{j}).(4\mathbf{i} + 2\mathbf{j})}{|2\mathbf{i} + 3\mathbf{j}| \times |4\mathbf{i} + 2\mathbf{j}|} = \dfrac{2}{\sqrt{13}\,\sqrt{20}} = 0.1240$
Hence ∠AOB = 83°.

Example 17

Find the relationship between p and q if the vectors $\mathbf{a} = p\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{b} = 2\mathbf{i} + q\mathbf{j}$ *are at right angles. Given that* $q = -2$, *find* $|\mathbf{a} + \mathbf{b}|$.

To be at right angles, $(p\mathbf{i} + 3\mathbf{j}).(2\mathbf{i} + q\mathbf{j}) = 0$.
Then $2p + 3q = 0$.
If $q = -2$, then $p = 3$.
$|\mathbf{a} + \mathbf{b}| = |3\mathbf{i} + 3\mathbf{j} + 2\mathbf{i} - 2\mathbf{j}| = |5\mathbf{i} + \mathbf{j}| = \sqrt{26}$

Example 18

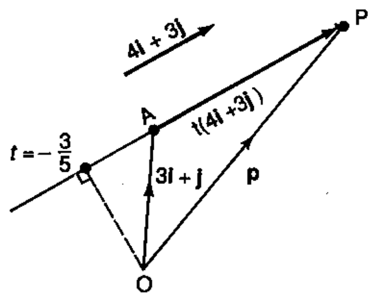
A straight line passes through the point A whose position vector is $3\mathbf{i} + \mathbf{j}$ *and is parallel to the vector* $4\mathbf{i} + 3\mathbf{j}$.

(a) If the position vector of any point P on the line is $\mathbf{p}$, *show that* $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$.

(b) Find the value of t for which $\overrightarrow{OP}$ *is perpendicular to the line.*

(c) Hence find the distance of the origin from the line.

Fig. 8.37



(a) In Fig. 8.37, A is the point with position vector $3\mathbf{i} + \mathbf{j}$.
$\overrightarrow{AP}$ is parallel to $4\mathbf{i} + 3\mathbf{j}$ so $\overrightarrow{AP} = t(4\mathbf{i} + 3\mathbf{j})$ where t is any number.
Then $\overrightarrow{OP} = \mathbf{p} = \overrightarrow{OA} + \overrightarrow{AP} = 3\mathbf{i} + \mathbf{j} + t(4\mathbf{i} + 3\mathbf{j})$.

(b) $\overrightarrow{OP} = (3 + 4t)\mathbf{i} + (1 + 3t)\mathbf{j}$
If OP is perpendicular to $4\mathbf{i} + 3\mathbf{j}$ then $(3 + 4t) \times 4 + (1 + 3t) \times 3 = 0$.
This gives $12 + 16t + 3 + 9t = 0$ or $t = -\frac{3}{5}$.

(c) When $t = -\frac{3}{5}$, the position vector of P $= 3\mathbf{i} + \mathbf{j} - \frac{3}{5}(4\mathbf{i} + 3\mathbf{j}) = \frac{3\mathbf{j}}{5} - \frac{4\mathbf{j}}{5}$.

The distance from O to the line will then be $|OP| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$.

Exercise 8.5 *(Answers on page 626.)*

For questions 1 – 7, take $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{d} = -4\mathbf{i}$.

1 Find the scalar products (a) **a.c**, (b) **b.d**, (c) **b.c**.

2 Which two of the vectors are perpendicular?

3 Evaluate $\mathbf{b}^2$.

4 Find the angles between (a) **b** and **c**, (b) **a** and **d**.

5 The vector $t\mathbf{i} + \mathbf{j}$ is perpendicular to $\mathbf{b} - \mathbf{a}$. Find the value of t.

6 Find the angle between $\mathbf{a} + \mathbf{b}$ and $\mathbf{c} + \mathbf{d}$.

7 Find the relation between m and n if $m\mathbf{a} + n\mathbf{b}$ is perpendicular to $\mathbf{c} - \mathbf{b}$.

8 Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 4\mathbf{i} + \mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 7\mathbf{j}$, verify that $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.

9 If $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{r} = 5\mathbf{i} + \mathbf{j}$, show that $\mathbf{p}.(\mathbf{q} - \mathbf{r}) = \mathbf{p}.\mathbf{q} - \mathbf{p}.\mathbf{r}$ and find the angle between **p** and **q**.

10 The position vectors of A and B are $4\mathbf{i} + 3\mathbf{j}$ and $7\mathbf{i} - \mathbf{j}$ respectively. Show that OA is perpendicular to AB and find $\angle$AOB.

11 The position vectors of A, B and C are $3\mathbf{i} + \mathbf{j}$, $4\mathbf{i} + 3\mathbf{j}$ and $6\mathbf{i} + 2\mathbf{j}$ respectively.
(a) Show the positions of the points on a diagram.
(b) Find $\angle$B.
(c) Hence find the area of $\triangle$ABC.

12 Two bodies are moving in a plane, one parallel to the vector $3\mathbf{i} - \mathbf{j}$, the other parallel to $-4\mathbf{i} + 2\mathbf{j}$. Find the angle between their paths.

13 If the vectors $t\mathbf{i} + 2\mathbf{j}$ and $t\mathbf{i} - 8\mathbf{j}$ are perpendicular, find the values of t.

14 If the vectors $2p\mathbf{i} - 3\mathbf{j}$ and $p\mathbf{i} + 6\mathbf{j}$ are perpendicular, find the values of p.

15 A(2,3), B(–1,4) and C(5,–2) are three points. Evaluate $\overrightarrow{BA}.\overrightarrow{BC}$ and hence find $\angle$ABC.

16 The position vectors of A, B and C are $2\mathbf{j}$, $3\mathbf{i} + 4\mathbf{j}$ and $5\mathbf{i}$ respectively. Find $\overrightarrow{BA}.\overrightarrow{BC}$ and hence find $\angle$ABC.

17 OABC is a parallelogram with $\overrightarrow{OA} = 4\mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OC} = -6\mathbf{i} + 4\mathbf{j}$. P and Q are the midpoints of BC and AB respectively.
(a) Find the position vectors of P and Q.
(b) Evaluate $\overrightarrow{OP}.\overrightarrow{OQ}$ and hence find $\angle$POQ.

18 OABC is a parallelogram where the position vectors of A and C are $3\mathbf{i} + 6\mathbf{j}$ and $-2\mathbf{i} + 4\mathbf{j}$ respectively. (a) Find the position vector of B. D is the midpoint of OC and E divides OA in the ratio 2:1. Find (b) the position vectors of D and E and (c) $\angle$BDE.

19 ABCD is a quadrilateral where the position vectors of A, B, C and D are **a**, **b**, **c** and **d** respectively.
(a) State in terms of **a**, **b**, **c**, **d** (i) $\overrightarrow{AB}$, (ii) $\overrightarrow{CD}$, (iii) $\overrightarrow{AC}$, (iv) $\overrightarrow{BD}$.
(b) If $(\mathbf{c} - \mathbf{d}) = k(\mathbf{b} - \mathbf{a})$ where $k > 1$, what type of quadrilateral is ABCD?

(c) State in terms of **a**, **b**, **c**, **d**, the condition for the diagonals to be at right angles.
(d) If $|\mathbf{d}-\mathbf{b}||\mathbf{c}-\mathbf{a}| = 2(\mathbf{d}-\mathbf{b}).(\mathbf{c}-\mathbf{a})$, what is the angle between the diagonals?

20 The position vectors of A, B and C are $3\mathbf{i}+\mathbf{j}$, $-\mathbf{i}-3\mathbf{j}$ and $5\mathbf{i}$ respectively. P is a point such that $\overrightarrow{AP} = k\overrightarrow{AB}$ where k is any number.
(a) Find the position vector of P in terms of k, **i** and **j**.
(b) Find the value of k if PC is perpendicular to AC.

21 If $\mathbf{r}_1 = 3\mathbf{i}$, $\mathbf{r}_2 = \mathbf{i}+\mathbf{j}$ and $\mathbf{r}_3 = -\mathbf{i}-3\mathbf{j}$, find the values of t so that $t\mathbf{r}_1 + \mathbf{r}_2$ will be perpendicular to $t\mathbf{r}_2 + \mathbf{r}_3$.

22 A and B have position vectors $3\mathbf{i}+6\mathbf{j}$ and $6\mathbf{i}+3\mathbf{j}$ respectively. C lies on OA where OC:CA = 1:2 and D lies on OB where OD:DB = 2:1.
(a) Find the position vectors of C and D.
(b) Find $\overrightarrow{CD}$ and $\overrightarrow{AB}$ in terms of **i** and **j**.
(c) Hence find the angle between $\overrightarrow{CD}$ and $\overrightarrow{AB}$.

23 The position vectors of A and B are $4\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+7\mathbf{j}$ respectively.
(a) Find $\angle$AOB.
(b) C lies on AB where AC:CB = 2:1. Find the position vector of C.
(c) Hence find $\angle$AOC.

24 The points A, B and C have position vectors $3\mathbf{i}+3\mathbf{j}$, $8\mathbf{i}+2\mathbf{j}$ and $\mu\mathbf{i}+11\mathbf{j}$ respectively, where μ is a positive number. D lies on BC where BD:DC = 1:2.
(a) Find the position vector of D in term of μ, **i** and **j**.
(b) Express $\overrightarrow{AD}$ in terms of μ, **i** and **j**.
(c) If AD is perpendicular to BC, find the value of μ.

25 The position vectors of A, B and C are $3\mathbf{i}+4\mathbf{j}$, $8\mathbf{i}-6\mathbf{j}$ and $m\mathbf{i}+n\mathbf{j}$ respectively, where m and n are numbers.
(a) Evaluate $(3\mathbf{i}+4\mathbf{j}).(m\mathbf{i}+n\mathbf{j})$.
(b) Find $|\overrightarrow{OA}|$ and $|\overrightarrow{OC}|$.
(c) Hence express cos $\angle$AOC and cos $\angle$BOC in terms of m and n.
(d) If $\angle$AOC = $\angle$BOC, find the relation between m and n.
(e) Hence find the equation of the line OC.

SUMMARY

- Magnitude of vector $\mathbf{a} = |\mathbf{a}| = a$.
- If $\mathbf{a} = k\mathbf{b}$, where k is a scalar (a number) $\neq 0$, then the vectors **a** and **b** are parallel and in the same direction if $k > 0$ but in opposite directions if $k < 0$.
  $|\mathbf{a}| = |k| \times |\mathbf{b}|$
- Conversely if **a** and **b** are parallel ($\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$), then $\mathbf{a} = k\mathbf{b}$. P, Q and R are collinear if $\overrightarrow{PQ} = k\overrightarrow{QR}$ (or $\overrightarrow{PQ} = k\overrightarrow{PR}$) and conversely.

- If $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\mathbf{a} + \mathbf{b}$ is the diagonal OC of the parallelogram OACB.
- $\mathbf{a} - \mathbf{b}$ is the vector from the end of $\mathbf{b}$ to the end of $\mathbf{a}$,
  $\mathbf{b} - \mathbf{a}$ is the vector from the end of $\mathbf{a}$ to the end of $\mathbf{b}$,
  where $\mathbf{a}$, $\mathbf{b}$ start from the same point.
- If $m\mathbf{a} + n\mathbf{b} = p\mathbf{a} + q\mathbf{b}$ then $m = p$ and $n = q$.
- The position vector of A is the vector $\overrightarrow{OA}$ where O is the origin.
- $\mathbf{i}$, $\mathbf{j}$ are unit vectors in the directions of the positive coordinate axes.
  Column vector form: $\binom{a}{b} = a\mathbf{i} + b\mathbf{j}$
  If $\mathbf{r} = a\mathbf{i} + b\mathbf{j} = \binom{a}{b}$, then $|\mathbf{r}| = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$ (check for the correct quadrant).
- The unit vector in the direction of $\mathbf{r}$ is $\hat{\mathbf{r}}$.
  If $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$, then $\hat{\mathbf{r}} = \frac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2 + b^2}}$.
- Scalar product of $\mathbf{a}$ and $\mathbf{b} = \mathbf{a}.\mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \cos\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

  $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

  Commutative law: $\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a}$
  Distributive law: $\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$.
  If $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a}.\mathbf{b} = ab$ (same direction), or $\mathbf{a}.\mathbf{b} = -ab$ (opposite directions).
  If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a}.\mathbf{b} = 0$.

  $$(a_1\mathbf{i} + b_1\mathbf{j}).(a_2\mathbf{i} + b_2\mathbf{j}) = \binom{a_1}{b_1}.\binom{a_2}{b_2} = a_1a_2 + b_1b_2.$$

REVISION EXERCISE 8 *(Answers on page 626.)*

A

1 Given $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = -5\mathbf{i} - 12\mathbf{j}$ and $\mathbf{c} = m\mathbf{i} + n\mathbf{j}$, calculate
(a) $\mathbf{a}.\mathbf{b}$, (b) the angle between $\mathbf{a}$ and $\mathbf{b}$.
(c) If $\mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c}$, find the relation between m and n.

2 The position vector of A relative to an origin O is $3\mathbf{i} + 5\mathbf{j}$. Given that $\overrightarrow{AB} = 8\mathbf{i} + 2\mathbf{j}$, evaluate $\overrightarrow{OA}.\overrightarrow{OB}$ and hence find angle AOB. (C)

3 The points A, B and C have position vectors $\mathbf{p} + \mathbf{q}$, $3\mathbf{p} - 2\mathbf{q}$ and $6\mathbf{p} + m\mathbf{q}$ relative to an origin O. Find the value of m for which A, B and C are collinear. (C)

4 The position vectors of A and B relative to an origin O are $6\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + p\mathbf{j}$ respectively. Express $\overrightarrow{AO}.\overrightarrow{AB}$ in terms of p and hence find (i) the value of p for which AO is perpendicular to AB, (ii) the cosine of $\angle OAB$ when $p = 6$. (C)

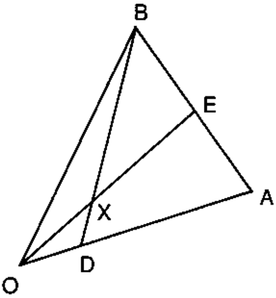
5 A, B and C are points with position vectors $4\mathbf{p} - \mathbf{q}$, $\mu(\mathbf{p} + \mathbf{q})$ and $\mathbf{p} + 2\mathbf{q}$ respectively, relative to an origin O. Obtain expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Given that B lies on AC, find the value of μ. (C)

6 (a) $\overrightarrow{OA}$ is perpendicular to $\binom{-4}{3}$ and $|\overrightarrow{OA}| = 15$. State $\overrightarrow{OA}$ in column vector form if A lies in the first quadrant.

(b) $|a\mathbf{i} + b\mathbf{j}| = 5$ and $a\mathbf{i} + b\mathbf{j}$ is perpendicular to $8\mathbf{i} - 6\mathbf{j}$. Find the value of a and of b.

7 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively relative to an origin O (Fig. 8.38). The point D is such that $\overrightarrow{OD} = p\overrightarrow{OA}$ and the point E is such that AE = qAB.

Fig. 8.38



The line segments BD and OE intersect at X. If OX = $\frac{2}{5}$OE and XB = $\frac{4}{5}$DB express $\overrightarrow{OX}$ and $\overrightarrow{XB}$ in terms of $\mathbf{a}$, $\mathbf{b}$, p and q and hence evaluate p and q. (C)

8 (a) Given that $\overrightarrow{OM} = \mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{ON} = \mathbf{i} + \mathbf{j}$, evaluate $\overrightarrow{OM}.\overrightarrow{ON}$ and hence calculate $\angle$MON to the nearest degree.

(b) The position vectors, relative to an origin O, of two points S and T are $2\mathbf{p}$ and $2\mathbf{q}$ respectively. The point A lies on OS and is such that OA = AS. The point B lies on OT produced and is such that OT = 2TB. The lines ST and AB intersect at R. Given that $\overrightarrow{AR} = \lambda\,\overrightarrow{AB}$ and that $\overrightarrow{SR} = \mu\,\overrightarrow{ST}$, express $\overrightarrow{OR}$ (i) in terms of $\mathbf{p}$, $\mathbf{q}$ and λ, (ii) in terms of $\mathbf{p}$, $\mathbf{q}$, μ. Hence evaluate λ and μ and express $\overrightarrow{OR}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. (C)

9 The position vectors of A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively relative to an origin O. Draw a diagram showing the positions of O, A, B and C given that (i) $\mathbf{a}.\mathbf{c} = 0$, (ii) $\mathbf{b} - \mathbf{a} = k(\mathbf{c} - \mathbf{a})$, (iii) $\mathbf{b}.(\mathbf{a} - \mathbf{c}) = 0$, (iv) $2|\mathbf{b}| = |\mathbf{a} - \mathbf{c}|$.
What is the position of B relative to A and C?

10 The position vectors, relative to an origin O, of three points A, B and C are $2\mathbf{i} + 2\mathbf{j}$, $5\mathbf{i} + 11\mathbf{j}$ and $11\mathbf{i} + 9\mathbf{j}$ respectively.

(i) Given that $\overrightarrow{OB} = m\overrightarrow{OA} + n\overrightarrow{OC}$, where m and n are scalar constants, find the value of m and of n.

(ii) Evaluate $\overrightarrow{AB}.\overrightarrow{BC}$ and state the deduction which can be made about $\angle$ABC.

(iii) Evaluate $\overrightarrow{AB}.\overrightarrow{AC}$ and **hence** find $\angle$BAC. (C)

11 The position vectors of A and B are $\mathbf{i} + 3\mathbf{j}$ and $-2\mathbf{i} + \mathbf{j}$ respectively.

(a) Evaluate $\overrightarrow{OA}.\overrightarrow{OB}$ and hence find $\angle$AOB.

C is a point whose position vector is given by $\overrightarrow{OC} = \overrightarrow{OA} + t\overrightarrow{AB}$.

(b) Find the values of t for which (i) OC is perpendicular to AB, (ii) $|\overrightarrow{OC}| = |\overrightarrow{AC}|$.

12 The position vectors of points A and B relative to an origin O are **a** and **b** respectively. The point P is such that $\overrightarrow{OP} = 4\overrightarrow{OB}$. The midpoint of AB is the point Q. The point R is such that $\overrightarrow{OR} = \frac{8}{5}\overrightarrow{OQ}$.
Find, in terms of **a** and **b**, the vectors $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{AR}$ and $\overrightarrow{RP}$. Hence show that R lies on AP and find the ratio AR:RP.
Given that the point S is such that $\overrightarrow{OS} = \mu\,\overrightarrow{OQ}$, find the value of μ such that PS is parallel to BA. (C)

13 P and Q have position vectors $2t\mathbf{i} + (t+1)\mathbf{j}$ and $(t+1)\mathbf{i} - (t+2)\mathbf{j}$ respectively. If $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$ show that $3t^2 - 4t - 4 = 0$ and hence find the possible values of t. For each one, calculate $\overrightarrow{OP}\,.\,\overrightarrow{OQ}$ and the angle POQ.

14 OABC is a quadrilateral with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$.
If $(\mathbf{a} + \mathbf{c}).(\mathbf{a} - \mathbf{c}) = 0$, what type of quadrilateral is OABC?
If, in addition, $\mathbf{a}.\mathbf{c} = 0$ what is the quadrilateral?

15 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. C is the midpoint of AB and D divides OB in the ratio 2:1. AD and OC intersect at P.

(a) Taking $\overrightarrow{OP} = p\overrightarrow{OC}$ and $\overrightarrow{AP} = q\overrightarrow{AD}$, find two vector expressions for $\overrightarrow{OP}$ and hence find the ratio OP:PC.

(b) E divides OB in the ratio 1:2 and AE meets OC at Q. By a similar method, find the ratio OQ:QC.

(c) Hence find the ratio OQ:QP:PC.

16 The position vectors of A, B and C are $-\mathbf{a} + 2\mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$ and $3\mathbf{a} + 5\mathbf{b}$ respectively. P divides BC in the ratio 3:4. Q lies on AB so that $AQ = \frac{2}{5}AB$. R lies on AC so that $\overrightarrow{CR} = 2\overrightarrow{AR}$. Find the position vectors of P, Q and R. Show that these points are collinear and state the ratio PQ:QR.

B

17 The points A and B have position vectors $4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{i} + t\mathbf{j}$ respectively.
If $\cos \angle AOB = \frac{2}{\sqrt{5}}$, find the values of t.

18 The position vector **p** of a point P is given by $\mathbf{p} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$.
Find the equation of the curve on which P lies for all values of θ.

19 If $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c}$, show that **a** is perpendicular to $\mathbf{b} - \mathbf{c}$.

20 In $\triangle$AOB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. The altitudes BD and OE intersect at H and $\overrightarrow{OH} = \mathbf{h}$.

(a) State $\overrightarrow{BH}$ in terms of **h** and **b**.

(b) Show that $(\mathbf{h} - \mathbf{b}).\mathbf{a} = 0$ and that $\mathbf{h}.(\mathbf{b} - \mathbf{a}) = 0$ and hence deduce that $\mathbf{b}.(\mathbf{h} - \mathbf{a}) = 0$.

(c) Hence state a geometrical result about the altitudes of a triangle.

21 If $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$, show that the area of the $\triangle$ABC is given by $\frac{1}{2}\sqrt{(ab)^2 - (\mathbf{a}.\mathbf{b})^2}$.
Hence find the area of $\triangle$ABC if the coordinates of A, B and C are (3,2), (–1,–1) and (5,–3) respectively.

22 The points A, B and C have position vectors **a**, **b** and **c** respectively. L, M and N are the midpoints of AB, BC and CA respectively. G lies on CL so that CG = 2GL. Find the position vector of G. Show that A, G and M are collinear and state the ratio AG:GM. (Similarly B, G and N are collinear and BG:GN = 2:1. G is called the **centroid** of the triangle ABC.)

Calculus (1): Differentiation

9

Calculus is a very important branch of Mathematics. It was developed by Newton (1642–1727) and Leibnitz (1646–1716) to deal with changing quantities. The gradient of a curve is an example of such a quantity and we begin with this.

GRADIENT OF A CURVE

The gradient of a straight line is constant. It is equal to the ratio $\frac{y\text{-step}}{x\text{-step}}$ between any two points of the line (see Chapter 1, page 8). On a curve however, the gradient is changing from one point to another. We define the gradient at any point on a curve therefore to be the gradient of the **tangent to the curve at that point (Fig.9.1).**

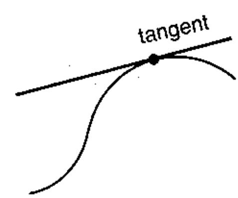


Fig.9.1

We now find a **gradient function,** derived from the function represented by the curve, using a method called a **limiting process.** Consider the simple quadratic curve $y = x^2$ (Fig.9.2) and take the point P(3,9) on that curve.

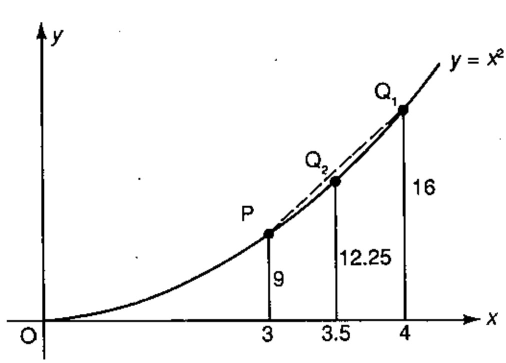


Fig.9.2

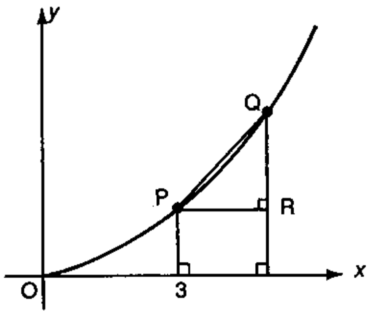
Now take a point near P, say $Q_1(4,16)$. The gradient of the line PQ_1 is $\frac{16-9}{1} = 7$ which only approximately equals the gradient of the tangent at P.

To get a better approximation we try again, this time with $Q_2(3.5,12.25)$ which is closer to P.

The gradient of $PQ_2 = \frac{3.25}{0.5} = 6.5$.

Now see what happens if we repeat this, taking positions of Q closer and closer to P, using a calculator (Fig.9.3).

Fig.9.3



Coordinates of Q	*QR*	*PR*	*Gradient*	
(3.3,10.89)	1.89	0.3	6.3	
(3.1,9.61)	0.61	0.1	6.1	
(3.01,9.0601)	0.0601	0.01	6.01	tending to 6
(3.001,9.006001)	0.006001	0.001	6.001	
(3.0001,9.00060001)	0.00060001	0.0001	6.0001	
.	.	.	.	
.	.	.	.	

The sequence of values suggests that as we continue, taking Q closer and closer to P, the gradient approaches 6. We say that 6 is the **limiting value or limit** of the sequence. As Q $\longrightarrow$ (tends to) P, the gradient of PQ $\longrightarrow$ 6 and **we take this limiting value as the gradient at P.**

Note that we cannot find this value *directly*. We have to use this limiting method. (We also have to be sure that there will be a limit but this will be assumed in our work).

Exercise 9.1 *(Answers on page 626.)*

1 Repeat the limiting process to find the gradient where $x = 2$ and $x = -1$ on the curve $y = x^2$.

2 Use the limiting method to find the gradient at the point where $x = 2$ on the curve $y = 2x^2$.

3 By the same method, find the gradient where $x = 4$ on the curve $y = \sqrt{x}$.

GENERAL METHOD FOR THE GRADIENT FUNCTION

To find the gradient at another point on the curve we must repeat the calculations. A better approach would be to find a formula for the gradient, using the same method.

In Fig.9.4, we take a general point P whose coordinates are (x, x^2).

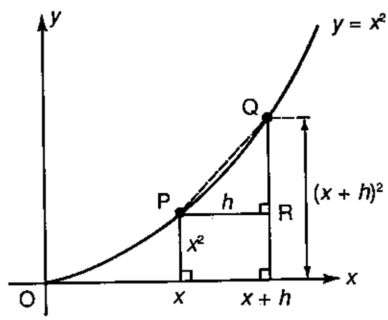


Fig.9.4

Now take a nearby point Q where $x_Q = x + h$. At present the value of h is not specified except that $h \neq 0$.

Then $y_Q = (x + h)^2$.

$$\text{Gradient of PQ} = \frac{\text{QR}}{\text{PR}}$$

$$= \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \frac{2xh + h^2}{h}$$

$$= 2x + h \quad (h \neq 0)$$

Now suppose Q moves closer and closer to P, i.e. $h \longrightarrow 0$. Then $2x + h \longrightarrow 2x$. The limiting value of $2x + h$ is $2x$ and we take this as the gradient at P. When $x = 3$, the gradient $= 6$, as we found before. When $x = 0$, the gradient is 0, which can be seen from the graph as this is the turning point.

The function $x \longmapsto 2x$ is the **gradient function** for the curve $y = x^2$. Each curve will have its own gradient function which we find by the limiting method, known as working from first principles.

Example 1

Find the gradient function for $y = 2x^2 - 3$ (Fig.9.5).

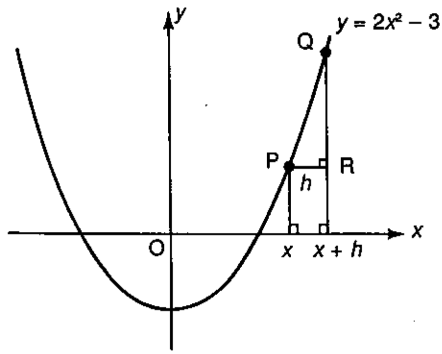


Fig. 9.5

P is a general point $(x, 2x^2 - 3)$.
Q is a nearby point with $x_Q = x + h$, $y_Q = 2(x + h)^2 - 3$.

$$\begin{aligned} QR &= 2(x + h)^2 - 3 - (2x^2 - 3) \\ &= 2x^2 + 4xh + 2h^2 - 3 - 2x^2 + 3 \\ &= 4xh + 2h^2 \end{aligned}$$

$$\frac{QR}{PR} = \frac{4xh + 2h^2}{h} = 4x + 2h$$

Now as $h \longrightarrow 0$, $4x + 2h \longrightarrow 4x$. The limiting value is $4x$. The gradient function is therefore $x \longmapsto 4x$.

Example 2

(a) *Find the gradient function for the curve $y = x^3 + 2x$ (Fig.9.6).*
(b) *Hence find the gradients at $x = 0$ and $x = -1$.*
(c) *Is there a value of x where the gradient is 0?*

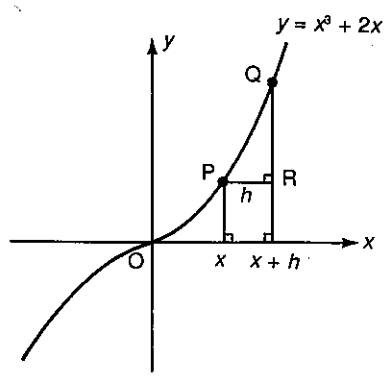


Fig.9.6

(a) Take P as $(x, x^3 + 2x)$ and a nearby point Q where $x_Q = x + h$, $y_Q = (x+h)^3 + 2(x+h)$.

Then $QR = (x+h)^3 + 2(x+h) - (x^3 + 2x)$
$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x$
$= 3x^2h + 3xh^2 + h^3 + 2h$

Gradient of PQ $= \frac{QR}{PR} = \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} = 3x^2 + 3xh + h^2 + 2$

As $h \longrightarrow 0$, $3xh$ and h^2 each $\longrightarrow 0$. The limiting value is $3x^2 + 2$ and the gradient function is $x \longmapsto 3x^2 + 2$.

(b) When $x = 0$, the gradient $= 2$; when $x = -1$, the gradient $= 5$.

(c) The equation $3x^2 + 2 = 0$ has no solution so the gradient of the curve is never 0.

NOTATION

If the equation of the curve is $y = f(x)$, we write the gradient function as $\mathbf{f'(x)}$.

If we take a point P on $y = f(x)$ (Fig.9.7), the coordinates of P are $(x, f(x))$. The coordinates of Q are $(x + h, f(x + h))$.

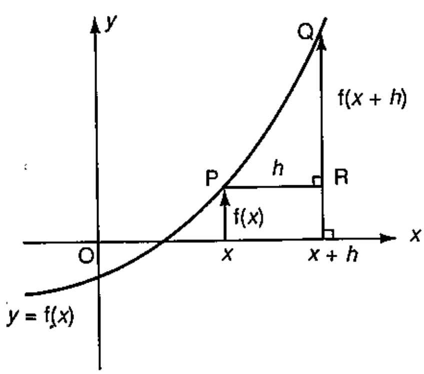


Fig.9.7

Then $QR = f(x+h) - f(x)$ and the gradient of PQ $= \frac{f(x+h) - f(x)}{h}$.

The limiting value of $\frac{f(x+h) - f(x)}{h}$ as $h \longrightarrow 0$ will be $f'(x)$.

We write this as $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\lim_{h \to 0}$ means 'take the limiting value when $h \longrightarrow 0$'.

Example 3

If $f(x) = \frac{1}{x-1}$ $(x \neq 1)$, *find* $f'(x)$.

$$f(x+h) - f(x) = \frac{1}{x+h-1} - \frac{1}{x-1}$$
$$= \frac{x-1-(x+h-1)}{(x+h-1)(x-1)}$$
$$= \frac{-h}{(x+h-1)(x-1)}$$

Then $\frac{f(x+h)-f(x)}{h} = \frac{-1}{(x+h-1)(x-1)}$

As $h \longrightarrow 0$, $x+h-1 \longrightarrow x-1$.

So the limiting value of $\frac{f(x+h)-f(x)}{h}$ will be $\frac{-1}{(x-1)^2}$ and $f'(x) = \frac{-1}{(x-1)^2}$.

Note that this is always negative, so the gradient on the curve is always negative, as seen from the sketch (Fig.9.8).

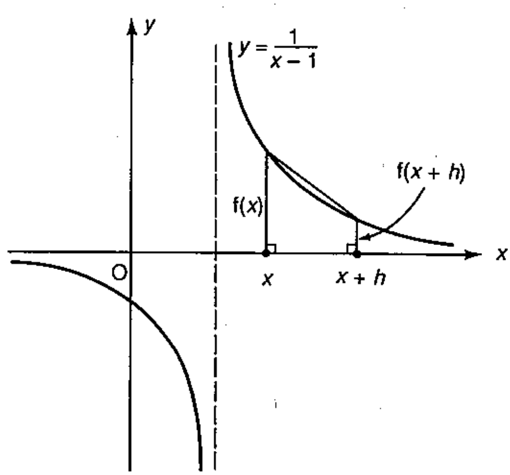


Fig.9.8

Exercise 9.2 *(Answers on page 627.)*

1 Find the gradient function for the following curves:

(a) $y = 3x^2$ (b) $y = 3x^2 - 1$ (c) $y = x^2 + x - 1$

(d) $y = 2 - x^2$ (e) $y = \frac{1}{x}$ $(x \neq 0)$ (f) $y = x^3 - x^2 + 1$

(g) $y = \frac{1}{x-1}$

2 For each of the curves in Question 1, find the gradient where (i) $x = 2$, (ii) $x = -1$, (iii) $x = 0$ (except curve (e)).

3 (a) Find the gradient function for $y = 3x^2 - 6x + 2$.

(b) For what value of x is the gradient 0?

(c) Hence find the minimum value of the function.

4 (a) If $f(x) = x^3 - x^2$, find $f'(x)$.
(b) For what values of x is the gradient on the curve $y = f(x)$ zero?
(c) Find the values of $f(x)$ at these points.

5 By finding the gradient function, show that the curve $y = 1 - 4x - x^2$ has a turning point where $x = -2$. Is this a maximum or minimum point?

6 Find the gradient function for the curve $y = \frac{1}{x} + 4x$ $(x \neq 0)$.
Hence find the values of x where the gradient on this curve is zero.

7 Find the gradient function for $y = ax^2 + bx + c$ where a, b and c are constants.

The δy, δx Notation for the Gradient Function

To find $f'(x)$, we took two points whose x-coordinates were x and $x + h$. We now introduce a new and important notation. Instead of h, we write δx (read *delta x*) which is *one* symbol for the change in x, called the **increment** in x.

We use the curve $y = x^2$ again (Fig.9.9). Now if x changes to $x + \delta x$, y will also change to $y + \delta y$, where δy is the corresponding increment in y.

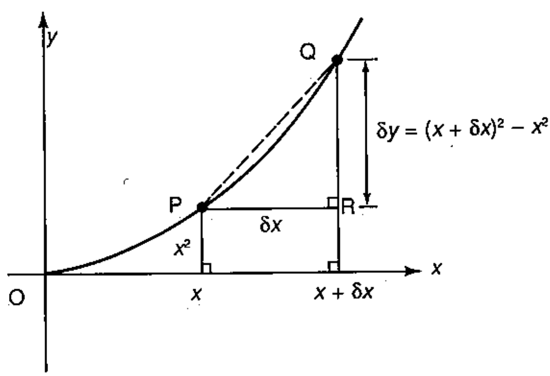


Fig.9.9

$PR = \delta x$, $QR = \delta y$.

The coordinates of Q are $(x + \delta x, y + \delta y)$ and so $y + \delta y = (x + \delta x)^2$.

$$\begin{aligned} QR &= (x + \delta x)^2 - x^2 \\ &= x^2 + 2x\delta x + (\delta x)^2 - x^2 \\ &= 2x\delta x + (\delta x)^2 \end{aligned}$$

The gradient of $PQ = \dfrac{\delta y}{\delta x} = \dfrac{2x\delta x + (\delta x)^2}{\delta x} = 2x + \delta x$

Now we let $\delta x \longrightarrow 0$. The limiting value of $\frac{\delta y}{\delta x}$ will be $2x$, so the gradient function is $2x$ as before.

The special feature of this notation is that we write the gradient as $\frac{dy}{dx}$ (read *dee y by dee x*) to symbolize the limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$.
(The curly δ is straightened to ordinary d to show that we have taken the limiting value).

So $$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

This δy, δx notation will be used from now on.

Note: $\frac{dy}{dx}$ is to be taken as one symbol and NOT as a fraction.

Example 4

If $y = 16x + \frac{1}{x^2}$, find $\frac{dy}{dx}$ from first principles.
Find the value of x where the gradient is 0.

$f(x) = 16x + \frac{1}{x^2}$ and $f(x + \delta x) = 16(x + \delta x) + \frac{1}{(x + \delta x)^2}$

Hence $f(x + \delta x) - f(x) = 16x + 16\delta x + \frac{1}{(x + \delta x)^2} - 16x - \frac{1}{x^2}$

$$= 16\delta x + \frac{x^2 - (x + \delta x)^2}{(x + \delta x)^2 x^2}$$

$$= 16\delta x + \frac{x^2 - x^2 - 2x\delta x - (\delta x)^2}{(x + \delta x)^2 x^2}$$

$$= 16\delta x - \frac{2x\delta x + (\delta x)^2}{(x + \delta x)^2 x^2}$$

Then $\frac{\delta y}{\delta x} = 16 - \frac{2x + \delta x}{(x + \delta x)^2 x^2}$

As $\delta x \longrightarrow 0$, $2x + \delta x \longrightarrow 2x$ and $x + \delta x \longrightarrow x$.

Then $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 16 - \frac{2x}{x^4} = 16 - \frac{2}{x^3}$

When the gradient is 0, $\frac{dy}{dx} = 16 - \frac{2}{x^3} = 0$ i.e. $16x^3 = 2$ or $x^3 = \frac{1}{8}$ giving $x = \frac{1}{2}$.

Exercise 9.3 *(Answers on page 627.)*

1 Find $\frac{dy}{dx}$ from first principles for

(a) $y = 3x^2 + 1$ (b) $y = 2 - 4x^2$ (c) $y = 4x^3$

(d) $y = x - \frac{1}{x}$ (e) $y = \frac{x^2}{2} - x + 3$ (f) $y = \frac{x^3}{3}$

2 What is the gradient of $y = 5$? Hence explain why $\frac{dy}{dx} = 0$ if $y = k$ (a constant).

3 Given $y = 2x^2 - 4x + 1$, find (a) $\frac{dy}{dx}$ and (b) the coordinates of the point on the curve where the gradient is (i) 0, (ii) -8.

4 (a) Find $\frac{dy}{dx}$ if $y = x^2 + ax + 3$ where a is a constant.

(b) Find the value of a if the gradient where $x = 3$ is 2.

MEANING OF $\frac{dy}{dx}$

The notation $\frac{dy}{dx}$ for the limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$ is appropriate as it is a reminder that $\frac{dy}{dx}$ is derived from $\frac{\delta y}{\delta x}$. We call this **differentiation** (as it uses the difference δx) and $\frac{dy}{dx}$ is called the **derivative** or the **differential coefficient** of y with respect to x. We shall use the abbreviation 'wrt' for 'with respect to'.

$\frac{dy}{dx}$ gives the gradient function for a curve and the value of $\frac{dy}{dx}$ at a given point is the gradient of the curve and therefore of the tangent there.

Now the gradient at a point measures the rate at which y is changing wrt x. The steeper the gradient the greater this rate of change. For example, on the curve $y = x^2$, the two quantities are each changing and the rate of change is $2x$. When $x = 3$, $y = 9$ but y is changing at that point 6 times as much as x is changing: $\frac{dy}{dx}$ measures this rate of change. This is what makes differentiation a powerful tool in Mathematics.

The idea and notation can be applied to any function. For example, if s is a function of t, $s = f(t)$, then $\frac{ds}{dt}$ is the rate of change of s wrt t.
If A is a function of r, $A = f(r)$, then $\frac{dA}{dr}$ is the rate of change of A wrt r.

Example 5

If $p = 3t^2 - 2t + 1$, find $\frac{dp}{dt}$.

p is a function of t so we take an increment δt in t.
The corresponding increment in p is δp.

$$\frac{dp}{dt} = \lim_{\delta t \to 0} \frac{\delta p}{\delta t}$$

$p + \delta p = 3(t + \delta t)^2 - 2(t + \delta t) + 1$

Now show that $\delta p = 6t\delta t + 3(\delta t)^2 - 2\delta t$

$\frac{\delta p}{\delta t} = 6t + 3\delta t - 2$ and the limiting value is $6t - 2$.

Hence $\frac{dp}{dt} = 6t - 2$.

NOTE ON INDICES

We shall be dealing with negative indices shortly so this note recalls the rules for indices.

To multiply powers of the same term, add the indices.

$x^3 \times x^2 = x \times x \times x \times x \times x = x^5 = x^{3+2}$

To divide powers of the same term, subtract the indices.

$x^5 \div x^2 = \frac{x \times x \times x \times x \times x}{x \times x} = x^3 = x^{5-2}$

$x \div x = 1 = x^{1-1} = x^0$ and $x^0 = 1$

$x \div x^3 = x^{1-3} = x^{-2}$ but $\frac{x}{x^3} = \frac{1}{x^2}$.

A negative index means the reciprocal: $x^{-n} = \frac{1}{x^n}$

THE DERIVATIVE OF ax^n

Here are listed some of the derivatives we have already found:

Function	*Derivative*
x^2	$2x$
$2x^2$	$4x$
$4x^3$	$12x^2$
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$

Can you see a pattern in these?
The derivative of x^2 is $2x = 2x^{2-1}$.
The derivative of $4x^3$ is $12x^2 = 4 \times 3x^{3-1}$.
The derivative of x^{-1} is $-x^{-2} = -1x^{-1-1}$.

We deduce the following rule (which we shall not prove):
to differentiate a single term, multiply by the index and then reduce the index by 1.
For example, to find the derivative of $5x^3$:

$$\underset{\text{the index 3 becomes a multiplier}}{}\quad 5 \times 3\, x^2 \longleftarrow \text{the new index is } 3 - 1 = 2$$

The derivative of ax^n is anx^{n-1}.

What about the derivative of a constant, say $y = 5$? This is 0, as the gradient is always 0.
The derivative of a constant is 0.

If $y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$
If $y = k$ (a constant), $\frac{dy}{dx} = 0$

THE DERIVATIVE OF A POLYNOMIAL

In Example 5, we saw that the derivative of the polynomial $3t^2 - 2t + 1$ was $6t - 2$. The derivative of $3t^2$ is $6t$, the derivative of $-2t$ is -2 and the derivative of 1 is 0. These are added to obtain the derivative of the polynomial.

The derivative of a polynomial is the sum of the separate derivatives of the terms.

This rule applies only to *polynomials* and does not apply to functions such as $\sqrt{3x - 1}$ or $\frac{x+2}{x-1}$ which are not polynomials.

Example 6

Differentiate wrt x (a) $3x^5 + 7$, (b) $x^3 - \frac{1}{2}x^2 - \frac{1}{x}$, (c) $(2x-3)^2$, (d) $\frac{4}{x^3}$, (e) $ax^3 + 2bx^2 - cx + 7$ where a, b and c are constants.

(a) If $y = 3x^5 + 7$, then $\frac{dy}{dx} = 3 \times 5x^{5-1} + 0 = 15x^4$.

Note: Do not write $3x^5 + 7 = 15x^4$. This is incorrect. Use a letter such as y for the function and then write $\frac{dy}{dx}$.

(b) If $y = x^3 - \frac{1}{2}x^2 - \frac{1}{x} = x^3 - \frac{1}{2}x^2 - x^{-1}$

then $\frac{dy}{dx} = 3x^2 - \frac{1}{2} \times 2x^1 - (-1)x^{-2} = 3x^2 - x + \frac{1}{x^2}$

Rewrite reciprocals such as $\frac{1}{x^2}$ in terms of negative indices before differentiating.

(c) Here we express $(2x-3)^2$ as a polynomial first by expansion.

$(2x-3)^2 = 4x^2 - 12x + 9$

If $y = 4x^2 - 12x + 9$, then $\frac{dy}{dx} = 8x - 12x^0 + 0 = 8x - 12$, as $x^0 = 1$.

(d) If $y = \frac{4}{x^3} = 4x^{-3}$, then $\frac{dy}{dx} = 4(-3)x^{-3-1} = -12x^{-4}$

which can be left in this form or written as $-\frac{12}{x^4}$.

(e) If $y = ax^3 + 2bx^2 - cx + 7$, then $\frac{dy}{dx} = 3ax^2 + 4bx - cx^0 + 0 = 3ax^2 + 4bx - c$.

Example 7

Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 9x + 6$ where the gradient is 0.

The gradient $= \frac{dy}{dx}$.

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

If the gradient = 0, then $\frac{dy}{dx} = 0$ and so $3(x-3)(x+1) = 0$ which gives $x = 3$ or $x = -1$.

When $x = 3$, $y = 3^3 - 3(3^2) - 9(3) + 6 = -21$: coordinates $(3,-21)$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 6 = 11$: coordinates $(-1,11)$

Example 8

(a) Differentiate $A = 2\pi r^2 + 2\pi rh$ wrt r, where h is a constant.

(b) If $T = \frac{(3p+1)^2}{p}$, find $\frac{dT}{dp}$ and the values of p for $\frac{dT}{dp} = 0$.

(a) $\frac{dA}{dr} = 2\pi \times 2r^1 + 2\pi hr^0 = 4\pi r + 2\pi h$

(b) Here we rewrite the expression as a polynomial first.

$$\frac{(3p+1)^2}{p} = \frac{9p^2+6p+1}{p} = 9p + 6 + \frac{1}{p} \qquad \text{(dividing each term in}$$
$$= 9p + 6 + p^{-1} \qquad \text{the numerator by } p)$$

Then $\frac{dT}{dp} = 9 + 0 + (-1)p^{-2} = 9 - \frac{1}{p^2}$

If $\frac{dT}{dp} = 0$, then $9 = \frac{1}{p^2}$ and $p = \pm\ \frac{1}{3}$.

Exercise 9.4 *(Answers on page 627.)*

1 Differentiate wrt x:

(a) $5x$ (b) $4x^2$ (c) 7

(d) $3x^2 - 5$ (e) $3x^2 - x - 1$ (f) $x^3 - x^2 - x - 1$

(g) $1 - 3x^2$ (h) $(x-1)^2$ (i) $\frac{4}{x}$ x^{-1}

(j) $3x^2 + \frac{1}{x}$ x^{-1} $6x - 1$ (k) $(x-2)^3$ (l) $(x - \frac{1}{x})^2$

(m) $\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}$ (n) $\frac{x^3 - 2x^2 + x - 3}{x}$ x^{-1} (o) $\frac{(2x-1)^2}{x^2}$

(p) $\frac{2x^2 - x - 1}{x^3}$

2 Differentiate $s = 3t^2 - 4t - 2$ wrt t. $(3x^2 - 4x + 1) \cdot x^{-2}$

3 If $W = 3r^3 - 2r^2 + r - 3$, find $\frac{dW}{dr}$.

4 Given $u = (3t-2)^2$, find $\frac{du}{dt}$.

5 $y = 4x^2 - \mu x + 6$. Given that the rate of change of y wrt x is 2 when $x = 1$, find the value of μ.

6 Find the gradient on the curve $y = 4x^3 - 10x + 1$ at the point where $x = -1$.

7 The gradient on the curve $y = ax^2 - 3x + 4$ where $x = -2$ is 13. Find the value of a.

8 Find the coordinates of the points on the curve $y = 3x^3 - 4x + 2$ where the gradient is 0.

9 Find the coordinates of the points on the curve $y = x^3 - 3x^2 + x - 5$ where the gradient is 1.
$(2, -7)$ $(0, -5)$

10 Find the values of z for which $\frac{dP}{dz} = 0$ where $P = 4z^3 - 2z^2 - 8z + 5$.

11 Show that there is only one point on the curve $y = x^4 - 32x + 10$ where the gradient is 0 and find its coordinates. $x^3 - 32 = 0$ $x^3 = 32$

12 Given that $u = 4t^3 + 3t^2 - 6t - 1$, find the values of t for which $\frac{du}{dt} = 12$.

13 Find the values of t for which $\frac{ds}{dt} = 0$ given that $s = 4t^3 + t^2 - 2t - 5$.

14 Given $v = 4s^2 - 12s - 7$, find $\frac{dv}{ds}$. For what value of s is $\frac{dv}{ds} = 4$?

15 Given that the curve $y = ax^2 + \frac{b}{x}$ has a gradient of 5 at the point $(1, 1)$, find the values of a and b. What is the gradient of the tangent to the curve at the point where $x = \frac{1}{2}$?

 $1 = a$ $b \cdot x^{-1}$

16 If the gradient on the curve $y = ax + \frac{b}{x}$ at the point $(-1,-1)$ is 5, find the values of a and b.

17 The curve given by $y = ax^3 + bx^2 + 3x + 2$ passes through the point (1,2) and the gradient at that point is 7. Find the values of a and b.

18 Given $y = 2x^3 - 3x^2 - 12x + 5$, find the domain of x for which $\frac{dy}{dx} \geq 0$.

19 The function P is given by $P = \frac{a}{t} + bt^2$ and when $t = 1$, $P = -1$. $-a.t^{-2} + 2bt$
The rate of change of P when $t = \frac{1}{2}$ is -5. Find the values of a and b.

20 Given that $R = mp^4 + np^2 + 3$, find $\frac{dR}{dp}$. When $p = 1$, $\frac{dR}{dp} = 12$ and when $p = \frac{1}{4}$, $\frac{dR}{dp} = -\frac{3}{4}$. Find the values of m and n.

COMPOSITE FUNCTIONS

In part (c) of Example 6, to find the derivative of $(2x - 3)^2$, we first expand it into a polynomial. Similarly, if we want to find the derivative of $y = (3x - 2)^5$, we first expand it into a polynomial. This would be rather lengthy so we look for a neater method. To do this we take $(3x - 2)^5$ as a **composite** or **combined** function.

The function $y = (3x - 2)^5$ can be built up from two simpler functions, $u = 3x - 2$ and then $y = u^5$. We call u the **core** function.

Now u is a function of x so $\frac{du}{dx} = 3$. y is a function of the core u so $\frac{dy}{du} = 5u^4$. To obtain $\frac{dy}{dx}$ from these two derivatives we use a rule for the derivative of composite functions (which we shall not prove):

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u and u is a function of x.

Note: du cannot be cancelled on the right hand side as these are not fractions but derivatives. However the notation suggests the result and is easy to remember.

Then $\frac{dy}{dx} = 5u^4 \times 3 = 15u^4 = 15(3x - 2)^4$.

Example 9

Find $\frac{dy}{dx}$ given that $y = (x^2 - 3x + 1)^4$.

Take $u = x^2 - 3x + 1$ as the core. Then $y = u^4$.

$\frac{dy}{du} = 4u^3$, differentiating y wrt the core u.

$\frac{du}{dx} = 2x - 3$, differentiating the core wrt x.

Multiply these two derivatives to obtain $\frac{dy}{dx}$:

$\frac{dy}{dx} = 4u^3 \times (2x - 3) = 4(2x - 3)(x^2 - 3x + 1)^3$.

7 + b

$-\frac{5}{3} - \frac{6}{5} = 6$

Example 10

Find $\frac{dy}{dx}$ *if* $y = (ax^2 + bx + c)^n$.

Take $u = ax^2 + bx + c$ and then $y = u^n$.

$\frac{dy}{du} = nu^{n-1}$ and $\frac{du}{dx} = 2ax + b$.

Then $\frac{dy}{dx} = nu^{n-1} \times (2ax + b) = n(2ax + b)(ax^2 + bx + c)^{n-1}$.

With practice, $\frac{dy}{dx}$ can be written down in two steps on one line.

Suppose $y =$ (**core**)^n (**core** being a function of x)

Step 1 *Step 2*

$$\underbrace{\frac{dy}{dx} = n(\textbf{core})^{n-1}}_{\text{derivative of (core)}^n \text{ wrt core}} \times \underbrace{\frac{d(\textbf{core})}{dx}}_{\text{derivative of core wrt } x}$$

Example 11

Differentiate $\frac{2}{x^2 - 3x + 1}$ *wrt x.*

Take $y = 2(\underbrace{x^2 - 3x + 1}_{\text{core}})^{-1}$

Then $\frac{dy}{dx} = 2(-1)(x^2 - 3x + 1)^{-2} \times \underbrace{(2x - 3)}_{\text{derivative of core wrt } x}$

$= \frac{-2(2x - 3)}{(x^2 - 3x + 1)^2}$

Example 12

Given that $s = 3t - \frac{1}{1 - 2t}$, *find (a)* $\frac{ds}{dt}$ *and (b) the values of s when* $\frac{ds}{dt} = \frac{25}{9}$.

(a) $s = 3t - (\underbrace{1 - 2t}_{\text{core}})^{-1}$

$\frac{ds}{dt} = 3 - (-1)(1 - 2t)^{-2}\underbrace{(-2)}_{\text{derivative of core wrt } t}$

$= 3 - \frac{2}{(1 - 2t)^2}$

(b) If $3 - \frac{2}{(1 - 2t)^2} = \frac{25}{9}$, then $\frac{2}{(1 - 2t)^2} = \frac{2}{9}$ i.e. $9 = (1 - 2t)^2$.

Hence $9 = 1 - 4t + 4t^2$ or $4t^2 - 4t - 8 = 0$

which gives $t^2 - t - 2 = 0$ or $(t - 2)(t + 1) = 0$ and $t = 2$ or -1.

When $t = 2$, $s = 6 - \frac{1}{1 - 4} = 6\frac{1}{3}$

and when $t = -1$, $s = -3 - \frac{1}{3} = -3\frac{1}{3}$.

Exercise 9.5 *(Answers on page 627.)*

1 Differentiate the following wrt x:

(a) $(x-3)^5$ (b) $(3x-1)^7$ (c) $(5-2x)^3$

(d) $(4x-5)^{10}$ (e) $(4x-3)^4$ (f) $(x^2-x+1)^3$

(g) $(3-x-2x^2)^5$ (h) $\frac{1}{x-2}$ (i) $\frac{4}{1-3x}$

(j) $\frac{4}{3+2x}$ (k) $(x-\frac{1}{x})^4$ (l) $\frac{1}{x^2+3}$

(m) $\frac{4}{x^2-x-1}$ (n) $(ax+b)^n$ (o) $\frac{2}{(2x-3)^4}$

(p) $\frac{1}{(1-3x-2x^2)^3}$ (q) $(2x-\frac{1}{2x})^3$

2 If $s=(2t-1)^3$, find (a) $\frac{ds}{dt}$ and (b) the value of t for which $\frac{ds}{dt}=24$.

3 If $v=(3t^2-2t+1)^2$ find the value of $\frac{dv}{dt}$ when $t=-1$.

4 Given that $A=\frac{t^2}{2}-\frac{(1-t)^3}{5}$, find $\frac{dA}{dt}$ and simplify the result. Hence find the value of t for which $\frac{dA}{dt}=1$.

5 If $s=\frac{3}{4-2r}$ find $\frac{ds}{dr}$ and the values of s when $\frac{ds}{dr}=\frac{1}{6}$.

6 The equation of a curve is $y=2x-\frac{4}{x+1}$. Find (a) $\frac{dy}{dx}$ and (b) the gradient of the curve when $x=-3$.

7 Find the gradient of the curve $y=\frac{3}{x^2-2x+1}$ where $x=2$.

8 If $y=\frac{1}{x+1}$, find the coordinates of the points where the gradient is $-\frac{1}{4}$.

9 If $y=\frac{1}{(x-3)^2}$, find the coordinates of the point where the gradient $=2$.

10 Given that $v=\frac{3}{1-4t}$, find (a) $\frac{dv}{dt}$, (b) the values of t when $\frac{dv}{dt}=3$.

11 If $y=3t+1+\frac{1}{1+2t}$, find (a) $\frac{dy}{dt}$ and (b) the values of t when $\frac{dy}{dt}=2\frac{1}{2}$.

12 When $x=1$, the gradient of the curve $y=\frac{1}{3+ax}$ is 2. Find the values of a.

13 Given that $L=\frac{1}{a+bx}$ and that $L=1$ and $\frac{dL}{dx}=3$ when $x=1$, find the values of a and b.

14 The curve $y=\frac{1}{a+bx}$ passes through the point $(1,-1)$ and its gradient at that point is 2. Find the values of a and b.

THE SECOND DIFFERENTIAL COEFFICIENT $\frac{d^2y}{dx^2}$

If y is a function of x, then $\frac{dy}{dx}$ is also a function of x (or a constant).

Hence we can differentiate $\frac{dy}{dx}$ wrt x. This gives the **second differential coefficient** $\frac{d(\frac{dy}{dx})}{dx}$ which is written as $\frac{d^2y}{dx^2}$ (read *dee two y by dee x two*) for brevity. The 2's are not squares but symbolize differentiating twice.

The square of $\frac{dy}{dx}$ is written as $(\frac{dy}{dx})^2$.

$\frac{d^2y}{dx^2}$ is sometimes also written as $f''(x)$ where $y=f(x)$.

Example 13

Find $\dfrac{d^2y}{dx^2}$ *and* $\left(\dfrac{dy}{dx}\right)^2$ *if*

(a) $y = 2x^3 - 3x^2 + 1$, *(b)* $y = (4x - 1)^3$, *(c)* $y = \dfrac{1}{2 - 3x}$.

(a) $\dfrac{dy}{dx} = 6x^2 - 6x$

$\dfrac{d^2y}{dx^2} = 12x - 6$

$\left(\dfrac{dy}{dx}\right)^2 = (6x^2 - 6x)^2$ which is quite different from $\dfrac{d^2y}{dx^2}$.

(b) $\dfrac{dy}{dx} = 3(4x - 1)^2 \times 4 = 12(4x - 1)^2$

$\dfrac{d^2y}{dx^2} = 24(4x - 1) \times 4 = 96(4x - 1)$

$\left(\dfrac{dy}{dx}\right)^2 = 144(4x - 1)^4$.

(c) $y = (2 - 3x)^{-1}$

$\dfrac{dy}{dx} = (-1)(2 - 3x)^{-2} \times (-3) = 3(2 - 3x)^{-2}$

$\dfrac{d^2y}{dx^2} = 3(-2)(2 - 3x)^{-3} \times (-3) = 18(2 - 3x)^{-3}$

$\left(\dfrac{dy}{dx}\right)^2 = \dfrac{9}{(2 - 3x)^4}$

As we shall see, $\dfrac{d^2y}{dx^2}$ has important applications. It is also possible to find further derivatives, such as $\dfrac{d^3y}{dx^3}$, etc. but we shall not use these.

Exercise 9.6 *(Answers on page 626.)*

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for

(a) $4x^3 - 5x^2 + 8$ (b) $(2x - 7)^3$ (c) $(1 - 4x)^4$ (d) $\frac{1}{x}$

(e) $x^2 - \frac{1}{x}$ (f) $\frac{3}{2 - x}$ (g) $x^4 - x^2 - \frac{1}{x^2}$

2 If $s = 3t^2 - \frac{2}{t^2}$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

3 If $y = (ax + 2)^2$ and $\frac{d^2y}{dx^2} = 18$, find the values of a.

4 $y = ax^3 + bx$. Given that $\frac{dy}{dx} = -1\frac{1}{4}$ and that $\frac{d^2y}{dx^2} = 3$ when $x = \frac{1}{2}$, find the values of a and b.

5 For the function $y = (ax + b)^2$, $\frac{dy}{dx} = -6$ when $x = \frac{1}{3}$ and $\frac{d^2y}{dx^2} = 18$.
Find the values of a and b.

6 If the gradient of the curve $y = 2x^3 + \mu x^2 - 5$ is -2 when $x = 1$, find the value of μ and the value of $\frac{d^2y}{dx^2}$ at that point.

7 If $y = \frac{1}{2-x}$, find $(\frac{dy}{dx})^2$ and $\frac{d^2y}{dx^2}$. Show that $y\frac{d^2y}{dx^2} = 2(\frac{dy}{dx})^2$.

8 If $s = 3t^3 - 30t^2 + 36t + 2$, find the values of t for which $\frac{ds}{dt} = 0$ and the value of t for which $\frac{d^2s}{dt^2} = 0$.

9 If $y = 2x^3 - 4x^2 + 9x - 5$, what is the range of values of x for which $\frac{d^2y}{dx^2} \geq 0$?

SUMMARY

- If $y = f(x)$, $f'(x)$ is the gradient function. The value of $f'(x)$ is the gradient at a given point.
- $f'(x) = \frac{dy}{dx} = \lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ where δx is the increment in x.
- $\frac{dy}{dx}$ is the derivative or differential coefficient of y wrt x. It measures the rate of change of y wrt x.
- If $y = ax^n$, $\frac{dy}{dx} = nax^{n-1}$; if $y = k$ (a constant), $\frac{dy}{dx} = 0$
- The derivative of a sum of terms or a polynomial is the sum of the derivatives of the separate terms.
- If $y = f(u)$ where u is a function of x, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
- $\dfrac{d(\frac{dy}{dx})}{dx}$ is the second differential coefficient of y wrt x, written as $\dfrac{d^2y}{dx^2}$ or as $f''(x)$ if $y = f(x)$.

REVISION EXERCISE 9 *(Answers on page 628.)*

1 Differentiate wrt x:

(a) $(x - 5)^3$

(b) $(1 - 2x)^5$

(c) $\frac{2}{1 - 4x}$

(d) $(2x^2 - 1)^3$

(e) $(1 - 3x - 2x^2)^3$

(f) $\frac{1}{x(1 - x)}$

(g) $(2x - \frac{1}{x})^3$

(h) $\frac{x^3 - 2x^2 + 1}{2x^2}$

(i) $\frac{(x - 1)(x + 4)}{x}$

(j) $\frac{1 - 3x}{2} - \frac{2}{1 - 3x}$

2 If $y = x^3 - 3x^2 + 7$, for what range of values of x is $\frac{dy}{dx} \leq 0$?

3 Find the gradient of the curve $y = 5 + 2x - 3x^2$ at each of the points where it meets the x-axis.

4 (a) Find the gradient of the curve $y = \frac{6}{x}$ where $x = 3$.
Hence find (b) the equation of the tangent at that point and (c) the coordinates of the points where this tangent meets the axes.
(d) Calculate the distance between these points.

5 Given that $s = 3t^2 - 4t + 1$, find the rate of change of s wrt t when $s = 5$.

6 If $y = 4x - \frac{1}{1-2x}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

7 For the function $y = 2x^3 - 4x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Hence find the value of $\dfrac{\frac{dy}{dx}}{\sqrt{1 + \frac{d^2y}{dx^2}}}$ when $x = 2$.

8 If $A = \frac{1}{1 + \mu r}$ where μ is a constant > 0 and the rate of change of A wrt r is $-\frac{5}{9}$ when $r = 0.4$, find the value of μ.

9 The tangent at the point (a,b) on the curve $y = 1 - x - 2x^2$ has a gradient of 7. Find the values of a and b.

10 The curve $y = \frac{a}{x} + bx$ (a, b constants) passes through the points A(1,–1) and B($4,-11\frac{1}{2}$). (a) Find the value of a and of b. (b) Show that the tangent to the curve at the point where $x = -2$ is parallel to AB.

11 (a) Show that the gradients of the tangents to the curve $y = x^2 - x - 2$ where the curve meets the x-axis are numerically equal.
(b) Find the equations of these tangents and show that they intersect on the axis of the curve.

12 The line $y = x + 1$ meets the curve $y = x^2 - x - 2$ at the points A and B. Find the gradients of the tangents to the curve at these points.

13 If $p = 2s^3 - s^2 - 28s$, find the values of s which make $\frac{dp}{ds} = 0$ and for these values of s find the value of $\frac{d^2p}{ds^2}$.

14 The gradient of the curve $y = ax^2 + bx + 2$ at the point (2,12) is 11. Find the values of a and b.

15 If $y = x^3 + 3x^2 - 9x + 2$, for what range of values of x is $\frac{dy}{dx}$ negative?

16 Given that $y = (x + 2)^2 - (x - 2)^3$, find the range of values of x for which $\frac{dy}{dx} \geq 0$.

17 If $y = \frac{A}{x} + Bx$, where A and B are constants, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y$.

Calculus (2): Applications of Differentiation

10

INCREASING AND DECREASING FUNCTIONS

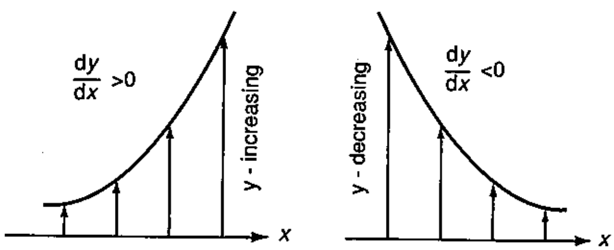


Fig. 10.1

On any stretch of the curve $y = f(x)$, where $\frac{dy}{dx} > 0$, the curve slopes upwards. Hence y *increases* as x *increases* and $f(x)$ is an **increasing** function.
Similarly if $\frac{dy}{dx} < 0$, y decreases as x increases and $f(x)$ is a **decreasing** function.
At any point where $\frac{dy}{dx} = 0$, $f(x)$ has a **stationary** value and is neither increasing nor decreasing. Such a point is a **stationary** point.

Example 1

For what range of values of x is the function $y = x^3 - 3x^2 - 9x + 4$ (a) decreasing, (b) increasing?

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$

(a) $\frac{dy}{dx} < 0$ for $-1 < x < 3$ and the function is decreasing in this interval (Fig.10.2).

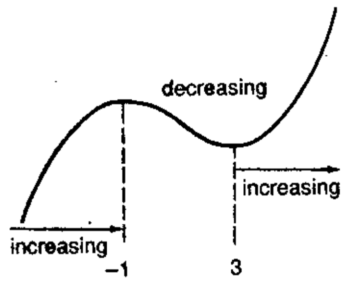


Fig. 10.2

(b) $\frac{dy}{dx} > 0$ for $x < -1$ or $x > 3$ and the function is increasing in these intervals.
The function has stationary values at $x = 3$ and at $x = -1$.

Example 2

(a) For what range of values of x is the function $y = x + \frac{1}{4x}$ increasing? (b) What are the coordinates of the stationary points?

(a) $\frac{dy}{dx} = 1 - \frac{1}{4x^2}$
$\frac{dy}{dx} > 0$ if $1 > \frac{1}{4x^2}$, i.e. $4x^2 > 1$ or $x^2 > \frac{1}{4}$.
Hence $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

This is illustrated in the sketch of the curve in Fig. 10.3.

(b) $\frac{dy}{dx} = 0$ where $1 - \frac{1}{4x^2} = 0$ i.e. $x^2 = \frac{1}{4}$ and $x = \pm\frac{1}{2}$.
So the coordinates of the stationary points A and B are $(\frac{1}{2},1)$ and $(-\frac{1}{2},-1)$.

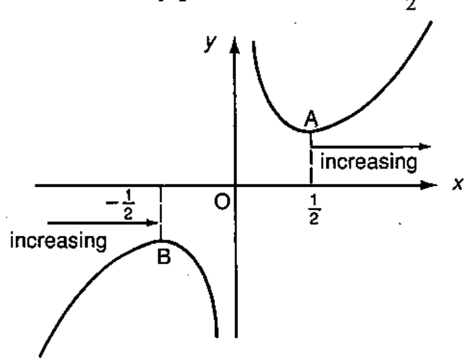


Fig. 10.3

TANGENTS AND NORMALS

As we have seen, if $y = f(x)$ is the equation of a curve, then $\frac{dy}{dx}$ gives the gradient of the tangent at any point (Fig. 10.4).

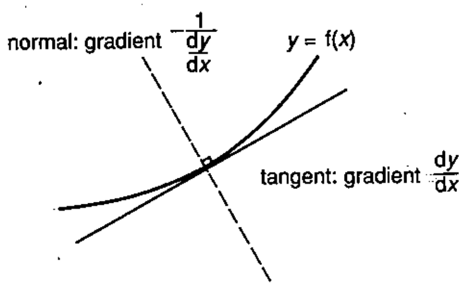


Fig. 10.4

Associated with the tangent is the **normal**, which is perpendicular to the tangent. The gradient of the tangent is $\frac{dy}{dx}$, so the gradient of the normal is $-\frac{1}{\frac{dy}{dx}}$.

Example 3

Find the equations of the tangent and the normal to the curve $y = x^2 - 2x - 3$ at the point where it meets the positive x-axis.

When $y = 0$, $x^2 - 2x - 3 = (x - 3)(x + 1) = 0$. So the curve meets the positive x-axis where $x = 3$.

$\frac{dy}{dx} = 2x - 2$ and when $x = 3$, $\frac{dy}{dx} = 4$.

We write this briefly as $\left(\frac{dy}{dx}\right)_{x=3} = 4$, meaning the *value* of $\frac{dy}{dx}$ when $x = 3$.

The equation of the tangent is then $y = 4(x - 3)$ i.e. $y = 4x - 12$.

The gradient of the normal $= -\frac{1}{4}$ so the equation of the normal is

$y = -\frac{1}{4}(x - 3)$ i.e. $4y + x = 3$.

Example 4

(a) *Find the equation of the normal to the curve $y = x + \frac{4}{x}$ at the point P where $x = 4$.*

(b) If the normal meets the curve again at N, find the coordinates of N (Fig.10.5).

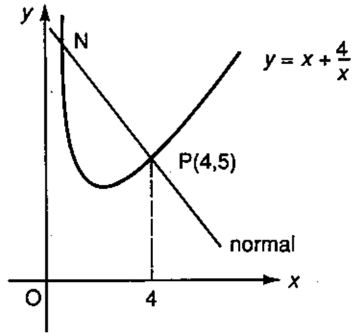


Fig.10.5

(a) The coordinates of P are $(4, 4 + \frac{4}{4})$ i.e. (4,5).

$\frac{dy}{dx} = 1 - \frac{4}{x^2}$ so $\left(\frac{dy}{dx}\right)_{x=4} = 1 - \frac{1}{4} = \frac{3}{4}$.

Hence the gradient of the normal $= -\frac{4}{3}$.

The equation of the normal is $y - 5 = -\frac{4}{3}(x - 4)$ i.e. $3y + 4x = 31$.

(b) To find where the normal meets the curve, we solve the simultaneous equations $3y + 4x = 31$ and $y = x + \frac{4}{x}$.

$x + \frac{4}{x} = \frac{31 - 4x}{3}$ i.e. $3x^2 + 12 = 31x - 4x^2$ or $7x^2 - 31x + 12 = 0$.

Then $(7x - 3)(x - 4) = 0$ which gives $x = 4$ or $\frac{3}{7}$.

$x = 4$ is the point P so the coordinates of N are $\left(\frac{3}{7}, \frac{3}{7} + \frac{4}{\frac{3}{7}}\right)$ i.e. $\left(\frac{3}{7}, \frac{205}{21}\right)$.

Example 5

(a) Find the x-coordinate of the point on the curve $y = 2x^3 + x^2 - 2x + 1$ where the curve is parallel to the line $y = 2x$.

(b) Is any part of the curve parallel to the line $y + 3x = 1$?

(a) $\frac{dy}{dx} = 6x^2 + 2x - 2$ and this must equal 2 (the gradient of $y = 2x$).

Then $6x^2 + 2x - 2 = 2$ which gives $3x^2 + x - 2 = 0$ or $(3x - 2)(x + 1) = 0$.

Hence $x = \frac{2}{3}$ or -1.

At these points the curve is parallel to $y = 2x$.

(b) If $\frac{dy}{dx} = 6x^2 + 2x - 2 = -3$ (the gradient of $y + 3x = 1$), then $6x^2 + 2x + 1 = 0$. But this equation has no real solutions. Hence the gradient of the curve is never equal to -3 and the curve is never parallel to $y + 3x = 1$.

Exercise 10.1 *(Answers on page 628.)*

1 Find the range of values of t for which V is increasing if $V = 4t^3 - 3t$.

2 In what interval must x lie if the function $y = x^4 - x^3$ is decreasing?

3 For what values of x is the function $y = \frac{x}{4} + \frac{1}{x}$ increasing? State the coordinates of the stationary points on the curve.

4 Find the range of values of x for which the function $y = 1 - x + 2x^2 - x^3$ is increasing.

5 Find the interval in which x lies if the function $y = 2x^3 + 3x^2 - 12x + 4$ is decreasing and the coordinates of the stationary points.

6 For what values of t is the function $s = 4 - 3t + 2t^2$ decreasing?

7 Find the equations of the tangent and the normal to the following curves at the given point:

(a) $y = x^2 - 2;\ x = -3$
(b) $y = 2x^3;\ x = 1$
(c) $y = 1 - x - 3x^2;\ x = -1$
(d) $y = 2x^3 - x - 1;\ x = -1$
(e) $y = \frac{4}{x};\ x = -2$
(f) $y = \frac{3}{x+1};\ y = 1$
(g) $y = \frac{1}{1-2x};\ y = -1$
(h) $y = 2x + \frac{4}{x};\ x = -2$
(i) $y = 2x^2 - 3;\ x = 2$
(j) $y = 1 - x - 3x^2 - x^3;\ x = -1$
(k) $y = 3 - \frac{2}{x};\ y = 7$
(l) $y = \frac{2}{1-2x};\ y = 1$

8 The tangent to the curve $y = x^2 - 2x + 3$ at a certain point is parallel to the line $y = x$. Find the equation of the tangent and the coordinates of the point where it meets the x-axis.

9 Find (a) the coordinates of the point on the curve $y = 3x^2 + 2x + 1$ where the tangent is parallel to the line $4x + y = 5$ and (b) the equation of the normal at that point.

10 (a) Find the equation of the tangent to the curve $y = x^3 - 2x^2 + x$ at the origin.
(b) At what point does it meet the curve again?

11 The normal to the curve $y = 2x - \frac{1}{1-x}$ where $x = 2$, meets the curve again at the point P. Find
(a) the equation of the normal,
(b) the coordinates of P,
(c) the equation of the tangent at P.

12 The normal to the curve $y = 2x - \frac{1}{x+1}$ at the point where $x = 1$ meets the curve again at a second point. Find the x-coordinate of this point.

13 A and B are points on the curve $y = 2x - \frac{6}{x}$ whose x-coordinates are 1 and 3 respectively. Find the equations of the tangents at A and B and the coordinates of the point where they intersect.

14 Show that the function $y = x^3 + x^2 + 5x + 6$ is always increasing.

15 If the function $y = x^3 + ax^2 + 3x - 1$ is always increasing, find the range of possible values of a.

16 (a) The normal at the point A(−1,2) on the curve $y = 3 - x^2$ meets the curve again at B. Find (i) the equation of the normal at A and (ii) the coordinates of B.
(b) Find the coordinates of the point C on the curve where the curve is parallel to the normal at A.

STATIONARY POINTS: MAXIMA AND MINIMA

As we have seen, a curve $y = f(x)$ has a **stationary** point where $\frac{dy}{dx} = 0$. There are three types of stationary point: **maximum, minimum** and **point of inflexion** (Fig.10.6).

Maximum and minimum points are also called **turning** points as the tangent 'turns round' at these points.

Fig. 10.6

> The essential condition for a stationary point is $\frac{dy}{dx} = 0$.

If this equation has solutions, they are the x-coordinates of the stationary points. We then test for the type of point. A curve may have one or more or none of these points.

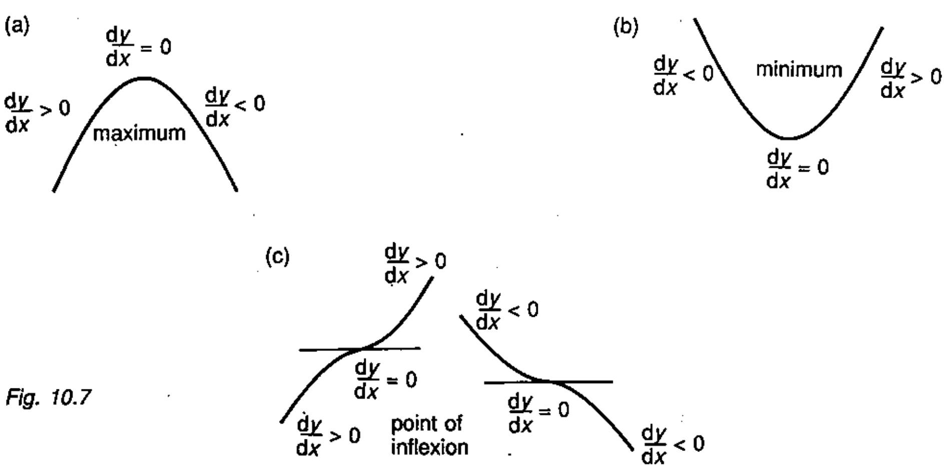


Fig. 10.7

Maximum point

$\frac{dy}{dx}$ passes from positive values through 0 to negative values (Fig. 10.7(a)).

Minimum point

$\frac{dy}{dx}$ passes from negative values through 0 to positive values (Fig. 10.7(b)).

Point of inflexion

$\frac{dy}{dx}$ has the same sign on each side of the zero value (Fig.10.7(c)).

Note that maximum and minimum apply only in the neighbourhood of the stationary point. The values of the function at this point are not necessarily the greatest and least values of the function overall.

Quadratic Function

We have already seen that the quadratic function $y = ax^2 + bx + c$ *always* has either a maximum (when $a < 0$) or a minimum point ($a > 0$).

As $\frac{dy}{dx} = 2ax + b$, this equation always has a solution.

Example 6

Which type of stationary point does $y = 1 - 2x - 2x^2$ have and what is the value of y at that point?

As $a = -2 < 0$, the curve has a maximum point.
$\frac{dy}{dx} = -2 - 4x$ and $\frac{dy}{dx} = 0$ gives $x = -\frac{1}{2}$.
So the curve has a maximum at $(-\frac{1}{2}, 1\frac{1}{2})$.

Example 7

Find the nature of the stationary points on the curve $y = 4x^3 - 3x^2 - 6x + 2$.

$\frac{dy}{dx} = 12x^2 - 6x - 6 = 6(2x^2 - x - 1) = 6(2x + 1)(x - 1)$

$\frac{dy}{dx} = 0$ when $x = -\frac{1}{2}$ or 1.

A simple test to decide on the nature of the stationary point is to examine the **sign** of $\frac{dy}{dx}$ on each side of that point.

Consider the signs of the factors $(2x + 1)$ and $(x - 1)$.

x	slightly $< -\frac{1}{2}$	$-\frac{1}{2}$	slightly $> -\frac{1}{2}$
sign of $(2x + 1)(x - 1)$	$(-)(-) = +$	0	$(+)(-) = -$
sketch of tangent			

The sketch of the curve around $x = -\frac{1}{2}$ is i.e. there is a maximum point at $x = -\frac{1}{2}$.

x	slightly < 1	1	slightly > 1
sign of $(2x + 1)(x - 1)$	$(+)(-) = -$	0	$(-)(-) = +$
sketch of tangent			

The sketch of the curve around $x = 1$ is i.e. there is a minimum point at $x = 1$. When $x = -\frac{1}{2}$, $y = 4(-\frac{1}{8}) - 3(\frac{1}{4}) - 6(-\frac{1}{2}) + 2 = 3\frac{3}{4}$ (a maximum value), and when $x = 1$, $y = -3$ (a minimum value).

Example 8

Examine the nature of the stationary point(s) on the curve $y = x^3 - x^2 + 5x - 1$.

$\frac{dy}{dx} = 3x^2 - 2x + 5$. For stationary points, $3x^2 - 2x + 5 = 0$. This equation has no solutions, so the curve has no stationary points.

Example 9

What type of stationary point(s) does the curve $y = x^3 - 3x^2 + 3x - 1$ have?

$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$

$\frac{dy}{dx} = 0$ gives $x = 1$. There is only one stationary point on the curve.

x	slightly < 1	1	slightly > 1
sign of $(x - 1)^2$	+	0	+
sketch of tangent			

The sketch of the curve is which is a point of inflexion.

The $\frac{d^2y}{dx^2}$ Test for Maxima and Minima

The sign test is adequate for simple functions but $\frac{d^2y}{dx^2}$ can be used to test for maxima and minima.

Around a maximum point, $\frac{dy}{dx}$ passes from positive to negative so it is a *decreasing* function (Fig.10.8). Hence the gradient of $\frac{dy}{dx}$, i.e. $\frac{d^2y}{dx^2}$, is negative at that point.

Around a minimum point, $\frac{dy}{dx}$ passes from negative to positive so it is an *increasing* function (Fig.10.9). Hence the gradient of $\frac{dy}{dx}$ i.e. $\frac{d^2y}{dx^2}$, is positive at that point.

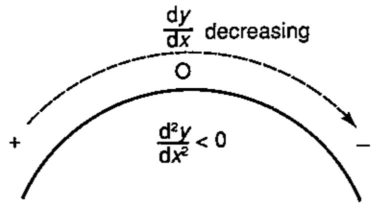


Fig. 10.8

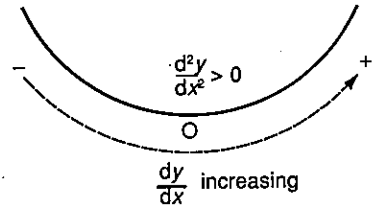


Fig. 10.9

Maximum point	Minimum point
$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$	$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

If $\frac{d^2y}{dx^2} = 0$, this test is indecisive. For such cases the sign test should be used (see Example 11).

Example 10

Find the type of stationary points on the curve $y = 4x + \frac{1}{x}$ and the coordinates of these points.

$\frac{dy}{dx} = 4 - \frac{1}{x^2}$

If $\frac{dy}{dx} = 0$, then $x^2 = \frac{1}{4}$ and $x = \pm\,\frac{1}{2}$.

$\frac{d^2y}{dx^2} = \frac{2}{x^3}$

When $x = +\frac{1}{2}$, $\frac{d^2y}{dx^2} > 0$ so this is a minimum point at $(\frac{1}{2}, 4)$.

When $x = -\frac{1}{2}$, $\frac{d^2y}{dx^2} < 0$ so this is a maximum point at $(-\frac{1}{2}, -4)$.

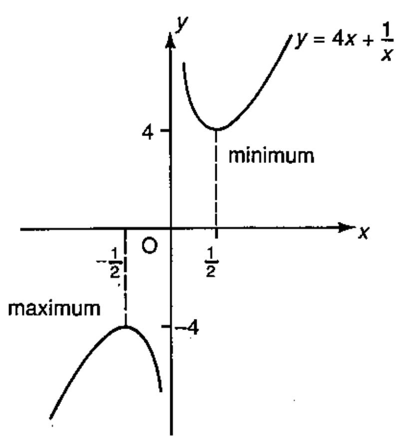


Fig. 10.10

Example 11

Find the nature of the stationary points on (a) $y = x^3$, (b) $y = x^4$.

(a) $\dfrac{dy}{dx} = 3x^2$ so $\dfrac{dy}{dx} = 0$ gives $x = 0$.

$\dfrac{d^2y}{dx^2} = 6x = 0$ when $x = 0$.

x	slightly < 0	0	slightly > 0
sign of $3x^2$	+	0	+
sketch of tangent	/	—	/

The sign test shows that this is a point of inflexion (Fig.10.11(a)).

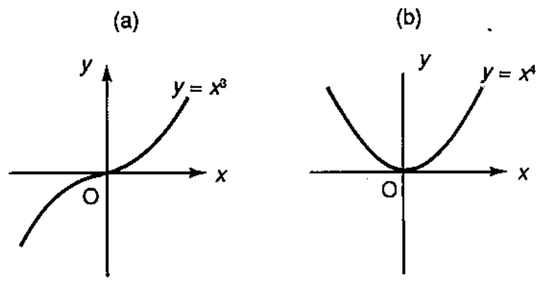


Fig. 10.11

(b) $\dfrac{dy}{dx} = 4x^3$ and again $\dfrac{dy}{dx} = 0$ gives $x = 0$.

$\dfrac{d^2y}{dx^2} = 12x^2 = 0$ when $x = 0$.

x	slightly < 0	0	slightly > 0
sign of $4x^3$	–	0	+
sketch of tangent	\	—	/

The sign test shows that this is a minimum (Fig.10.11(b)).

Exercise 10.2 *(Answers on page 628.)*

1 For each of the following functions, find (i) the x-coordinates and the nature of the stationary points (if any) and (ii) the value of the function at these points.

(a) $7 - 6x - x^2$ (b) $x^2 - 3x$

(c) $x^3 - 3x - 2$ (d) $x^5 - 2$

(e) $2x^3 + 3x^2 - 36x + 4$ (f) $2x^3 - x^2 + 1$

(g) $3x^4 - 4x^3 + 1$

(h) $x + \frac{25}{x}$

(i) $3x - \frac{1}{x}$

(j) $x^3 - 6x^2 + 12x + 2$

(k) $x^3 + 3x - 2$

(l) $x^3 + 3x^2 + 4x + 4$

(m) $x^2 + \frac{16}{x^2}$

(n) $3x^2 - 5x - 2$

(o) x^6

(p) $x^3 + x^2 + 1$

(q) $x^3 - x^2 - 5x - 1$

(r) $x^3 + 3x^2 + 3x - 4$

(s) $x - \frac{1}{9x}$

(t) $x + \frac{1}{x-1}$

(u) $x^2 + \frac{16}{x}$

(v) $2x - \frac{6}{(5-3x)}$

2 The function $y = ax^3 - 12x + 2$ has a turning point where $x = 2$. Find (a) the value of a, (b) the nature of this turning point.

3 The function $y = 2x^3 + ax^2 - 12x - 4$ has a minimum point where $x = 1$. Find (a) the value of a, (b) the position of the maximum point.

4 The function $y = x^3 + ax^2 - 7x - 1$ has a stationary value where $x = 1$. Find (a) the value of a and (b) the type and position of the stationary points.

5 Find (a) the positions and nature of the stationary points on the curve $y = x^3 - 2x^2 + 1$ and (b) the coordinates of the point where the gradient on the curve is a minimum.

6 For what value of t is $s = t^3 - 9t^2 + 15t - 10$

(a) a maximum,

(b) a minimum?

For what value of t is $\frac{ds}{dt}$ a minimum?

7 Given that $v = 1 - t + 2t^2 - t^3$, find the value of t for which $\frac{dv}{dt}$ is a maximum and explain why it is a maximum.

8 The function $y = ax^3 + bx^2 - 12x + 13$ passes through the point (1,0) and has a stationary point where $x = -1$. Find

(a) the value of a and of b,

(b) the type and position of the stationary points.

9 Find the value of x for which $y = 4x^3 - x^2 - 2x + 1$ has

(a) a maximum,

(b) a minimum value.

Hence find the values of θ for the function $T = 4\cos^3\theta - \cos^2\theta - 2\cos\theta + 1$ at its maximum and minimum values.

10 For the function $A = \pi r^3 - 6r^2 + 3$, find, in terms of π, the values of r at the stationary points, and find which type each point is.

11 If $y = 4x^3 + 3ax^2 + 48x - 3$, in what interval must a *not* lie if y has stationary points? If $a = 10$, find the x-coordinates and the nature of the stationary points.

12 Find the type and position of the stationary point(s) on the curve $y = \frac{1}{x-1} + \frac{1}{2-x}$.

MAXIMUM AND MINIMUM PROBLEMS

The methods we have learnt can be used to find the maximum and minimum values of a quantity which varies under certain conditions.

Example 12

Two numbers x and y are connected by the relation $x + y = 6$. Find the values of x and y which give a stationary point of the function $T = 2x^2 + 3y^2$ and determine whether they make T a maximum or minimum.

We must express T in terms of *one* of the variables x or y.
Choosing x, $y = 6 - x$ and $T = 2x^2 + 3(6 - x)^2$.
For a stationary point, we put $\frac{dT}{dx} = 0$.
Then $\frac{dT}{dx} = 4x - 6(6 - x) = 10x - 36 = 0$ and so $x = 3.6$ and $y = 2.4$.
To decide whether this gives a maximum or minimum we find $\frac{d^2T}{dx^2}$.
$\frac{d^2T}{dx^2} = 10$ which is positive.
Hence T will have a minimum value when $x = 3.6$, $y = 2.4$.

Example 13

A cylindrical can (with lid) of radius r cm is made from 300 cm² of thin sheet metal.

(a) Show that its height, h cm, is given by $h = \frac{150 - \pi r^2}{\pi r}$.

(b) Find r and h so that the can will contain the maximum possible volume and find this volume.

(a) The surface area A of a cylinder radius r, height h is given by
$A = 2\pi r^2 + 2\pi rh = 300$.
Hence $2\pi rh = 300 - 2\pi r^2$ and $h = \frac{150 - \pi r^2}{\pi r}$

(b) The volume $V = \pi r^2 h$ and V is to be maximized. We must express V in terms of one variable and so we substitute for h from (a).
Then $V = \pi r^2 \frac{150 - \pi r^2}{\pi r} = r(150 - \pi r^2) = 150r - \pi r^3$.
To find the maximum value of V, we set $\frac{dV}{dr} = 0$.
$\frac{dV}{dr} = 150 - 3\pi r^2 = 0$ and so $3\pi r^2 = 150$ giving $r = \sqrt{\frac{50}{\pi}} \approx 4$ cm.
Checking that this is a maximum, $\frac{d^2V}{dr^2} = -6\pi r$ which is < 0.

From (a), when $r = 4$ cm, $h = \frac{150 - 50}{\pi \times 4} \approx 8$ cm.

Hence to obtain the maximum volume, the radius = 4 cm and the height = 8 cm.
The maximum volume is then $\pi r^2 h = \pi 4^2 \times 8 = 402$ cm³.

[Note that the height = the diameter. A can of this shape will give maximum volume for a given surface area.]

Example 14

The length of a closed rectangular box is 3 times its width (Fig.10.12). If its volume is 972 cm³, find the dimensions of the box if the surface area is to be a minimum.

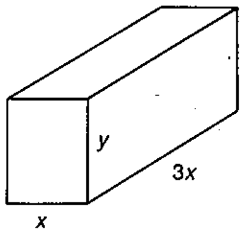


Fig. 10.12

Take the width as x cm, length $3x$ cm and let the height be y cm for the moment.

The volume $V = 3x^2y = 972$ i.e. $x^2y = 324$ (i)

The surface area $A = 6x^2 + 6xy + 2xy = 6x^2 + 8xy$ (ii)

We must now express A in terms of one variable.

From (i), $y = \frac{324}{x^2}$

and so $A = 6x^2 + 8x\frac{324}{x^2} = 6x^2 + \frac{2592}{x}$, and $\frac{dA}{dx} = 12x - \frac{2592}{x^2}$.

To minimize A, we set $\frac{dA}{dx} = 0$.

Then $12x - \frac{2592}{x^2} = 0$ giving $12x^3 = 2592$ or $x^3 = 216$. Hence $x = 6$.

To verify that this is a minimum, $\frac{d^2A}{dx^2} = 12 + \frac{2 \times 2592}{x^3}$ which will be positive.

From (i), when $x = 6$, $y = 324 \div 36 = 9$.

Hence the dimensions are 18 cm by 6 cm by 9 cm for the minimum surface area.

Example 15

Triangle ABC is isosceles with AB = AC = 20 cm and BC = 24 cm (Fig.10.13). A rectangle PQRS is drawn inside the triangle with PQ on BC, and S and R on AB and AC respectively.

(a) If PQ = 2x cm, show that the area A cm² of the rectangle is given by $A = \frac{8x(12 - x)}{3}$.

(b) Hence find the value of x for which A is a maximum.

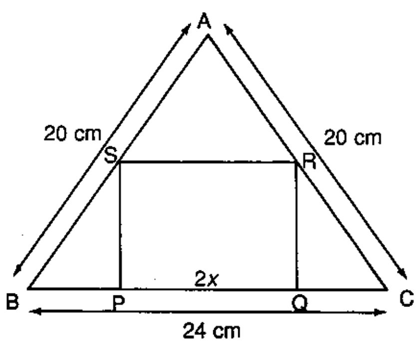


Fig. 10.13

(a) We need to know the height of the rectangle.
Let SP = y cm (Fig.10.14).
If M is the midpoint of BC, then BM = 12, BP = $12 - x$ and, by Pythagoras' Theorem, AM = 16.

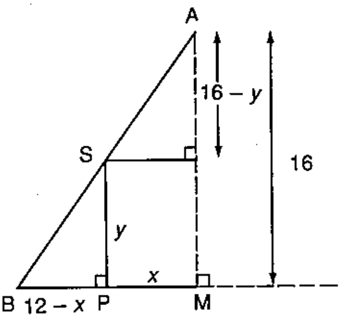


Fig. 10.14

From $\triangle$SBP, $\tan \angle SBP = \frac{y}{12 - x}$ and from $\triangle$ABM, $\tan \angle SBP = \frac{16}{12} = \frac{4}{3}$.

Then $\frac{y}{12 - x} = \frac{4}{3}$ so $y = \frac{4(12 - x)}{3}$.

Hence $A = 2xy = \frac{8x(12 - x)}{3}$.

(b) $A = \frac{96x - 8x^2}{3} = 32x - \frac{8x^2}{3}$ and $\frac{dA}{dx} = 32 - \frac{16x}{3}$.

When $\frac{dA}{dx} = 0, x = 6$.

$\frac{d^2A}{dx^2} = -\frac{16}{3}$ which is negative thus confirming that this gives a maximum.

Exercise 10.3 *(Answers on page 629.)*

1 Given that $x + y = 8$, find the minimum value of $x + y^2$.

2 Find the minimum value of $x^2 - xy + y^2$ given that $x + y = 10$.

3 x and y are numbers such that $x + y = 4$. Find the minimum value of $x^2 + xy + 2y^2$.

4 Given that $u = 3 + 4t^2 - 2t^3$, find the maximum value of u for the domain $0 \le t \le 2$, showing that it is a maximum.

5 If $s = 7 + 8t + 5t^2 - t^3$, find the value of t which gives a minimum value of s, showing that it is a minimum.

6 What is the minimum value of $x + \frac{1}{x}$ if $x > 0$?

7 If $R = \frac{V^2}{4} + \frac{500}{V}$, find the value of V for which R is a minimum.

8 A rectangular box, with a lid, is made from thin metal. Its length = $2x$ cm and its width = x cm. If the box must have a volume of 72 cm^3,
(a) show that the area A cm^2 of metal used is given by $A = 4x^2 + \frac{216}{x}$,
(b) find the value of x so that A is a minimum.

9 The cost \$$C$ of running a boat on a trip is given by $C = 4v^2 + \frac{1000}{v}$ where v is the average speed in km h^{-1}. Find the value of v for which the cost is a minimum.

10 It is estimated that the load L which can safely be placed on a beam of width x, length y and height h is given by $L = \frac{4xy^2}{h}$. If $h = 30$ and $x + y = 15$, find the greatest load that the beam can bear.

11 A piece of wire of length 20 cm is formed into the shape of a sector of a circle of radius r cm and angle θ radians.

(a) Show that $\theta = \frac{20 - 2r}{r}$ and that the area of the sector is $r(10 - r)$ cm^2.

(b) Hence find the values of r and θ to give the maximum area.

12 A cylinder is placed inside a circular cone of radius 18 cm and height 12 cm so that its base is level with the base of the cone, as shown in Fig.10.15.

(a) If the radius of the cylinder is r cm, show that its height h cm is given by $h = \frac{2}{3}(18 - r)$.

(b) Hence find the value of r to give the maximum possible volume of the cylinder and find this volume in terms of π.

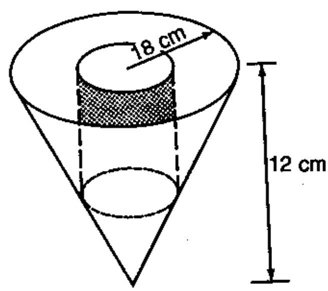


Fig. 10.15

13 A straight line passes through the point (2,3) and its gradient is m. It meets the positive x– and y– axes at A and B respectively.

(a) State the equation of the line in terms of m.

(b) Show that OA = $2 - \frac{3}{m}$ and find a similar expression for OB.

(c) Show that the area of $\triangle$OAB = $6 - \frac{9}{2m} - 2m$.

(d) Hence find the value of m for which this area is a minimum, showing that it is a minimum.

14 From a rectangular piece of thin cardboard 16 cm by 10 cm, the shaded squares each of side x cm are removed (Fig.10.16). The remainder is folded up to form a tray.

(a) Show that the volume V cm^3 of this tray is given by $V = 4(x^3 - 13x^2 + 40x)$.

(b) Hence find a possible value of x which will give the maximum value of V.

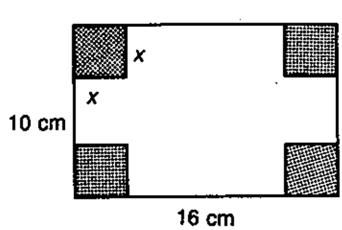


Fig. 10.16

15 The cost of making x articles per day is $\$(\frac{1}{2}x^2 + 50x + 50)$ and the selling price of each one is $\$(80 - \frac{1}{4}x)$. Find

(a) the daily profit in terms of x,

(b) the value of x to give the maximum profit.

16 Ship A is at O at noon and is sailing due East at 10 km h^{-1} (Fig.10.17). At that time, ship B is 100 km due South of O and is sailing at 20 km h^{-1} due North.

(a) State the distances in km of A and B from O after t hours.

(b) Show that the distance S km between A and B is then given by
$S^2 = 500t^2 - 4000t + 10\ 000$.

(c) Find the value of t for which S^2 is a minimum and hence find the minimum distance between the ships.

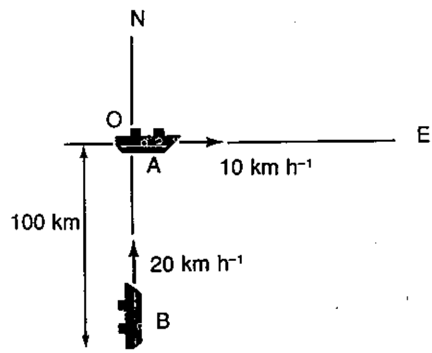


Fig. 10.17

17 The dimensions of a cylinder of radius r are such that the sum of its length and its circumference is 8π cm.

(a) Show that its length is $\pi(8 - 2r)$ cm.

(b) Hence state its volume in terms of r and find the value of r which gives the maximum volume.

18 In Fig.10.18, ABCD is a rectangle which fits inside the semicircle of radius 10 cm and centre O.

(a) If AB = $2x$ cm, show that the area A cm^2 of the rectangle is given by
$A^2 = 4x^2(100 - x^2)$.

(b) Find the value of x which makes A^2 a maximum.

(c) Hence find the maximum area of the rectangle.

$80x - \frac{1}{4}x^2$

$8x$

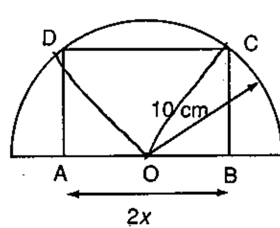


Fig. 10.18

~~8x~~

$8x\cdot(100-x^2)+4x^2(-2x) = 0$

$800x - 16x^3 = 0$

$x = 0$

$x^2 = 50$

$x = 5\sqrt{2}$

D

19 In Fig.10.19, ABCD is a rectangle where AB = 9 m and AD = 6 m. CE = 4 m and FE is parallel to AD. X is a point on FE where XF = x m and M is the midpoint of BC. Find

(a) AX^2 and XM^2 in terms of x,

(b) the value of x for which $AX^2 + XM^2$ is a minimum.

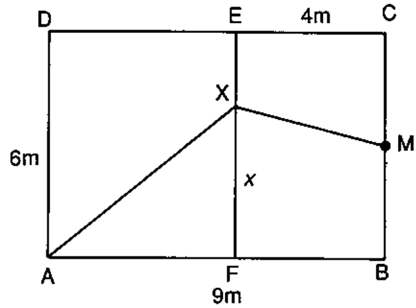


Fig 10.19

20 In $\triangle ABC$, $\angle BAC = 60°$, AB = 4 cm and AC = 2 cm. P lies on AB extended where BP = x cm, while Q lies on AC extended where CQ = y cm. Given that $x + y = 10$, show that $PQ^2 = 3x^2 - 24x + 112$ and find the value of x which will make PQ^2 a minimum. State the ratio of BC:PQ in that case.

21 The position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ of two points A and B are given by $\mathbf{r}_A = 2t\mathbf{i} + (1 + t)\mathbf{j}$ and $\mathbf{r}_B = (t + 1)\mathbf{i} - (t + 2)\mathbf{j}$.

(a) Find the values of t for which OA is perpendicular to OB where O is the origin.

(b) Find the vector $\overrightarrow{AB}$ in terms of t.

(c) Find the value of t for which $|\overrightarrow{AB}|^2$ is a minimum.

(d) Hence find the shortest distance between A and B.

22 A can is in the shape of a closed cylinder with a hemisphere at one end (Fig.10.20). Its volume is 45π cm^3. Taking r cm as the radius of the cylinder and h cm as its height, show that

(a) $r^2h + \frac{2r^3}{3} = 45$,

(b) the external surface area A of the can is given by $A = \frac{5\pi r^2}{3} + \frac{90\pi}{r}$

(c) Hence find the value of r for which A is a minimum and find the minimum value of A.

(Volume of a sphere = $\frac{4\pi r^3}{3}$, surface area of a sphere = $4\pi r^2$).

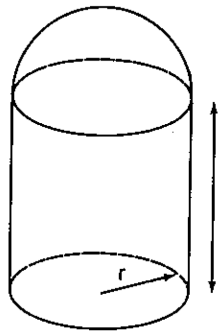


Fig. 10.20

23 ABC is an isosceles triangle with AB = AC = 10 cm and $\angle$BAC = 60°. A particle P starts from B and moves along BA at a speed of 2 cm s^{-1}. Another particle Q starts from A at the same time and moves along AC at a speed of 4 cm s^{-1}.

(a) Write down the distances of P and Q from A at time t seconds after the start. Find

(b) an expression for PQ^2 in terms of t and

(c) the value of t for which PQ^2 is a minimum.

(d) Hence find the minimum length of PQ.

24 Fig.10.21 shows a framework in the shape of a rectangular box made from straight pieces of wire. The total length of these pieces is 60 cm.

(a) Show that $y = (15 - 5x)$ cm.

(b) Find an expression for the volume enclosed by the framework in terms of x and hence find (c) the value of x which makes this volume a maximum and (d) the maximum volume.

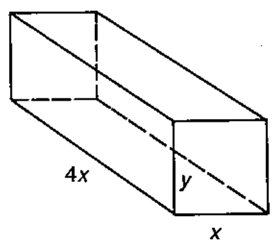


Fig. 10.21

25 A piece of wire 48 cm long is divided into two parts. One part is formed into the shape of a circle of radius r cm while the other part is formed into a square of side x cm.

(a) Show that $r = \frac{24 - 2x}{\pi}$.

(b) Find an expression in terms of x for the total area A of the two shapes and hence calculate (correct to 3 significant figures) the value of x for which A is a minimum.

26 In $\triangle$ABC, $\angle$A = 60° and AB = x cm, AC = y cm where $x + 2y = k$ (a constant). Find an expression for BC^2 in terms of x and k and hence find the ratio $x{:}y$ for which BC^2 is a minimum.

27 ABCD is a square of side 10 cm. P lies on BC where BP = x cm and Q lies on CD where CQ = $\frac{3x}{2}$ cm. (a) Find an expression in terms of x for the area of $\triangle$APQ and hence (b) find the value of x which makes this area a minimum.

28 A rectangular box has a square cross–section and the sum of its length and the perimeter of this cross–section is 2 m. If the length of the box is x m, show that its volume V m³ is given by $V = \frac{x(2 - x)^2}{16}$.

Hence find the maximum volume of the box.

29 Fig. 10.22 shows part of the parabola $y = 8x - x^2$ with a rectangle ABCD which fits between the curve and the x-axis. Taking AB = $2x$ show that (a) OB = $x + 4$ and (b) the area of ABCD = $32x - 2x^3$ units². Hence find the value of x which makes this area a maximum and state the maximum area.

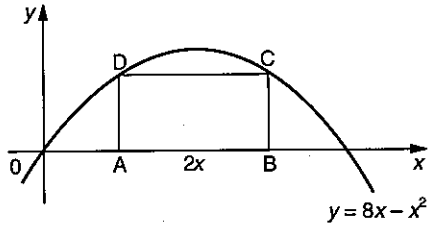


Fig. 10.22

VELOCITY AND ACCELERATION

A common rate of change is the **speed** of a moving body. This is the rate of change of distance travelled with respect to time. The average speed is

$$\frac{\text{distance travelled}}{\text{time taken}}$$

Speed is usually measured in m s^{-1} but also in cm s^{-1} or km h^{-1}.

If the *direction* is to be taken into account, then we speak of the **velocity** of the body. The magnitude of the vector velocity is the speed.

Now if the time $\longrightarrow 0$, we shall have the limiting value of the average speed, i.e. the speed at a particular instant or the **instantaneous** speed. So if s is the distance travelled in time t and s is a function of t, then $\frac{ds}{dt}$ will give the speed v at a given instant.

$$v = \frac{ds}{dt} \text{ where } s \text{ is a function of } t$$

If v itself is changing, then we have the rate of change of speed v with respect to t, called the **acceleration** (***a***).

Now $$a = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2s}{dt^2} \text{ where } v \text{ is a function of } t$$

Acceleration is the rate of *increase* of the velocity with respect to time and hence its standard unit is metres per second *per second*, written m s^{-2}.

A positive acceleration means that the speed is increasing, while a negative acceleration (or a **deceleration** or **retardation**) means that the speed is decreasing.

If the distance s is measured from a fixed point O, its value at any time t is also called the **displacement** of the particle from O. This is its actual distance *from* O at time t which is **not** necessarily the same as the distance *travelled* up to time t. This is illustrated in Example 16.

Example 16

A particle starts from a point O and moves in a straight line so that its distance s cm from O after time t seconds is given by $s = 2t^2 - \frac{t^3}{6}$. *Find*

(a) its initial velocity and acceleration,
(b) the time after the start when it comes to a momentary halt,
(c) its distance from O at this time.
(d) What maximum velocity does it reach before that time?
(e) After what time does the particle pass through O again?

(A 'particle' means a body small enough for its dimensions to be ignored.)

If $s = 2t^2 - \frac{t^3}{6}$, then the velocity $v = \frac{ds}{dt} = 4t - \frac{t^2}{2}$ (i)

and the acceleration $a = \frac{dv}{dt} = 4 - t$ (ii)

(a) When $t = 0$ (the start), $v = 0$ and $a = 4$. The particle starts from rest (motionless) with an acceleration of 4 cm s^{-2}.
From (ii), note that the acceleration decreases to 0 in the first 4 seconds and then becomes negative.

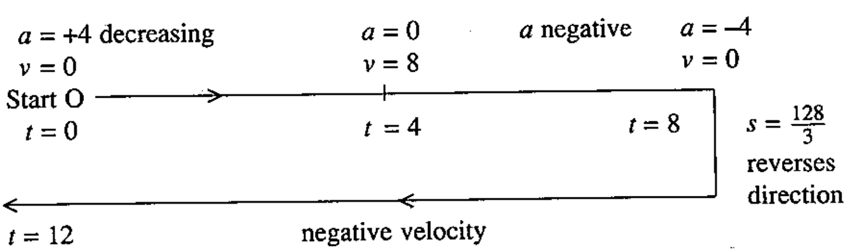
(b) From (i), $v = 0$ when $4t - \frac{t^2}{2} = 0$ i.e. $t(4 - \frac{t}{2}) = 0$ which gives $t = 0$ (the start) or $t = 8$.
At $t = 8$, $a = 4 - 8 = -4$ so the particle stops and instantly reverses direction, moving back towards O. Such a position, where $v = 0$ but $a \neq 0$, is called 'instantaneous rest'.

(c) When $t = 8$, $s = 2(8)^2 - \frac{8^3}{6} = \frac{128}{3}$ cm.

(d) The maximum velocity occurs when $\frac{dv}{dt} = 0$.
From (ii), this occurs when $t = 4$ and v is then $4(4) - \frac{16}{2} = 8$ cm s^{-1}.

(e) $s = 0$ when $2t^2 - \frac{t^3}{6} = 0$ i.e. $t^2(2 - \frac{t}{6}) = 0$ which gives $t = 0$ or or $t = 12$.
Hence the particle passes through O again after 12 seconds, now moving in the reverse direction.

The following diagram shows the features of the motion.



At time $t = 8$, the displacement = distance travelled = $\frac{128}{3}$.
At time $t = 12$, the displacement = 0 but the distance travelled was $\frac{256}{3}$. The particle reversed during that time.

Example 17

The distance s m of a particle moving in a straight line measured from a fixed point O on the line is given by $s = t^2 - 3t + 2$ where t is the time in seconds from the start. Find

(a) its initial distance from O,
(b) its initial velocity and in which direction,
(c) its initial acceleration,
(d) the times when it passes through O and with what velocity,
(e) when and where it is at instantaneous rest.

(a) At the start, $t = 0$. Then $s = 2$ m. The particle starts 2 m from O.

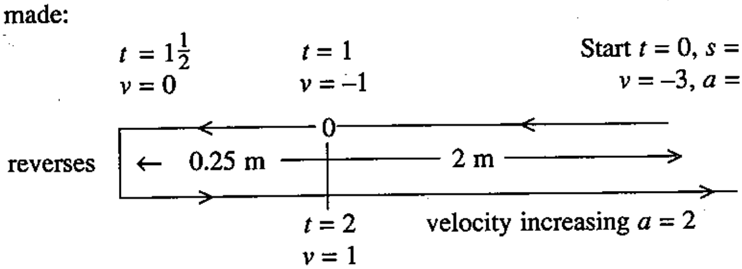
(b) $v = \dfrac{ds}{dt} = 2t - 3$
When $t = 0$, $v = -3$, i.e. in the direction towards O.

(c) $a = \dfrac{dv}{dt} = 2$
The acceleration is constant i.e. 2 m s^{-2}.

(d) $s = 0$ when $t^2 - 3t + 2 = (t - 2)(t - 1) = 0$ i.e. when $t = 2$ or 1.
When $t = 1$, $v = -1$ and when $t = 2$, $v = 1$.

(e) The particle is at instantaneous rest when $v = 0$, i.e. when $t = 1\frac{1}{2}$ seconds.
Then $s = (1\frac{1}{2})^2 - 3(1\frac{1}{2}) + 2 = -0.25$ m.

Putting these facts together, the following diagrammatic representation of the motion can be made:



Exercise 10.4 *(Answers on page 629.)*

1 A particle, moving in a straight line, starts from rest and its displacement s m from a fixed point of the line is given by $s = t^2 - kt$ where k is a constant and t is the time (in seconds) after the start. If it comes to instantaneous rest after 2 seconds, find
(a) the value of k,
(b) the initial velocity of the particle.

2 The distance s m of a particle moving in a straight line measured from a fixed point O on the line is given by $s = t^2 - 2t$ where t is the time in seconds after the start.
(a) What is the initial velocity of the particle?
(b) When is the particle at instantaneous rest?
(c) When does it pass through O for the second time?
(d) What is the acceleration of the particle?

3 For a particle moving in a straight line, its displacement s m from a point O on the line is given by $s = t^2 - 5t + 6$, where t is the time in seconds from the start. Find
(a) the initial distance of the particle from O,
(b) its initial velocity,
(c) when it is at instantaneous rest,
(d) at what time(s) after the start it passes through O.
(e) the distance travelled in the first 3 seconds.

4 A small body moves along the x-axis so that its distance x from the origin at time t s is given by $x = 2t^3 - 15t^2 + 24t + 20$. Find
(a) the velocity with which it starts,
(b) when it is at instantaneous rest,
(c) the minimum distance of the body from the origin.
(d) Between what times is the particle moving towards the origin?
(e) What is its acceleration at the times in (d)?

5 A particle moves in a straight line. Its displacement s m from a fixed point on the line is given by $s = t^2 - 4t - 5$, at a time t after the start, where $t \geq 0$. Find
(a) where the particle starts and its initial velocity,
(b) when and where it comes to instantaneous rest,
(c) when it passes through the fixed point,
(d) its acceleration.

6 A particle moves along the x-axis and its x-coordinate at time t s after the start is given by $x = 2t^3 - 9t^2 + 12t - 1$ for $t \geq 0$.
(a) Find its x-coordinate and velocity at the start.
(b) At what times does the particle come to instantaneous rest?
(c) What is its maximum velocity in the direction of the negative x-axis?
(e) When is its acceleration zero?

7 The velocity v cm s^{-1} of a particle moving in a straight line is given by $v = 6t - kt^2$, where k is a constant and t s is the time from the start. If its acceleration is 0 when $t = 1$, find
(a) the value of k,
(b) the time when the particle comes to instantaneous rest,
(c) the maximum velocity of the particle.

SMALL INCREMENTS: APPROXIMATE CHANGES

Given a function $y = f(x)$, suppose x is changed by an increment δx to become $x + \delta x$. Then y changes by an increment δy. We can find an approximate value for δy in a simple way using $\frac{dy}{dx}$, provided **δx is small.**

In Fig. 10.23, A is the point on $y = \mathrm{f}(x)$ where $x = k$. AB = δx and BC = δy. AT is the tangent at A and the gradient of this tangent = $\left(\frac{dy}{dx}\right)_{x=k}$

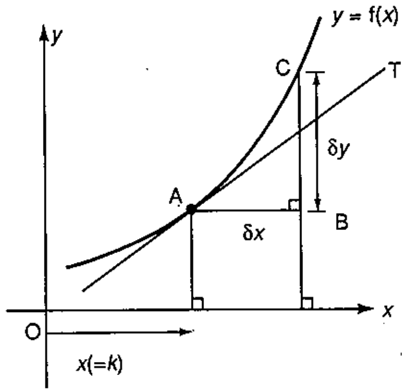


Fig. 10.23

Now if δx is small, we can take $\frac{BC}{AB} = \frac{\delta y}{\delta x}$ to be approximately equal to the gradient of the tangent at A.

Then $\frac{\delta y}{\delta x} \approx \left(\frac{dy}{dx}\right)_{x=k}$ and so

$$\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$$

Example 18

If $y = \frac{27}{x^2}$, find the approximate change in y if x is increased from 3 to 3.01.

Here $k = 3$.

$\frac{dy}{dx} = -\frac{54}{x^3}$ and so $\left(\frac{dy}{dx}\right)_{x=3} = -2.$

Then $\delta y \approx \left(\frac{dy}{dx}\right)_{x=3} \times \delta x = -2 \times 0.01 = -0.02.$

Note that the negative value indicates a *decrease* in the value of y.

Example 19

Given that $T = x^3 - 2x^2 + 1$ and x is decreased from 2 to 1.985, find the new value of T approximately.

$\frac{dT}{dx} = 3x^2 - 4x$ so $\left(\frac{dT}{dx}\right)_{x=2} = 4.$

Then $\delta T \approx \left(\frac{dT}{dx}\right)_{x=2} \times \delta x$

$= 4 \times (-0.015)$ (as x was decreased)

$= -0.06$

Hence the new value of $T = 2^3 - 2(2^2) + 1 - 0.06 = 0.94.$

Example 20

The volume V of a sphere is given by $V = \frac{4\pi r^3}{3}$ *where r is the radius.*

(a) State an expression for the approximate change in V if r is changed by a small amount δr.

(b) Hence find the approximate percentage change in V if r is increased by 1%.

(a) $\frac{dV}{dr} = 4\pi r^2$

$\delta V \approx \left(\frac{dV}{dr}\right) \times \delta r = 4\pi r^2 \times \delta r.$

(b) If r is increased by 1% then $\delta r = 0.01r$.

The percentage increase in $V \approx \frac{\delta V}{V} \times 100\% = \frac{4\pi r^2 \delta r}{\frac{4}{3}\pi r^3} \times 100\%$

$= \frac{3}{r} \times 0.01r \times 100\% = 3\%$

Example 21

If $y = 3 - x + 2x^2$ *and x is increased from 4 by an amount* $\frac{r}{100}$ *where r is small, find in terms of r*

(a) the approximate change in y,

(b) the percentage change in y.

(a) $\frac{dy}{dx} = -1 + 4x$ and $\left(\frac{dy}{dx}\right)_{x=4} = 15.$

Then $\delta y \approx 15 \times \frac{r}{100} = \frac{3r}{20}.$

(b) The original value of y was $3 - 4 + 32 = 31$ and the percentage change in

$y = \frac{\delta y}{y} \times 100\% \approx \frac{\frac{3r}{20}}{31} \times 100\% = \frac{15r}{31}\%.$

Example 22

If $y = 2x^2 - 3x + 1$, *find the positive value of x for which y = 3. Hence find the approximate increase in x which will change y from 3 to 3.015.*

When $y = 3$, $2x^2 - 3x + 1 = 3$ so $2x^2 - 3x - 2 = 0$ or $(2x + 1)(x - 2) = 0$ giving $x = 2$ (positive value).

$\frac{dy}{dx} = 4x - 3$ and $\left(\frac{dy}{dx}\right)_{x=2} = 5$; $\delta y = 0.015.$

Then substituting in $\delta y \approx \left(\frac{dy}{dx}\right)_{x=2} \times \delta x,$

$$0.015 = 5\delta x \text{ giving } \delta x \approx 0.003.$$

Exercise 10.5 *(Answers on page 629.)*

1 If $y = x^2 - x + 1$, find the approximate change in y when x is increased from 4 to 4.025.

2 Given the function $y = x^3 + x^2 - 4$, x is increased from 4 to 4.05. What is the approximate change in y?

3 If $z = 2x^2 - 7$, find the approximate change in z when x is decreased from 4 to 3.99.

4 Given that $y = (x + 2)^5$, find the approximate change in y when x is increased from 2 to 2.005.

5 Given that $y = (x^2 - x - 1)^4$, find the change in y following an increase in x from 2 to 2.01.

6 For the function $T = \frac{5}{s + 1}$, find the new value of T approximately due to an increase in s from 9 to 9.1.

7 $P = (1 - \frac{1}{x})^3$. When $x = 2$, it is decreased by 3%. Find the approximate percentage change in P.

8 Find the approximate change in T for the function $T = 4 + 3u - 2u^2$ when u is increased by 5% from the value of 2.

9 The radius of a circle is increased by 5%. Calculate the approximate percentage increase in
(a) the circumference,
(b) the area of the circle.

10 A piece of wire of length 20 cm is shaped into the form of a sector of a circle of radius r cm and angle θ radians.
(a) Show that the area A cm^2 of the sector is given by $A = r(10 - r)$.
(b) If r is increased by 2% when $r = 2.5$ cm, find the approximate percentage change in A.

11 The height of a cone is 20 cm but the radius of its circular base is increased from 10 cm to 10.01 cm. Find the approximate change in the volume of the cone in terms of π.

12 If $y = x^3 - 3x^2$, find, in terms of k, (a) the approximate increase in y if x is increased from 4 to $4 + k$, where k is small and (b) the approximate percentage change in y.

13 Each side of a cube is increased by p% where p is small. What is the approximate percentage increase in the volume of the cube in terms of p?

14 $y = x^2 - \frac{1}{1 - x}$. If x is increased from 3 to 3.001, find the approximate change in y.

15 If x is decreased from 5 to 4.98 in the function $y = \frac{2}{x - 1}$, what is the approximate percentage change in y?

16 Find the positive value of x when $y = 4$ for the function $y = x^2 - 5x - 2$.
Hence find the approximate change in x when y changes from 4 to 4.02.

17 The y-coordinate of a point in the first quadrant on the curve $y = 3x^2 - 8x - 1$ is 2. Find its x-coordinate. What is the approximate change in x if the point is moved to a position on the curve where $y = 2.04$?

18 For the function $y = 3x^2 + ax + b$, where a and b are constants, when x changes from 2 to 2.02, y changes from 2 to 2.12 approximately. Find the values of a and b.

19 In an experiment to find the values of T from the formula $T = \frac{2}{x^2+4}$, values of x are read from a measuring device. A value of x is read as 2.04 but should be 2. What is the approximate error in the value of T?

20 U is calculated from the formula $U = \frac{2}{x-1}$. Measurements of x are taken but they are liable to an error of ±1.5%. When x is measured as 3, what are the greatest and least values of U?

21 Given that $v = \frac{1}{u} + \frac{1}{1-u}$, find the approximate change in v when u is increased from 2 to 2.04.

22 For the function $A = \frac{1}{(r-1)^2}$, a small change in r when $r = 2$ $(r-1)^2$ produces a 2% reduction in the value of A. Find the change in r approximately.

Connected Rates of Change

Example 23

Some oil is spilt onto a level surface and spreads out in the shape of a circle. The radius r cm of the circle is increasing at the rate of 0.5 cm s^{-1}. At what rate is the area of the circle increasing when the radius is 5 cm?

The rate of change of the radius wrt time (t) = $\frac{dr}{dt} = 0.5$.

We wish to find the rate of change of the area A i.e. $\frac{dA}{dt}$.

We can find a link between these two rates by using the rule for the differential coefficient of combined functions i.e. $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$.

We know that $A = \pi r^2$ so $\frac{dA}{dr} = 2\pi r = 2\pi \times 5$.

Then $\frac{dA}{dt} = 2\pi \times 5 \times 0.5 = 15.7$ cm^2 s^{-1}.

This method can always be used to compare the rates of change of two connected quantities x and y with respect to a third quantity. The relation between x and y gives $\frac{dy}{dx}$.

Example 24

Water is emptied from a cylindrical tank of radius 20 cm at the rate of 2.5 litres s^{-1} and fresh water is added at the rate of 2 litres s^{-1} (Fig. 10.24). At what rate is the water level in the tank changing?

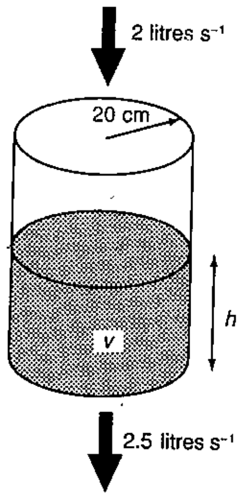


Fig. 10.24

Let the depth of water be h cm at time t s.
The rate of the change of the volume V cm^3 of the water
$= \frac{dV}{dt} = +2000 - 2500 = -500$ cm^3 s^{-1}
(1 litre = 1000 cm^3; so 2000 cm^3 of water flowing in and 2500 cm^3 of water flowing out, per second).

The rate of change of the water level $= \frac{dh}{dt}$ which we have to find.

Then $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$.

Since $V = \pi r^2 h$ and $\frac{dV}{dh} = 400\pi$,

therefore $-500 = 400\pi \times \frac{dh}{dt}$ giving $\frac{dh}{dt} = \frac{-500}{400\pi} = -0.40$ cm s^{-1}

The water level is *falling* at the rate of 0.4 cm s^{-1}.

Example 25

A hollow circular cone is held upside down with its axis vertical (Fig. 10.25). Liquid is added at the constant rate of 20 cm^3 s^{-1} but leaks away through a small hole in the vertex at the constant rate of 15 cm^3 s^{-1}. At what rate is the depth of the liquid in the cone changing when it is 12 cm?

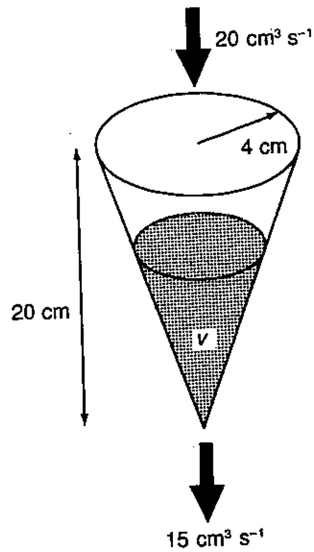


Fig. 10.25

If V cm³ is the volume of the liquid in the cone at time t s, then

$\frac{dV}{dt} = +20 - 15 = 5$ cm³ s⁻¹.

Let h cm be the height of the liquid at time t s. We have to find $\frac{dh}{dt}$.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ so we find a relation between V and h.

Taking r as the radius of the water surface, from Fig. 10.26 we have:

$\tan\theta = \frac{4}{20} = \frac{r}{h}$ so $r = \frac{h}{5}$ and $V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{75}$.

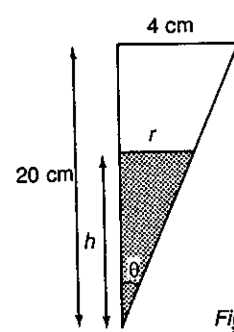


Fig. 10.26

Then $\frac{dV}{dh} = \frac{\pi h^2}{25}$.

So $\frac{dV}{dt} = 5 = \frac{\pi \times 12^2}{25} \times \frac{dh}{dt}$

and $\frac{dh}{dt} = \frac{25 \times 5}{\pi \times 144} \approx 0.28$ cm s⁻¹.

The depth of liquid is rising at the rate of 0.28 cm s⁻¹.

Note that in this example the rate of change of the depth depends on how much liquid is already in the cone as the cross-section is not constant.

Example 26

The pressure P units and the volume V m³ of a quantity of gas stored at a constant temperature in a cylinder are related by Boyle's Law $PV = k$ (a constant). At a certain time, the volume of gas in the cylinder is 30 m³ and its pressure is 20 units. If the gas is being compressed at the rate of 6 m³ s⁻¹, at what rate is the pressure changing?

$PV = k$ so $k = 20 \times 30 = 600$ units m^3.

The relation between P and V is $PV = 600$ or $P = \frac{600}{V}$.

Now $\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$ and $\frac{dP}{dV} = -\frac{600}{V^2}$

We are given that $\frac{dV}{dt} = -6$ (decreasing).

So $\frac{dP}{dt} = -\frac{600}{30^2} \times (-6) = 4$ units per second (increasing).

Example 27

A street lamp is 8 m high. A man of height 1.6 m walks along the street away from the lamp at a steady rate of 1 m s^{-1}. At what rate is the length of his shadow changing?

In Fig. 10.27, L is the lamp and OL = 8 m.
MN = 1.6 m is the man and MS = s m his shadow.

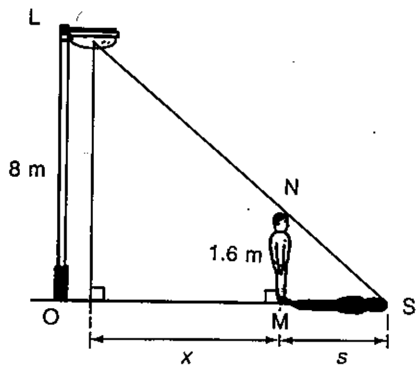


Fig. 10.27

Let OM = x m.

Now $\frac{dx}{dt}$ = rate at which the man is walking = 1 m s^{-1}. We require $\frac{ds}{dt}$.

$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$ so we find the relation between s and x.

$\tan \angle NSM = \frac{1.6}{s} = \frac{8}{x+s}$ which gives $s = \frac{x}{4}$.

Then $\frac{ds}{dx} = \frac{1}{4}$.

Hence $\frac{ds}{dx} = \frac{1}{4} \times 1 = 0.25$ m s^{-1} which is the rate at which the length of his shadow is increasing.

Exercise 10.6 *(Answers on page 630.)*

1 At what rate is the area of a circle decreasing when its radius is 8 cm and decreasing at 0.4 cm s^{-1}?

2 The area of a circle is decreasing at the rate of 2 cm^2 s^{-1}. How fast is the radius decreasing when the area is 9π cm^2?

3 The radius r cm of a sphere is 10 cm and it is increasing at the rate of 0.25 cm s^{-1}. At what rate is (a) the volume, (b) the surface area, increasing?
(For a sphere, volume $= \frac{4\pi r^3}{3}$ and surface area $= 4\pi r^2$).

4 A spherical balloon is being inflated by blowing in 2×10^3 cm^3 of air per second. At what rate is its radius increasing when its diameter is 20 cm?

5 ABC is a triangle with $\angle B = 90°$ and AB has a fixed length of 8 cm.
The length of BC is increasing at 0.5 cm s^{-1}. At what rate is the area of the triangle increasing?

6 A closed cylinder is of fixed length 10 cm but its radius is increasing at the rate of 1.5 cm s^{-1}. Find the rate of increase of its total surface area when the radius is 4 cm. (Leave the answer in terms of π).

7 A circular cylinder has a diameter of 40 cm and is being filled with water at the rate of 1.5 litres s^{-1}. At what rate is the water level rising?

8 The length of each side of a cubical framework of straight wires is expanding at the rate of 0.02 m s^{-1}. At what rate in cm^3 s^{-1} is the volume of the framework changing when each side is 0.2 m long?

9 x and y are connected by the equation $y = \frac{x^2 - 3}{x}$. If x is changing at a rate of 0.3 units per second, find the rate of change of y when $x = 3$.

10 $y = (2r^2 - r + 1)^3$ and $x = 4r$. At what rate is y changing with respect to x when $r = 0.5$?

11 The height of a cone remains constant at 20 cm. The radius of the base is 5 cm and is increasing at 0.2 cm s^{-1}. At what rate is the volume of the cone changing?

12 The volume V cm^3 of liquid in a container is given by $V = 2x^3 - 4x^2 + 5$ where x cm is the depth of the liquid. At what rate is the volume increasing when $x = 4$ and is increasing at the rate of 1.5 cm s^{-1}?

13 Liquid escapes from a circular cylinder of radius 5 cm at a rate of 50 cm^3 s^{-1}. How fast is the level of the liquid in the cylinder falling?

14 A hollow cone of radius 15 cm and height 25 cm, is held vertex down with its axis vertical. Liquid is poured into the cone at the rate of 500 cm^3 s^{-1}. How fast is the level of the liquid rising when the radius of its surface is 10 cm?

15 In an electrical circuit the resistance $R = \frac{10}{I}$ where I is the current flowing in the circuit. If I is increasing at 0.05 units per second, what is the rate of change of R when $R = 5$ units?

16 Two quantities p and q are related by the equation $(p - 1)(q + 2) = k$ where k is a constant. When $p = 5$ units, q is 7 units and q is changing at the rate of 0.04 units per second. Find the rate at which p is changing.

17 Water is being poured into a cylinder of radius 10 cm at a rate of 360 cm^3 s^{-1} but leaks out at a rate of 40 cm^3 s^{-1}. At what speed is the water level changing?

18 In Fig. 10.28, the sides of the rectangle ABCD are 18 cm and 10 cm. The rectangle KLMN lies inside ABCD and the shaded area has a width of x cm at each side.

(a) Express the shaded area in terms of x.

(b) If the shaded area is $\frac{8}{15}$ of the area of ABCD, find the value of x.

(c) The area of KLMN varies as x decreases at a constant rate of 0.25 cm s^{-1}. Find the rate at which the shaded area is decreasing when it is $\frac{8}{15}$ of the area of ABCD.

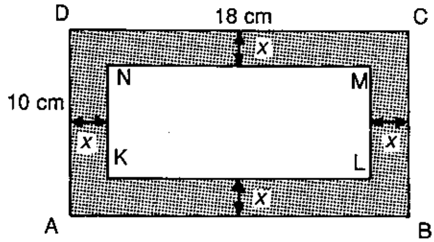


Fig. 10.28

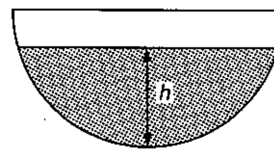


Fig. 10.29

19 A hemispherical bowl contains liquid as shown in Fig. 10.29. The volume V cm^3 of liquid is given by $V = \frac{1}{3}\pi h^2(24 - h)$ where h is the greatest depth of the liquid in cm. If liquid is poured into the bowl at the rate of 100 cm^3 s^{-1}, at what rate is the greatest depth of the liquid increasing when it is 2 cm? (Leave the answer in terms of π).

20 Sand falls on to level ground at a rate of 1000 cm^3 s^{-1} and piles up in the form of a circular cone whose vertical angle is 60°.

(a) Given that $\tan 30° = \frac{1}{\sqrt{3}}$, show that the radius r of the base is given by $r = \frac{h}{\sqrt{3}}$ where h is the height.

(b) Show that the volume V of the pile is $\frac{\pi h^3}{9}$.

(c) Hence find the rate at which the height of the pile is increasing when $h = 20$ cm.

21 In Fig. 10.30, ABC is an isosceles triangle where AC = BC = 13 cm and AB = 10 cm. PQ moves towards AB at a steady rate of 0.5 cm s^{-1} keeping parallel to AB. If PQ is x cm from C, show that

(a) PQ = $\frac{5x}{6}$ cm,

(b) the shaded area = $\frac{5}{12}(144 - x^2)$ cm^2.

(c) Hence find the rate at which the shaded area is decreasing when PQ is half way towards AB from C.

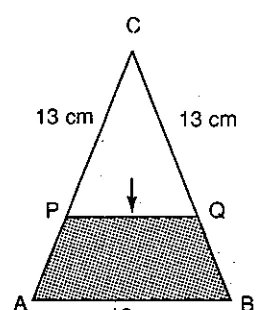


Fig. 10.30

22 (a) If $\frac{dL}{dt} = k$ where k is a number, show that $\frac{dL^2}{dt} = 2kL$.
(b) ABC is a triangle in which $\angle CAB = 60°$ and AB is of fixed length 5 cm. If AC = 8 cm, show that BC = 7 cm.
(c) Taking AC = x cm and L = length of BC, find an expression for L^2 in terms of x.
(d) Find $\frac{dL^2}{dt}$ when $x = 8$ and is increasing at 1 cm s^{-1}.
(e) Hence, using (a) find the rate at which the length of BC is changing.

SUMMARY

- As x increases,

  $y = f(x)$ is increasing for $\frac{dy}{dx} > 0$

  $y = f(x)$ is decreasing for $\frac{dy}{dx} < 0$
- Gradient of tangent to $y = f(x)$ is $\frac{dy}{dx}$.

  Gradient of normal is $-\frac{1}{\frac{dy}{dx}}$.
- For a stationary point (maximum, minimum or point of inflexion), $\frac{dy}{dx} = 0$.

  The stationary point is maximum if $\frac{d^2y}{dx^2} < 0$, minimum if $\frac{d^2y}{dx^2} > 0$.

  If $\frac{d^2y}{dx^2} = 0$, use the sign test.
- If distance s is a function of time t, then velocity $v = \frac{ds}{dt}$,

  acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
- If $y = f(x)$ and x is changed from a value k by a small increment δx,

  $\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \times \delta x$
- If $y = f(x)$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

REVISION EXERCISE 10 *(Answers on page 630.)*

A

1 Find the range of values of x for which the function $y = x^3 - 6x^2 - 15x + 3$ is increasing.

2 For what value of x does the function $y = 4x^3 - 6x^2 - 9x + 5$ have a minimum stationary point?

3 The area of a circle increases from 25π to 25.5π. Calculate the approximate increase in the radius.

4 Variables x and y are related by the equation $y = \frac{2x-6}{x}$.
(i) Obtain an expression for $\frac{dy}{dx}$ and hence find an expression for the approximate increase in y as x increases from 4 to $4 + p$, where p is small.

(ii) Given that x and y are functions of t and that $\frac{dy}{dt} = 0.4$, find the corresponding rate of change of x when $y = 1$. (C)

5 The area, A cm^2, of the image of a rocket on a radar screen is given by the formula $A = \frac{12}{r^2}$, where r km is the distance of the rocket from the screen. The rocket is approaching at 0.5 km s^{-1}. When the rocket is 10 km away, at what rate is the area of the image changing? When A is changing at 0.096 cm^2 s^{-1}, how far away is the rocket? (C)

6 A piece of wire, 60 cm long, is bent to form the shape shown in Fig. 10.31. This shape consists of a semicircular arc, radius r cm, and three sides of a rectangle of height x cm.

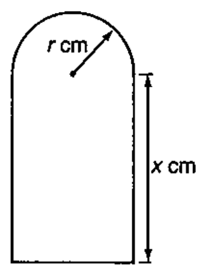


Fig. 10.31

Express x in terms of r and hence show that the area enclosed, A cm^2, is given by $A = 60r - 2r^2 - \frac{\pi r^2}{2}$.

Hence determine, to 3 significant figures, the value of r for which A is either a maximum or a minimum. Determine whether this value of r makes A a maximum or a minimum. (C)

7 If $y = 10 - x + 5x^2$, find the approximate percentage change in y when x is increased by $p\%$ (p small) when $x = 4$.

8 Under a heating process, the length, x cm, of each side of a metal cube increases from an initial value of 9.9 cm at a constant rate of 0.005 cm s^{-1}. Express the volume, V cm^3, and the surface area, A cm^2, of the cube in terms of x.

Write down expressions for $\frac{dV}{dx}$ and $\frac{dA}{dx}$.

Hence find (i) the rate at which V is increasing when the cube has been heated for 20 s, (ii) the approximate increase in A as x increases from 10 to 10.001 cm. (C)

9 $R = \frac{V^3}{25} + \frac{10\,800}{V}$. Find the value of V for which R is least.

10 A piece of wire, 100 cm in length, is divided into two parts. One part is bent to form an equilateral triangle of side x cm and the other is bent to form a square of side y cm. Express y in terms of x and hence show that A cm^2, the total area enclosed by the two shapes, is such that

$A = \frac{\sqrt{3}x^2}{4} + \frac{(100 - 3x)^2}{16}$.

Calculate the value of x for which A has a stationary value.
Determine whether this value of x makes A a maximum or a minimum. (C)

11 Show that the equation of the normal to the curve $y = 2x + \frac{6}{x}$ at the point (2,7) is $y + 2x = 11$. Given that this normal meets the curve again at P, find the x-coordinate of P. (C)

12 The diagram shows a solid body which consists of a right circular cylinder fixed, with no overlap, to a rectangular block. The block has a square base of side $2x$ cm and a height of x cm. The cylinder has a radius of x cm and a height of y cm. Given that the total volume of the solid is 27 cm^3, express y in terms of x.
Hence show that the total surface area, A cm^2, of the solid is given by

$$A = \frac{54}{x} + 8x^2.$$

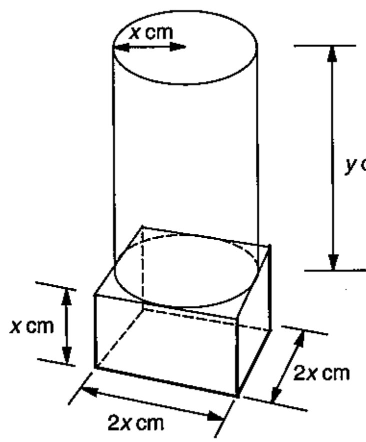


Fig. 10.32

Find
(i) the value of x for which A has a stationary value,
(ii) the value of A and of y corresponding to this value of x.
Determine whether the stationary value of A is a maximum or a minimum. (C)

13 Fig.10.31 shows part of the curve $y = 4 + 3x - x^2$ and the line $2y - 2 = x$. OB = b and BCD is parallel to the y-axis.
(a) Express the length of CD in terms of b.
(b) Hence find the value of b for which the length of CD is a maximum.

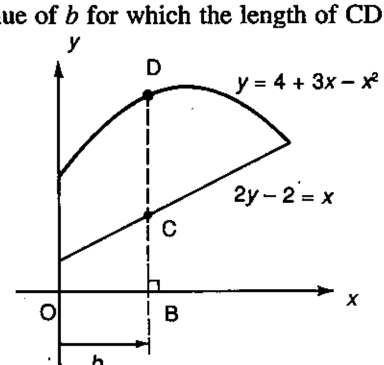


Fig. 10.33

14 A circular cylinder of height $2h$ cm is fitted inside a sphere of radius 10 cm. Find an expression for the radius of the cylinder in terms of h and hence find the maximum volume of the cylinder.

15 A point moves on the x-axis and its position at time t is given by $x = t(t^2 - 6t + 12)$. Show that its velocity at the origin is 12 and find its position when it comes to instantaneous rest. If v is its velocity and a its acceleration at time t, show that $a^2 = 12v$.

16 A piece of wire, of fixed length L cm, is bent to form the boundary OPQO of a sector of a circle (Fig. 10.34). The circle has centre O and radius r cm. The angle of the sector is θ radians.

Show that the area A cm², of the sector is given by $A = \frac{1}{2}rL - r^2$.

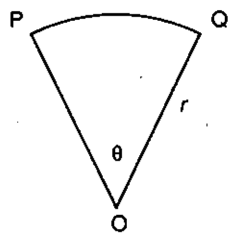


Fig. 10.34

(a) Find a relationship between r and L for which A has a stationary value and find the corresponding value of θ. Determine the nature of this stationary value.
(b) Show that, for this value of θ, the area of the triangle OPQ is approximately 45.5% of the area of the sector OPQ. (C)

17 A line of gradient m ($m < 0$) passes through the point (3,2) and meets the axes at P and Q. Find the coordinates of P and Q in terms of m and show that the area of $\triangle$POQ is $6 - \frac{2}{m} - \frac{9m}{2}$. Hence find the minimum area of $\triangle$POQ.

18 A particle is travelling in a straight line and its distance s cm from a fixed point on the line after t seconds is given by $s = 12t - 15t^2 + 4t^3$. Find.
(a) the velocity and acceleration after 3 seconds,
(b) the distance between the two points where it is at instantaneous rest.

19 A rectangular box without a lid is made from thin cardboard. The sides of the base are $2x$ cm and $3x$ cm and the height of the box is h cm. If the total surface area is 200 cm², show that

$$h = \frac{20}{x} - \frac{3x}{5}$$

and hence find the dimensions of the box to give the maximum volume.

20 Show that the height of a circular cone of volume V and radius r is given by $\frac{3V}{\pi r^2}$. If V remains constant but r is increased by 2%, find the approximate percentage change in h.

21 A particle P travels in a straight line so that its distance, s metres, from a fixed point O is given by $s = 11 + 6t^2 - t^3$ where t is the time in seconds measured from the start of the motion. Calculate

(i) the velocity of P after 3 seconds,

(ii) the velocity of P when its acceleration is instantaneously zero,

(iii) the average velocity of P over the first two seconds.

22 In Fig. 10.35, ABCD is a rectangle with AB = 6 cm and AD = 8 cm. DE = x cm. EC meets AB produced at F. Find the value of x which gives the minimum area of $\triangle$AFE and show that it is a minimum.

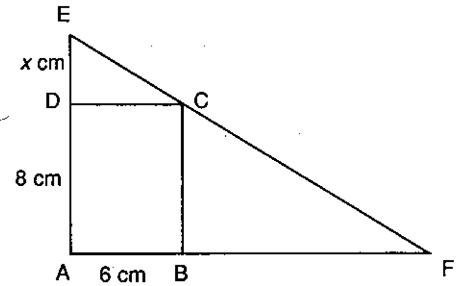


Fig. 10.35

B

23 Given the function $y = ax^3 + bx^2 + cx + d$, find the values of a, b, c and d if the curve

(i) passes through the point (0,–3),

(ii) has a stationary point at (–1,1),

(iii) the value of $\frac{d^2y}{dx^2} = 2$ when $x = 1$.

24 Find the nature of the stationary points on the curve $y = 3x^4 + 4x^3 + 2$.

25 A cylinder of radius r cm is placed upright inside a cone so that the top of the cylinder is 4 cm above the top of the cone as in Fig. 10.36. The cone has a radius of 6 cm and a height of 18 cm. The part of the cylinder inside the cone is h cm deep.

(a) Show that $h + 3r = 18$.

(b) Find an expression in terms of r for the volume of the cylinder.

(c) Hence find the value of h for which the volume of the cylinder is a maximum.

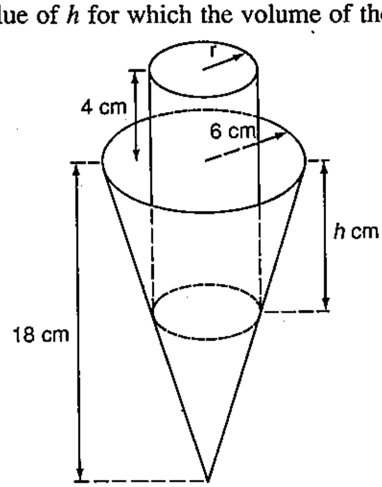


Fig. 10.36

26 (a) If $\frac{1}{u} + \frac{1}{v} = 2$, show that $u = \frac{v}{2v-1}$ and that this equals $\frac{1}{2}(1 + \frac{1}{2v-1})$.

(b) If v is increased by 2% when it is 2, find the percentage change in u.

27 A water trough 100 cm long has a cross section in the shape of a vertical trapezium ABCD as shown in Fig. 10.37. AB = 30 cm and AD and BC are each inclined at 60° to the horizontal. The trough is placed on level ground and is being filled at the rate of 10 litres s^{-1}.

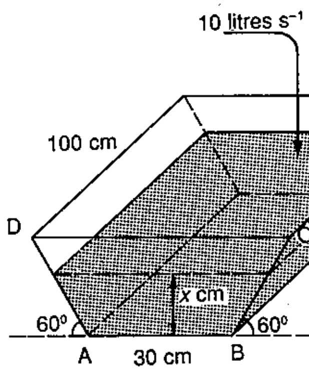


Fig. 10.37

(a) Given that $\tan 60° = \sqrt{3}$, show that the volume V cm³ of water in the trough when it is x cm deep is given by $V = 100x(30 + \frac{x}{\sqrt{3}})$.

(b) Hence calculate the rate at which the water level is rising when $x = 15$ cm.

28 A point A moves along the positive x-axis away from the origin O at a speed of 4 cm s^{-1} where OA > 5 cm. B is a fixed point on the positive y-axis where OB = 20 cm. P is a fixed point on the positive x-axis where OP = 5 cm and Q lies on the line joining B and A with PQ parallel to the y-axis.

(a) Show that when OA = x cm, PQ = $20(1 - \frac{5}{x})$ cm.

(b) Hence find the speed of Q along PQ as A moves when (i) $x = 12$ cm, (ii) $x = 20$ cm.

(c) Obtain an expression in terms of x for the acceleration of Q along PQ.

29 In $\triangle$OAB, $\angle$AOB = 60°, OA = 10 cm and OB = 4 cm. P lies on OA where OP = x cm and Q lies on OB. Given that the area of $\triangle$OPQ is twice that of $\triangle$OAB, find in terms of x, (a) OQ, (b) PQ². Hence find the value of x which will make PQ² a minimum and the corresponding length of OQ.

30 Find the point of intersection P of the curves $y^2 = 4x$ and $4y = x^2$ and sketch the parts of these curves which lie between the origin O and P. A lies on $y^2 = 4x$ with x-coordinate 2. B is a variable point (x,y) on the curve $4y = x^2$, lying between O and P. Find an expression for the area of $\triangle$OAB and hence find the maximum area of this triangle.

Calculus (3) : Integration

11

ANTI-DIFFERENTIATION

If we differentiate $y = 3x^2 - 4x + 3$, we obtain $\frac{dy}{dx} = 6x - 4$.

Supposing we were given $\frac{dy}{dx} = 6x - 4$, can we do a reverse operation, i.e. **anti-differentiate**, to find y?

This is easily done for a *single term* as follows.

Start with $y = \frac{ax^{n+1}}{n+1}$. (You will see why we choose this in a moment).

Then $\frac{dy}{dx} = \frac{a(n+1)x^n}{n+1} = ax^n$.

So if we are given $\frac{dy}{dx} = ax^n$, then $y = \frac{ax^{n+1}}{n+1}$.

To obtain this result, the index (n) has been increased by 1 to $n + 1$, and we then divide by the *new* index. Here is the rule for single terms:

If $\frac{dy}{dx} = ax^n$, $y = \frac{ax^{n+1}}{n+1}$ provided $n \neq -1$

This process of anti-differentiation is actually called **integration**. We **integrate** ax^n wrt x. ax^n is the **integrand** and the result is called the **integral**. A notation for this will be given later.

Example 1

Integrate wrt x (a) x^3 (b) $2x^2$ (c) $4x$ (d) 7 (e) $\frac{3}{x^2}$

We show here the steps taken to obtain the integral. With practice these would not be written down, only the result.

(a) Increase the index by 1 to 4: then divide by 4.

Result $\frac{x^4}{4}$.

(b) New index is 3: then divide by 3. The factor 2 is left as it is.

Result $\frac{2x^3}{3}$.

(c) $4x = 4x^1$. New index is 2, divide by 2.

Result $\frac{4x^2}{2} = 2x^2$.

Always simplify when possible.

(d) $7 = 7x^0$. New index is 1, divide by 1.

Result $\frac{7x^1}{1} = 7x$.

(e) $\frac{3}{x^2} = 3x^{-2}$. New index is $-2 + 1 = -1$. Divide by -1.

Result $\frac{3x^{-1}}{-1} = -\frac{3}{x}$.

Now check each result by differentiation and verify that the original expression is recovered.

Before we go further there is one important point to note. This is discussed in the next section.

THE ARBITRARY CONSTANT: INDEFINITE INTEGRAL

If $y = x^2 - 3x + c$ where c is any constant, then $\frac{dy}{dx} = 2x - 3$.

Now if we start with $\frac{dy}{dx} = 2x - 3$, then $y = x^2 - 3x$.

But this is not the original expression. The constant c is missing and so it must be added to the result. The correct result is $x^2 - 3x + c$. c is called the **arbitrary constant** as its value is not known, unless we are given further information. **It must always be added to an integral**. Such an integral is called an **indefinite integral**.

It is easy to get confused between differentiation and integration. It may help to remember:

Differentiation : multiply by the index and then **D**ecrease the index.

Integration : **I**ncrease the index and then divide by the new index.

As in differentiation, the integral of a sum of terms is the sum of the separate integrals. So we can integrate for example $x^3 - 3x^2 + 1$ or $(x + 2)^2$, provided it is expanded first, but not $\frac{1}{x+1}$.

Example 2

Integrate wrt x (a) $2x^3 - 3x + 1$ (b) $(2x - 3)^2$ (c) $\frac{x^4 - 3x + 1}{2x^3}$

(a) Integrate each term: $\frac{2x^4}{4} - \frac{3x^2}{2} + \frac{x^1}{1} + c = \frac{x^4}{2} - \frac{3x^2}{2} + x + c$

(b) Expand first: $4x^2 - 12x + 9$

Now integrate: $\frac{4x^3}{3} - \frac{12x^2}{2} + 9x + c = \frac{4x^3}{3} - 6x^2 + 9x + c$

(c) Divide by $2x^3$ first: $\frac{x}{2} - \frac{3x^{-2}}{2} + \frac{x^{-3}}{2}$

Now integrate: $\frac{x^2}{2(2)} - \frac{3x^{-1}}{2(-1)} + \frac{x^{-2}}{2(-2)} + c = \frac{x^2}{4} + \frac{3}{2x} - \frac{1}{4x^2} + c$

Notation

The symbol for integration is $\int$. For example we write $\int 3x\ dx$. This means that the integrand $3x$ is to be integrated wrt x.

So $$\int 3x\ dx = \frac{3x^2}{2} + c \quad \text{and} \quad \int u^2\ du = \frac{u^3}{3} + c.$$

If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\ dx + c$.

Example 3

Find

(a) $\int x^5\ dx$, *(b)* $\int dx$, *(c)* $\int 2t^3\ dt$, *(d)* $\int (s^2 - 2s + 3)\ ds$, *(e)* $\int (p-1)(2-p)\ dp$

(a) $\int x^5\ dx = \frac{x^6}{6} + c$

(b) $\int dx$ means $\int 1\ dx = \int x^0\ dx = x + c$

(c) Here the variable is t: $\int 2t^3\ dt = \frac{2t^4}{4} + c = \frac{t^4}{2} + c$.

(d) If the integrand is a polynomial, it must be placed in brackets between the $\int$ sign and ds.

$$\int (s^2 - 2s + 3)\ ds = \frac{s^3}{3} - \frac{2s^2}{2} + 3s + c = \frac{s^3}{3} - s^2 + 3s + c$$

(e) Expand first.

$$\int (-2 + 3p - p^2)\ dp = -2p + \frac{3p^2}{2} - \frac{p^3}{3} + c$$

Note that an integral such as $\int 4x\ dy$ is not possible unless x can first be expressed in terms of y.

Exercise 11.1 *(Answers on page 630.)*

1 Find the indefinite integrals wrt x of:

(a) $4x$ (b) $4x^3$ (c) -7
(d) $3x^2$ (e) $3 - x$ (f) $4x^4$
(g) $2x^5$ (h) $x^2 - 3$ (i) $1 - x - x^2$
(j) $x^2 - \frac{x}{4}$ (k) $1 - 3x - 4x^2$ (l) $x^5 - 3x^2$

(m) $\frac{2}{x^3}$ (n) $(x+2)^2$ (o) $(x-1)^3$

(p) $1-\frac{1}{3x^4}$ (q) $(2-x)^2$ (r) $x^2-\frac{1}{x^2}$

(s) $(x+2)(x-3)$ (t) $x+\frac{1}{x^2}$ (u) $\frac{x+1}{3x^3}$

(v) $(x^2+2)(x-1)$

2 (a) $\int (x-4)\,dx$ (b) $\int \frac{3}{y^2}\,dy$

(c) $\int \frac{dx}{3}$ (i.e. $\int \frac{1}{3}\,dx$) (d) $\int (2-\frac{1}{x^2})\,dx$

(e) $\int (3x-2)\,dx$ (f) $\int \frac{2t}{3}\,dt$

(g) $\int \frac{4}{y^2}\,dy$ (h) $\int (\frac{1}{u^2}+\frac{2}{u^3})\,du$

3 Find

(a) $\int (\frac{u+1}{u^3})\,du$ (b) $\int (3r-2)^2\,dr$

(c) $\int (3p-2)(p-3)\,dp$ (d) $\int (1-x)^3\,dx$

(e) $\int (\frac{4t^3-3t^2-1}{3t^2})\,dt$ (f) $\int (s-\frac{1}{2s})^2\,ds$

(g) $\int (1-4s)(2+3s)\,ds$ (h) $\int (\frac{x+1}{2})^2\,dx$

(i) $\int (x^3-\frac{x}{2})\,dx$ (j) $\int (1-2y)^2\,dy$

(k) $\int (2t^3-4t+\frac{1}{3})\,dt$ (l) $\int (2x+\frac{3}{2x})^2\,dx$

(m) $\int (\frac{2t^4+t^2-2}{t^4})\,dt$ (n) $\int p(2p+3)(3p-2)\,dp$

The Integral $\int \frac{1}{x}\,dx$

If we use the rule for this integral, $\int x^{-1}\,dx$, the result is $\frac{x^{-1+1}}{0}$ which is not possible. Hence $\int \frac{1}{x}\,dx$ is an exception to the rule.

In Part II of this book, we shall see that a special function is created for this integral.

APPLICATIONS OF INTEGRATION

Example 4

Find y given that $\frac{dy}{dx} = 2x-3$ and that $y=-4$ when $x=1$.

If $\frac{dy}{dx} = 2x-3$, then $y = \int (2x-3)\,dx = x^2-3x+c$.

This is the indefinite integral and is illustrated in Fig.11.1. For all values of c, the family of curves $y = x^2-3x+c$ are parallel, one vertically above the other. The equations could be $y = x^2-3x$ or $y = x^2-3x+5$ etc.

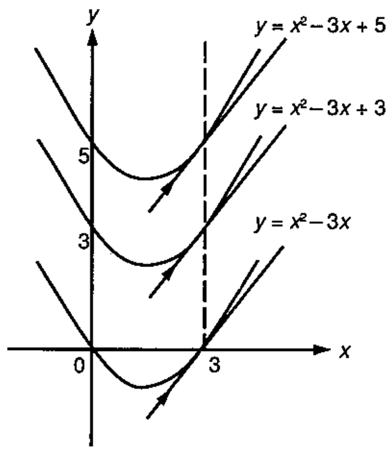


Fig.11.1

For any given value of x (say 3), the gradient on each curve at $x = 3$ is 3 as $\frac{dy}{dx} = 2x - 3$ and the tangents at these points are parallel.
Further information is therefore needed to identify a particular member of the family. In this example, we have this information to find c.

When $x = 1$, $y = 1 - 3 + c = -4$ so $c = -2$.
Hence $y = x^2 - 3x - 2$.

Example 5

The gradient of the tangent at a point on a curve is given by $x^2 + x - 2$. Find the equation of the curve if it passes through (2,1).

Gradient = $\frac{dy}{dx} = x^2 + x - 2$.

Then $y = \int (x^2 + x - 2)\,dx = \frac{x^3}{3} + \frac{x^2}{2} - 2x + c$.

When $x = 2$, $y = \frac{8}{3} + \frac{4}{2} - 4 + c = 1$.

Hence $c = \frac{1}{3}$.

The equation of the curve is $y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + \frac{1}{3}$ or $6y = 2x^3 + 3x^2 - 12x + 2$.

Example 6

A curve has a turning point at the point (–1,1). If the gradient is given by $6x^2 + ax - 12$, find the value of a and the equation of the curve.

$\frac{dy}{dx} = 6x^2 + ax - 12$

When $x = -1$, $\frac{dy}{dx} = 0$.

Then $6 - a - 12 = 0$ giving $a = -6$.

So $\frac{dy}{dx} = 6x^2 - 6x - 12$.

Hence $y = \int (6x^2 - 6x - 12)\, dx = 2x^3 - 3x^2 - 12x + c$.

When $x = -1$, $y = -2 - 3 + 12 + c = 1$, so $c = -6$.

The equation is $y = 2x^3 - 3x^2 - 12x - 6$.

Example 7

For a curve $y = f(x)$, $\frac{d^2y}{dx^2} = 6x - 2$. Given that $y = 11$ and $\frac{dy}{dx} = 10$ when $x = 2$, find the equation of the curve.

$\frac{d^2y}{dx^2}$ is obtained by differentiating $\frac{dy}{dx}$ wrt x. Then $\frac{dy}{dx}$ is found by integrating $\frac{d^2y}{dx^2}$ wrt x.

$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}\, dx = \int (6x - 2)\, dx = 3x^2 - 2x + c$

But $\frac{dy}{dx} = 10$ when $x = 2$.

Then $12 - 4 + c = 10$ giving $c = 2$.

$\frac{dy}{dx} = 3x^2 - 2x + 2$

Now we integrate again to find y.

$y = \int (3x^2 - 2x + 2)\, dx = x^3 - x^2 + 2x + c_1$

When $x = 2$, $y = 8 - 4 + 4 + c_1 = 11$ so $c_1 = 3$.

Hence the equation is $y = x^3 - x^2 + 2x + 3$.

Example 8

A particle moves in a straight line so that its velocity v m s^{-1} at time t s from the start is given by $v = t^2 - 2t - 3$ $(t \geq 0)$.

If it started 3 m from a fixed point O of the line, find

(a) the value of t when it is at instantaneous rest,

(b) its distance from O at that time,

(c) for what values of t its acceleration is positive.

(a) It is instantaneously at rest when $v = 0$.

$v = (t-3)(t+1)$ so $t = 3$ (-1 not being allowed).

(b) $v = \frac{ds}{dt} = t^2 - 2t - 3$

Then $s = \int (t^2 - 2t - 3)\,dt = \frac{t^3}{3} - t^2 - 3t + c$.

But $s = 3$ when $t = 0$, so $c = 3$.

Hence $s = \frac{t^3}{3} - t^2 - 3t + 3$.

When $t = 3$, $s = 9 - 9 - 9 + 3 = -6$ m.

(c) $a = \frac{dv}{dt} = 2t - 2$. Hence $a > 0$ when $t > 1$.

Example 9

For a particle moving in a straight line, its acceleration a m s^{-2} is given by $a = t - \frac{5}{2}$ where t is the time in s from the start.
Given that its velocity v at the start was 3 m s^{-1}, find (a) an expression for v in terms of t, (b) the time t when the particle is at instantaneous rest. (c) If the particle started from a fixed point O on the line, how far is it from O after 2 s?

(a) $a = \frac{dv}{dt} = t - \frac{5}{2}$.

Then $v = \int (t - \frac{5}{2})\,dt = \frac{t^2}{2} - \frac{5t}{2} + c$

When $t = 0$, $v = 3$.

So $c = 3$ and $v = \frac{t^2}{2} - \frac{5t}{2} + 3$

$= \frac{t^2 - 5t + 6}{2} = \frac{(t-3)(t-2)}{2}$

(b) $v = 0$ when $t = 3$ or $t = 2$.

(c) $v = \frac{ds}{dt} = \frac{t^2}{2} - \frac{5t}{2} + 3$ so $s = \int (\frac{t^2}{2} - \frac{5t}{2} + 3)\,dt$

$= \frac{t^3}{6} - \frac{5t^2}{4} + 3t + c_1$

When $t = 0$, $s = 0$ so $c_1 = 0$.

Therefore $s = \frac{t^3}{6} - \frac{5t^2}{4} + 3t$ and if $t = 2$, $s = 2\frac{1}{3}$ m.

Exercise 11.2 *(Answers on page 631.)*

1 A curve is given by $\frac{dy}{dx} = 2x - 1$. If it passes through the point (2,6), find its equation.

2 If a curve is given by $\frac{dy}{dx} = x(x-1)$, find (a) its equation if it passes through the point (1,0) and (b) the nature and coordinates of its turning points.

3 Given that $\frac{dy}{dx} = 1 - 5x$ and that $y = -5$ when $x = 2$, find the value of y when $x = 1$.

4 The rate of change of a quantity P is given by $\frac{dP}{dt} = t + 2$. If $P = 5$ when $t = 2$, find the value of P when $t = 3$.

5 The velocity v m s^{-1} of a particle P moving in a straight line at time t s is given by $v = 2t^2 - 3t$. Find an expression for its distance s m from a fixed point O on the line if OP = 4 m when $t = 1$ and its acceleration at that time.

6 Given that $\frac{d^2y}{dx^2} = 2x + 1$ and that $\frac{dy}{dx} = y = 3$ when $x = -1$, find y in terms of x.

7 Given that $\frac{d^2y}{dx^2} = 3$, find y in terms of x if $\frac{dy}{dx} = 4$ and $y = 6$ when $x = 2$.

8 A curve has gradient $x^2 - 4x + 3$ at the point (x,y) on the curve and it passes through the point $(3,-1)$. Find (a) its equation and (b) the types and coordinates of its turning points.

9 For the function $y = f(x)$, $\frac{dy}{dx} = x^2 + kx$ where k is a constant. If y has a turning point at the point $(3,-2)$, find the value of k and the value of y when $x = 4$.

10 If $\frac{dy}{dt} = 1 - \frac{1}{t^2}$, find the value of y when $t = 4$ if $y = 4$ when $t = 1$.

11 If $\frac{dy}{dx} = 6x^2 + 4x - 5$ and $y = 10$ when $x = 2$, find the value of y when $x = 3$.

12 The rate of change of a quantity L with respect to t is given by $\frac{dL}{dt} = 3t - 2$. If $L = 3$ when $t = 2$, find the value of L when $t = 4$.

13 A curve passes through the point $(1,0)$ and its gradient at any point (x,y) on the curve is $3x^2 - 2x - 1$. Find (a) the equation of the curve, and (b) the coordinates of the points where y has a maximum and minimum value, identifying each one. (c) For what range of values of x is the gradient on the curve decreasing?

14 A small body moves in a straight line so that its velocity v m s^{-1} at time t s is given by $v = t^3 - 6t^2 + 9t + 2$. Find (a) the times when its acceleration is zero, and (b) its distance from a fixed point on the line when $t = 2$ given that it started from this point. (c) For how long was its acceleration negative?

15 The velocity v m s^{-1} of a particle moving in a straight line is given by $v = t^2 - 4t$ where t is the time in seconds after starting from a fixed point O on the line. Find
(a) the time when the particle is instantaneously at rest,
(b) its velocity and acceleration at the start,
(c) its distance from O when $t = 3$.

16 The velocity v m s^{-1} of a particle moving in a straight line at time t seconds is given by $v = 1 + \frac{9}{t^2}$ for $1 \le t \le 3$.
When $t = 3$, the particle is 6 m from a fixed point on the line.
(a) Find an expression in terms of t for its distance from this fixed point.
(b) How far does it travel between $t = 1$ and $t = 3$?

17 Given that $\frac{d^2y}{dt^2} = 3 - kt$ where k is a constant and that $\frac{dy}{dt} = -6$ when $t = -1$ and 9 when $t = 2$, find the value of k. If $y = -\frac{5}{6}$ when $t = 1$, find y in terms of t.

18 A particle passes a fixed point O on a straight line with a velocity of 10 m s^{-1} and moves on the line with an acceleration of $(4 - t)$ m s^{-2} at time t s after passing through O. Find

(a) its velocity when $t = 4$,

(b) the distance of the particle from O when $t = 2$.

19 A quantity u varies with respect to t so that $\frac{du}{dt} = a + bt$ where a and b are constants. Given that it has a maximum value of $5\frac{1}{2}$ when $t = 1$ and that its rate of change when $t = 2$ is -3, find u in terms of t.

Area Under a Curve

An important application of integration is in finding the area under a given curve $y = f(x)$. Up to now, such areas could only be found approximately, for example, by counting squares or by the trapezium rule. Using calculus, we can now find the exact value of areas bounded by curves.

Fig.11.2 shows part of a curve $y = f(x)$. The shaded area A lies between the curve and the x-axis, bounded by the ordinates at a and b. This area is called the area under the curve between a and b.

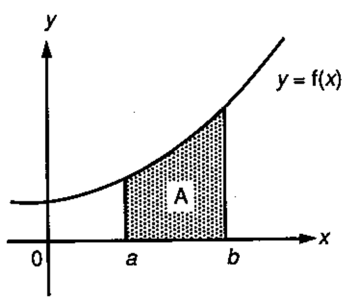


Fig.11.2

We now show a method of finding A. For the moment we can only deal with areas which lie above the x-axis.

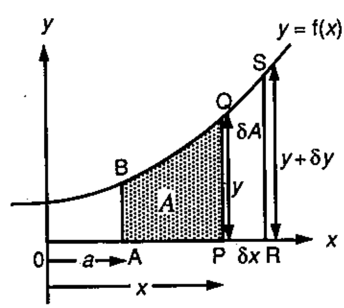


Fig.11.3

Let P be a point on the x-axis where OP = x (Fig.11.3). PQ = y, OA = a and AB is perpendicular to the x-axis. The shaded area under the curve from a to x is A.

Now take an increment δx in x to reach R and draw the ordinate RS. RS $= y + \delta y$ and the increment in $A = \delta A$ = the area PRSQ.

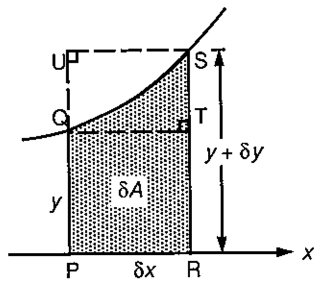


Fig.11.4

QT and US are drawn parallel to the x-axis (Fig.11.4).
Area of rectangle PRTQ $< \delta A <$ area of rectangle PRSU.

i.e. $y\delta x < \delta A < (y + \delta y)\, \delta x$ so $y < \dfrac{\delta A}{\delta x} < y + \delta y$.

Now if $\delta x \to 0$, $\delta y \to 0$ and $y + \delta y \to y$.

The left hand term of the inequality remains fixed at y but the right hand term $\to y$. Hence $\dfrac{\delta A}{\delta x} \to y$ and in the limit $\dfrac{dA}{dx} = y$.

We then have $$A = \int y\, dx + c = \int f(x)\, dx + c$$

We can find c from the fact that $A = 0$ when $x = a$.

Example 10

Find the area under the curve $y = x^2 + 2$ between $x = 1$ and $x = 3$ (Fig.11.5).

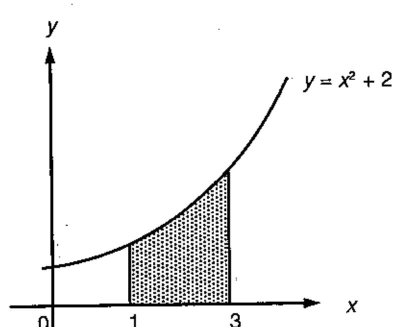


Fig.11.5

$A = \int y\, dx = \int (x^2 + 2)\, dx = \dfrac{x^3}{3} + 2x + c$

Now when $x = 1$, $A = 0$.

Hence $0 = \frac{1}{3} + 2 + c$ giving $c = -2\frac{1}{3}$.

Then $A_1^x = \frac{x^3}{3} + 2x - 2\frac{1}{3}$ where A_1^x means the area between the ordinates 1 and x $(x > 1)$.
We want the area from 1 to 3. So put $x = 3$.
$A_1^3 = 9 + 6 - 2\frac{1}{3} = 12\frac{1}{3}$ units².
If we had required the area from 1 to 2, i.e. A_1^2, then we put $x = 2$ etc.

Example 11

Find the area under the curve $y = 2 + x - x^2$.

The curve meets the x-axis where $y = 0$,
i.e. where $(2 - x)(1 + x) = 0$
giving $x = -1$ or $x = 2$ (Fig.11.6).

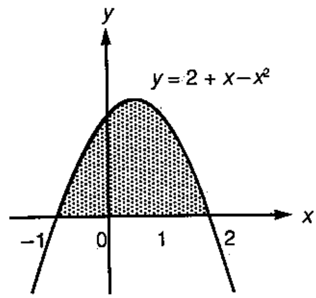


Fig.11.6

So $A = \int (2 + x - x^2)\,dx = 2x + \frac{x^2}{2} - \frac{x^3}{3} + c$.
But $A = 0$ when $x = -1$.
Hence $0 = -2 + \frac{1}{2} + \frac{1}{3} + c$ giving $c = 1\frac{1}{6}$.
Then $A_{-1}^x = 2x + \frac{x^2}{2} - \frac{x^3}{3} + 1\frac{1}{6}$.
Now put $x = 2$ to obtain the required area.
$A_{-1}^2 = 4 + 2 - \frac{8}{3} + 1\frac{1}{6} = 4\frac{1}{2}$ units².

DEFINITE INTEGRALS

We can shorten the above process by using the concept of a **definite integral.**

Suppose A_a^x is the area under $y = f(x)$ from a to x (Fig.11.7).

Then $A = \int f(x)\,dx = g(x) + c$ where $g(x)$ is the indefinite integral of $f(x)$.

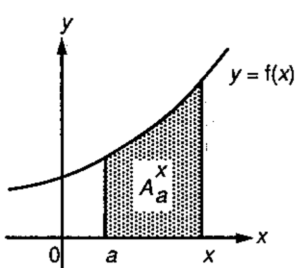


Fig.11.7

Now when $x = a$, $A = 0$ so $0 = g(a) + c$ giving $c = -g(a)$.
Hence $A_a^x = g(x) - g(a)$.

Now we put $x = b$

and $A_a^b = \mathrm{g(b)} - \mathrm{g(a)}$
$\quad = $ (value of the integral at b) – (value of the integral at a)

We write this as $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ and it is called the **definite integral** of f(x) wrt x between the **limits** a (the lower limit) and b (the upper limit). The arbitrary constant c disappears in the subtraction.

Hence if $y = \mathrm{f}(x)$, the area under the curve between the ordinates a and b, where $a < b$, is

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{g}(b) - \mathrm{g}(a)$$

where g(x) is the indefinite integral of f(x).

At present this is only true if $\mathrm{f}(x) \geq 0$. We investigate what happens if $\mathrm{f}(x) < 0$ later.

Example 12

Find the area under the curve $y = x^3 + 2$ from $x = -1$ to $x = 2$ (Fig.11.8).

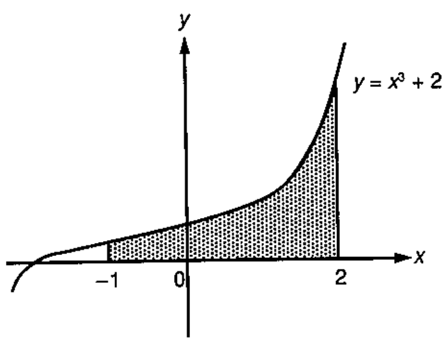


Fig.11.8

$$A = \int_1^2 y\,\mathrm{d}x$$

$= \int_1^2 (x^3 + 2)\mathrm{d}x \qquad \mathrm{g}(x) = \frac{x^4}{4} + 2x + c$

$= \left[\frac{x^4}{4} + 2x\right]_{-1}^{2}$ upper limit / lower limit

g(x) is placed in square brackets with the limits at the right. The arbitrary constant is not included.

$= (\frac{2^4}{4} + 2 \times 2) \quad - \quad (\frac{(-1)^4}{4} + 2 \times (-1))$

substitute **upper** limit to obtain g(b), the value when $b = 2$

substitute **lower** limit to obtain g(a), the value when $a = -1$

$= \quad 8 \quad - \quad (\frac{1}{4} - 2)$

$= \quad 8 \quad - \quad (-1\frac{3}{4})$

$= 9\frac{3}{4}$ units^2

Example 13

(a) Find the coordinates of (i) the point A where the curves $y = (x + 1)^2$ and $y = (x - 3)^2$ intersect and (ii) the points where the curves meet the x-axis.
(b) Hence find the area of the region enclosed by the curves and the x-axis.

(a) (i) The curves meet where $(x + 1)^2 = (x - 3)^2$ i.e. where $x = 1$.
Hence the coordinates of A are (1,4).

(ii) $y = (x + 1)^2$ meets $y = 0$ where $x = -1$, i.e. the point (–1,0).
$y = (x - 3)^2$ meets $y = 0$ where $x = 3$, i.e. the point (3,0).

(b) The curves are shown in Fig.11.9. The area required is divided into two parts because the boundary changes at A ($x = 1$).

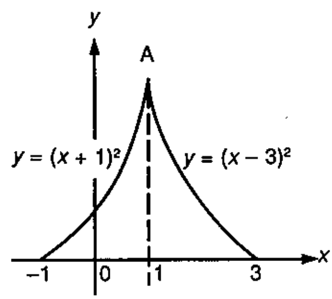


Fig.11.9

$$\text{Total area} = \int_{-1}^{1} (x + 1)^2\,dx + \int_{1}^{3} (x - 3)^2\,dx$$
$$= \int_{-1}^{1} (x^2 + 2x + 1)\,dx + \int_{1}^{3} (x^2 - 6x + 9)\,dx$$
$$= \left[\frac{x^3}{3} + x^2 + x\right]_{-1}^{1} + \left[\frac{x^3}{3} - 3x^2 + 9x\right]_{1}^{3}$$
$$= \underbrace{(\tfrac{1}{3} + 1 + 1)}_{\text{substitute upper limit 1}} - \underbrace{(-\tfrac{1}{3} + 1 - 1)}_{\text{substitute lower limit }-1} + \underbrace{(9 - 27 + 27)}_{\text{substitute upper limit 3}} - \underbrace{(\tfrac{1}{3} - 3 + 9)}_{\text{substitute lower limit 1}}$$
$$= \frac{7}{3} + \frac{1}{3} + 9 - \frac{19}{3}$$
$$= \frac{16}{3}\ \text{units}^2$$

In the next two examples, only the *value* of the definite integral is to be found.

Example 14

Find $\int_{-4}^{-2}(x - \frac{1}{x^2})\,dx$.

$$\int_{-4}^{-2}(x - \frac{1}{x^2})\,dx = \left[\frac{x^2}{2} + \frac{1}{x}\right]_{-4}^{-2}$$

$$= \underbrace{\left(\frac{(-2)^2}{2} + \frac{1}{-2}\right)}_{\text{substitute the upper limit } (-2)} - \underbrace{\left(\frac{(-4)^2}{2} + \frac{1}{-4}\right)}_{\text{substitute the lower limit } (-4)}$$

$$= (2 - \tfrac{1}{2}) - (8 - \tfrac{1}{4})$$

$$= -6\tfrac{1}{4}$$

Example 15

Evaluate $\int_{-2}^{0}(1 - t - t^2)\,dt$.

$$\int_{-2}^{0}(1 - t - t^2)\,dt = \left[t - \frac{t^2}{2} - \frac{t^3}{3}\right]_{-2}^{0}$$

$$= [0] - [-2 - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}]$$

$$= 0 - (-1\tfrac{1}{3}) = 1\tfrac{1}{3}$$

Example 16

The volume V of the liquid in a container leaks out at the rate of $30t$ cm^3 s^{-1} where t is the time in seconds. Find the amount of liquid lost in the third second.

$\frac{dV}{dt} = -30t$ (decreasing).

The third second is between $t = 2$ and $t = 3$.
So we find the (value of V when $t = 3$) – (value of V when $t = 2$) using a definite integral.

Change of volume $= \int_2^3 \frac{dV}{dt}\,dt = \int_2^3 (-30t)\,dt = \left[-15t^2\right]_2^3 = -135 - (-60)$

$= -75$

Hence 75 cm^3 of liquid was lost in that time.

Example 17

The velocity v of a particle moving in a straight line is given by $v = t^2 - 3t$ where t is the time after the start. What is the displacement of the particle between the times $t = 2$ and $t = 4$?

$v = \frac{ds}{dt} = t^2 - 3t$ so $s = \int (t^2 - 3t)\,dt$

The displacement is the distance between the positions of the particle at times $t = 2$ and $t = 4$, so it is the value of the definite integral

$$\int_2^4 (t^2 - 3t)\,dt = \left[\frac{t^3}{3} - \frac{3t^2}{2}\right]_2^4$$

$$= \left(\frac{64}{3} - \frac{48}{2}\right) - \left(\frac{8}{3} - \frac{12}{2}\right) = \frac{2}{3}$$

Note: As we have seen in Chapter 10, this is not necessarily equal to the actual distance travelled by the particle. It may have gone, for example, 8 units to the left followed by $8\frac{2}{3}$ units to the right.

Example 18

(a) Show from a diagram that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \text{ where } a < b < c.$$

(b) Given that $\int_2^5 f(x)\,dx = 10$, find (if possible) the values of

(i) $\int_2^5 2f(x)\,dx,$

(ii) $\int_2^5 [f(x) + 1]\,dx,$

(iii) $\int_2^3 [f(x) - 2]\,dx + \int_3^5 f(x)dx$

(iv) $\int_2^5 [1 - \frac{1}{2}f(x)]\,dx,$

(v) $\int_2^5 [f(x)]^2\,dx$

(a)

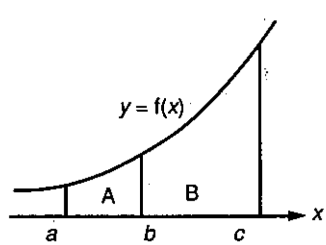


Fig.11.10

From Fig.11.10, $\int_a^c f(x)\,dx$ = area A + area B

$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

(b) (i) $\int_2^5 2f(x)\,dx = 2\int_2^5 f(x)\,dx = 2 \times 10 = 20$

(ii) $\int_2^5 [f(x) + 1]\,dx = \int_2^5 f(x)\,dx + \int_2^5 1\,dx$
$= 10 + [x]_2^5$
$= 10 + (5) - (2) = 13$

(iii) This equals $\int_2^3 f(x)\,dx - \int_2^3 2\,dx + \int_3^5 f(x)\,dx$
$= \int_2^5 f(x)\,dx - [2x]_2^3$
$= 10 - 2 = 8$

(iv) $\int_2^5 [1 - \frac{1}{2}f(x)]\,dx = \int_2^5 1\,dx - \frac{1}{2}\int_2^5 f(x)\,dx$
$= 3 - 5 = -2$

(v) Not possible, as f(x) is not known.

Exercise 11.3 *(Answers on page 631.)*

1 (a) $\int_{-2}^{1} dx$

(b) $\int_{-1}^{0} x\,dx$

(c) $\int_{-2}^{1} x^2\,dx$

(d) $\int_{0}^{3} (2x - 1)\,dx$

(e) $\int_{-2}^{-1} (1 - x)\,dx$

(f) $\int_{-2}^{2} x^3\,dx$

(g) $\int_{1}^{3} \frac{1}{x^2}\,dx$

(h) $\int_{-3}^{-1} (x - x^2)\,dx$

(i) $\int_{-1}^{2} (p - 2)(p - 3)\,dp$

(j) $\int_{a}^{b} dx$

(k) $\int_{t}^{2t} (u^2 - 3)\,du$

(l) $\int_{1}^{3} \frac{x - 3}{x^3}\,dx$

(m) $\int_{-2}^{0} t(t^2 - 1)\,dt$

(n) $\int_{-3}^{-2} (\frac{1}{x^2} - \frac{1}{x^3})\,dx$

(o) $\int_{-2}^{-1} 3x^2\,dx$

(p) $\int_{1}^{3} x^3\,dx$

(q) $\int_{0}^{3} x(3 - x)\,dx$

(r) $\int_{-4}^{0} (3 - 2x)\,dx$

(s) $\int_{1}^{2} (x + 1)(x + 2)\,dx$

(t) $\int_{1}^{2} (x - \frac{1}{x^2})\,dx$

(u) $\int_{-2}^{2} (x^3 - x^2 + x)\,dx$

2 If $\int_0^a (x - 4)\,dx = 10$, find the value of a.

3 Given that $\int_0^t (2x - 1)\,dx = 12$, find the values of t.

4 Given that $\int_1^2 (x + p)\,dx = 3$, find the value of p.

5 If $\int_0^3 (t^2 + \mu)\, dt = 3$, find the value of μ.

6 Find the value of u if $\int_u^{2u} \frac{1}{x^2}\, dx = \frac{1}{4}$.

7 Find the areas under the following curves between the coordinates given:

(a) $y = 4 - x^2$; $x = -2$, $x = 0$ (b) $y = x(3 - x)$; $x = 0$, $x = 3$

(c) $y = \frac{1}{2}x^2$; $x = 1$, $x = 2$ (d) $y = 3 - 2x - x^2$; $x = -3$, $x = 1$

(e) $y = 2 - x^2$; $x = -1$, $x = 1$ (f) $2y = 1 + x^2$; $x = -2$, $x = 1$

(g) $y = x^3 + 2$; $x = 0$, $x = 2$ (h) $y = x^2 - x - 2$; $x = -3$, $x = -1$

8 Find the area bounded by the curve $y = 2x - 2x^2$ and the positive x– and y–axes.

9 Find the area under the curve $y = x^2 + 3$ between the ordinates (i) $x = 0$ and $x = 2$, (ii) $x = -2$ and $x = 2$. Using a sketch of the curve explain the relation between the two areas.

10 The area under the curve $y = x^2 + ax - 5$ between the lines $x = 1$ and $x = 3$ is $14\frac{2}{3}$. Find the value of a.

11 If the area under the curve $y = \frac{x^2}{3}$ between $x = 2$ and $x = k$, where k is a constant, is 8 times the area under the curve between $x = 1$ and $x = 2$, find the value of k.

12 Given that $\frac{dA}{dt} = 2t^2 - t + 5$, find the change in the value of A between $t = 1$ and $t = 3$.

13 If $\frac{dT}{dt} = t^2 - t + 1$, find the change in T as t changes from 1 to 2.

14 The rate of change of a quantity P is given by $\frac{dP}{dt} = \frac{10}{t^2} + t$ for $t > 2$. Find the change in the value of P when t increases from 3 to 5.

15 If $\frac{d^2y}{dx^2} = 2x - 1$, find the increase in y as x increases from 2 to 4 given that $\frac{dy}{dx} = 6$ when $x = 2$.

16 The curve $y = ax^2 + bx + c$ passes through the points (0,–2) and (1,–3) and its gradient where $x = 2$ is 5. Find (a) the value of a, of b and of c and (b) the area under the curve between the lines $x = 2$ and $x = 3$.

17 (a) If $\int_0^a (x - 1)\, dx = \frac{1}{2} \int_0^a (x + 1)\, dx$, find the value of a.

(b) Given that $\int_1^6 f(x)\, dx = 7$, evaluate (i) $\int_1^6 3f(x)\, dx$, (ii) $\int_1^6 [2 - f(x)]\, dx$,

(iii) $\int_1^3 [f(x) - 2]\, dx + \int_3^6 f(x)\, dx$

18 A particle starts from a fixed point O and moves in a straight line. Its distance s from O at time t seconds from the start is given by $s = \frac{1}{3}t^3 - 2t^2 + 3t$. Find an expression for the velocity v of the particle in terms of t. At what times from the start is the particle at instantaneous rest? What is its displacement between those times?

19 Fig.11.11 shows part of the line $y + 2x = 5$ and the curve $y = x(4 - x)$, which meet at A.

(a) Find the coordinates of A.

(b) Hence find the area of the shaded region.

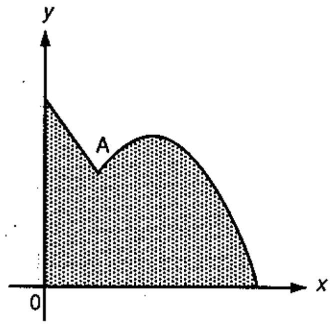


Fig.11.11

20 The curve $y = 4 - x^2$ meets the positive x-axis at B and the curve $y = x(4 - x)$ meets the positive x-axis at C. The curves intersect at A. Find

(a) the coordinates of A, B and C,

(b) the area of the region ABC bounded by the curves and the x-axis.

21 (a) If $y = x^2 - 4x + 4$, find (i) where the curve meets the y-axis and (ii) the x-coordinate m of the minimum point on the curve.

(b) Sketch on the same diagram the graph of $y = 4 - x^2$ for $-2 \leq x \leq 0$ and the graph of $y = x^2 - 4x + 4$ for $0 \leq x \leq m$.

(c) Hence find the total area under the two curves.

Further Notes on Areas

I Area between a curve and the y-axis

The area between $y = f(x)$ and the x-axis for $a \leq x \leq b$ is $\int_a^b y \, dx$.

Similarly, the area between $y = f(x)$ and the y-axis is $\int_c^d x \, dy$ where c and d are the limits on the y-axis and the equation of the curve is expressed in the form $x = g(y)$ (Fig.11.12).

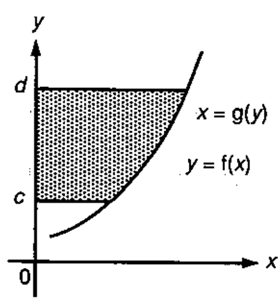


Fig.11.12

Example 19

Fig.11.13 shows part of the curve $(y-1)^2 = x-1$. Find the area of the shaded region.

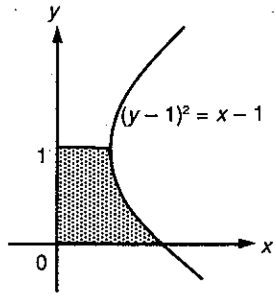


Fig.11.13

The area $= \int_0^1 x\,dy$ as 0 and 1 are the limits for y.

The equation of the curve is rewritten as $x = (y-1)^2 + 1 = y^2 - 2y + 2$.

So the area is $\int_0^1 (y^2 - 2y + 2)\,dy = \left[\frac{y^3}{3} - y^2 + 2y\right]_0^1$

$$= (\tfrac{1}{3} - 1 + 2) - (0) = 1\tfrac{1}{3} \text{ units}^2$$

II Area under the x-axis

The curve $y = x^2 - 3x + 2 = (x-2)(x-1)$ meets the x-axis where $x = 1$ and $x = 2$ (Fig.11.14).

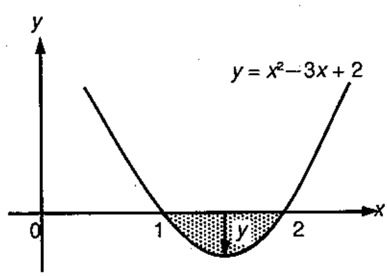


Fig.11.14

For all points in the domain $1 < x < 2$, y will be negative. So $\int y\,dx$ will also be negative for this domain.

$$\int_1^2 y\,dx = \int_1^2 (x^2 - 3x + 2)\,dx = \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2$$

$$= (\tfrac{8}{3} - 6 + 4) - (\tfrac{1}{3} - \tfrac{3}{2} + 2)$$

$$= \tfrac{2}{3} - \tfrac{5}{6}$$

$$= -\tfrac{1}{6} \text{ which is negative as expected.}$$

The numerical value of $\int y\,dx$ is $-\int y\,dx = \frac{1}{6}$ and this is the area below the x-axis.

If part of a curve lies below the x-axis, the area between that part and the x-axis is $-\int y\,dx$ (Fig.11.15).

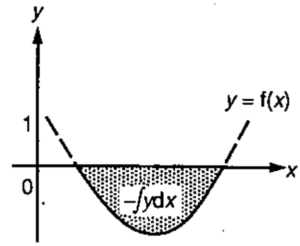


Fig. 11.15

If a curve lies partly above and partly below the x-axis (Fig.11.16), the total area will be $\int_a^b y\,dx - \int_b^c y\,dx$.

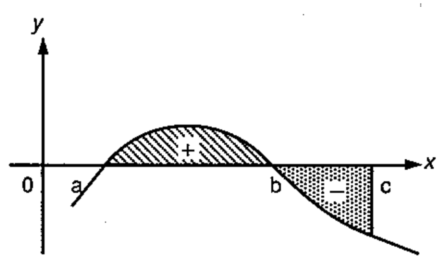


Fig. 11.16

The two parts are evaluated separately. Hence a sketch of the curve must be made to check if any part is below the x-axis.

Similarly the area of a region on the left of the y-axis will be negative. Its numerical value is $-\int x\,dy$.

Example 20

Find the area between the curve $y = x(x - 2)$ and the x-axis from $x = -1$ to $x = 2$.

The curve meets the x-axis at $x = 0$ and $x = 2$ (Fig.11.17).

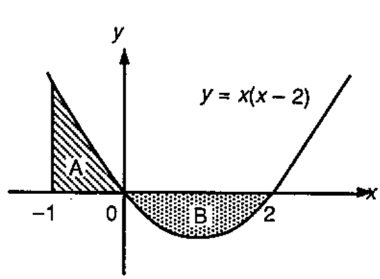


Fig.11.17

Area $A = \int_{-1}^{0} (x^2 - 2x)\,dx = \left[\frac{x^3}{3} - x^2\right]_{-1}^{0} = (0) - (-\frac{1}{3} - 1) = 1\frac{1}{3}$

Area B $= -\int_{0}^{2} (x^2 - 2x)\,dx = -\left[\frac{x^3}{3} - x^2\right]_{0}^{2} = -\left[\left(\frac{8}{3} - 4\right) - (0)\right] = 1\frac{1}{3}$

Hence the total area $= 1\frac{1}{3} + 1\frac{1}{3} = 2\frac{2}{3}$

Note that $\int_{-1}^{2} (x^2 - 2x)\,dx = \left[\frac{x^3}{3} - x^2\right]_{-1}^{2}$

$$= (\tfrac{8}{3} - 4) - (-\tfrac{1}{3} - 1) = 0$$

which is the correct value for the *integral* but not for the *area.*

III Area between two curves

Example 21

Find the area enclosed by the curve $y = 5 + x - x^2$ and the line $y = x + 4$.

First we find where these intersect:

$5 + x - x^2 = x + 4$

i.e. $x^2 = 1$ giving $x = 1$ or -1.

Then we require the area of the shaded region in Fig.11.18 with limits $x = -1$ and $x = 1$.

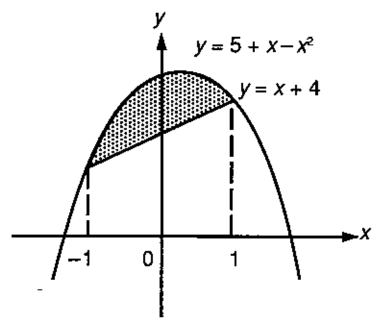


Fig.11.18

Now the area under the curve

$$= \int_{-1}^{1} (5 + x - x^2)\,dx$$

and the area under the line

$$= \int_{-1}^{1} (x + 4)\,dx.$$

Hence the shaded area $= \int_{-1}^{1} (5 + x - x^2)\,dx - \int_{-1}^{1} (x + 4)\,dx.$

Since these two definite integrals have the same limits they can be combined into one definite integral:

$$\int_{-1}^{1} [(5 + x - x^2) - (x + 4)]\,dx = \int_{-1}^{1} (5 + x - x^2 - x - 4)\,dx$$

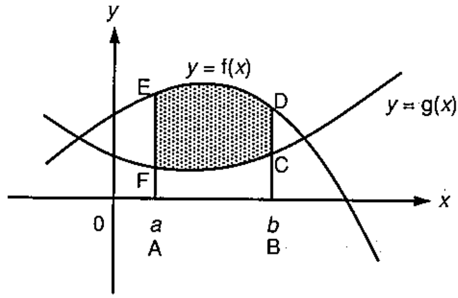
$$= \int_{-1}^{1} (1 - x^2)\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= (1 - \tfrac{1}{3}) - (-1 + \tfrac{1}{3}) = \tfrac{4}{3}$$

In Fig.11.19, $y = f(x)$ and $y = g(x)$ are two curves such that $f(x) > g(x)$ for $a \le x \le b$.

Fig. 11.19



Then $\int_a^b f(x)\,dx$ = area ABDE and $\int_a^b g(x)\,dx$ = area ABCF.

Hence the area between the curves i.e. the shaded area

= ABDE – ABCF

$= \int_a^b [f(x) - g(x)]\,dx$

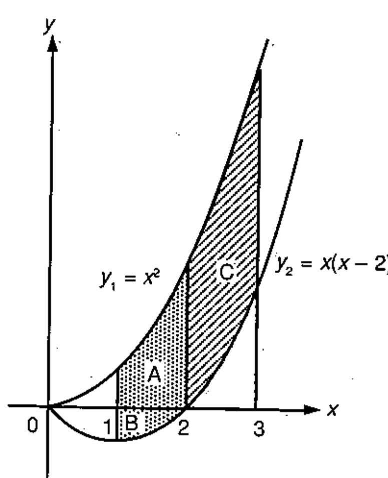
> Area between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$
>
> $= \int_a^b [f(x) - g(x)]\,dx$
>
> where $f(x) > g(x)$

This rule is still true if parts of either curve are below the x-axis (provided $f(x) > g(x)$) as the next example shows.

Example 22

Find the area enclosed by the curves $y_1 = x^2$ and $y_2 = x^2 - 2x$ and the lines $x = 1$ and $x = 3$ (Fig.11.20).

Fig.11.20



We require the shaded area.
The curve y_2 meets the x-axis at $x = 0$ and $x = 2$.
Consider the regions A, B and C.

$A = \int_1^2 y_1 \, dx$.

$B = -\int_1^2 y_2 \, dx$ (as y_2 is below the x-axis in this interval).

So $\int_1^2 y_1 \, dx - \int_1^2 y_2 \, dx = A + B$.

Hence the rule is true for areas crossing the x-axis.

$C = \int_2^3 y_1 \, dx - \int_2^3 y_2 \, dx$ (as both y_1 and y_2 are above the x-axis).

Hence the total shaded area A + B + C

$$= \int_1^2 (y_1 - y_2) \, dx + \int_2^3 (y_1 - y_2) \, dx$$
$$= \int_1^3 (y_1 - y_2) \, dx$$
$$= \int_1^3 (x^2 - x^2 + 2x) \, dx$$
$$= \left[x^2\right]_1^3 = 9 - 1 = 8 \text{ units}^2$$

Example 23

The tangents at $x = 0$ and $x = 3$ on the curve $y = 2x - x^2 - 1$ meet at T.
(a) Find the equations of these tangents and the coordinates of T.
(b) Calculate the area of the region bounded by the curve and the tangents.

The curve and the tangents are shown in Fig.11.21.

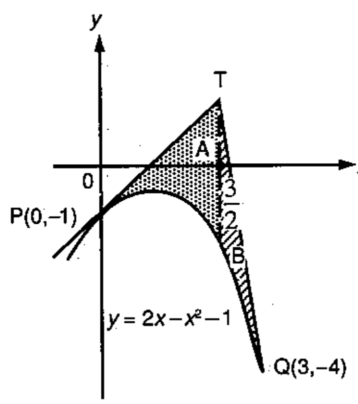


Fig.11.21

If $y = 2x - x^2 - 1$, then $\frac{dy}{dx} = 2 - 2x$.

(a) The tangent at P(0,–1) has gradient 2.
Hence its equation is $y + 1 = 2x$.

The tangent at Q(3,–4) has gradient $2 - 6 = -4$.
Hence its equation is $y + 4 = -4(x - 3)$ i.e. $y = -4x + 8$.

These lines intersect where $2x - 1 = -4x + 8$ i.e. $x = 1\frac{1}{2}$.
When $x = 1\frac{1}{2}$, $y = 2x - 1 = 2$.
So the coordinates of T are $(1\frac{1}{2}, 2)$.

(b) The shaded area is divided into two parts A and B as the boundary line changes at T $(x = \frac{3}{2})$.

$$\text{Area of A} = \int_0^{3/2} [(2x - 1) - (2x - x^2 - 1)]\, dx$$
$$= \int_0^{3/2} x^2\, dx = \frac{27}{24} = \frac{9}{8}$$
$$\text{Area of B} = \int_{3/2}^{3} [(-4x + 8) - (2x - x^2 - 1)]\, dx$$
$$= \int_{3/2}^{3} (x^2 - 6x + 9)\, dx$$
$$= \left[\frac{x^3}{3} - 3x^2 + 9x\right]_{3/2}^{3}$$
$$= (9 - 27 + 27) - (\tfrac{27}{24} - \tfrac{27}{4} + \tfrac{27}{2}) = \frac{27}{24} = \frac{9}{8}$$

Hence the total shaded area = $\frac{9}{4}$ units^2.

Exercise 11.4 *(Answers on page 631.)*

1 Find the area of the region bounded by the curve $y = x^2 - 9$ and the x-axis.

2 Calculate the area enclosed by the curve $y = 3x - x^2$, the x-axis and the lines $x = -1$, $x = 2$.

3 Find where the curve $y = x^2 - x - 1$ meets the line $y = 5$. Hence find the area of the region bounded by the curve and the line $y = 5$.

4 Part of the curve $y = x(x - 1)(x - 2)$ is shown in Fig.11.22. Find the values of a and b. Hence find the area of the region enclosed by the curve and the x-axis from $x = 0$ to $x = b$.

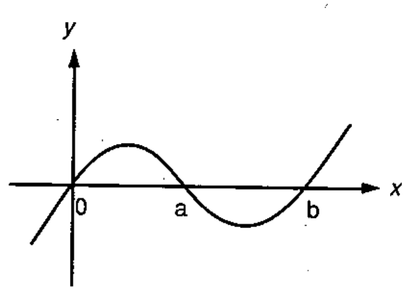


Fig.11.22

5 Find the area of the region enclosed by the following curves or lines:

(a) $y = 2x$, $y = x^2$

(b) $y = x^2$, $y = 4$

(c) $y = x^2 - 2$, $y = \frac{1}{2}x^2$

(d) $y = x^2$, $y = x^3$

(e) $y = 2x^2$, $y = x + 1$

(f) $y = x(2 - x)$, $y = x$

(g) $y = 2 - x^2$, $y = -2$

(h) $y = x^2 + 3$, $y = 5 - x$

(i) $y = x^2 - 1$, $y = x + 1$

6 In Fig.11.23, the curve $(y - 1)^2 = x + 4$ meets the y–axis at A and B.
(a) Find the coordinates of A and B and (b) calculate the areas of
(i) the shaded region P, (ii) the shaded region Q.

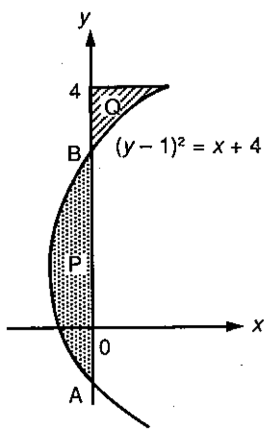


Fig.11.23

7 For a curve, $\frac{dy}{dx} = 2x + k$ where k is a constant, and the curve has a turning point where $x = 2$.

(a) If it passes through the point $(-1,8)$, find its equation.

(b) The line $y = x + 3$ meets the curve at points A and B. Find the coordinates of A and B.

(c) Hence find the area of the region enclosed by the curve and the line.

8 (a) Sketch the curve $y = x(3 - x)$.

(b) Find the equation of the normal to the curve at the origin and the x-coordinate of the point where this normal meets the curve again.

(c) Find the area of the region bounded by the curve and the normal.

9 The normal at the point $A(x = 0)$ on the curve $y = 2 - x - x^2$ meets the curve again at B. Find (a) the coordinates of B, and (b) the area of the region bounded by the curve and the normal.

10 Fig.11.24 shows part of the curve $y = 1 - \frac{1}{x^2}$.
Find (a) the coordinates of the point A where the curve meets the x-axis and (b) the equation of the tangent to the curve at A. (c) The line through B(2,0) parallel to the y-axis meets the curve at C and the tangent at T. Find the ratio of the areas of the shaded regions ABC and ACT.

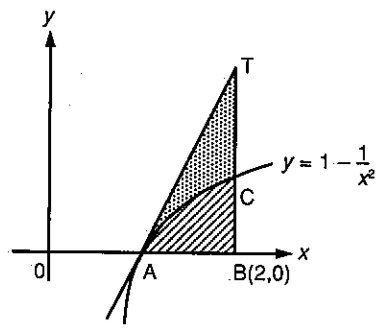


Fig.11.24

11 Fig.11.25 shows part of the curve $y = x^2$ and the line $y = 4$. The line AB is drawn through A(0,2) with gradient -1 to meet the curve at B. Find (a) the coordinates of B, and (b) the ratio of the shaded areas P and Q.

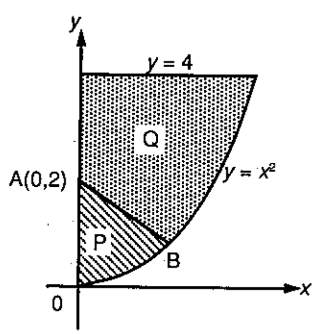


Fig.11.25

12 (a) Sketch the curve $y = x(4 - x)$.
(b) Find the equations of the tangents to the curve at the origin O and at the point where $x = 3$.
(c) If these tangents meet at T, find the x-coordinate of T and the area of the region enclosed by the tangents and the curve.

13 Fig.11.26 shows part of the curve $y = x^2 - x + 2$ and a line UV.

(a) Find the coordinates of U and V and the equation of UV.

(b) Hence find the area of the shaded region .

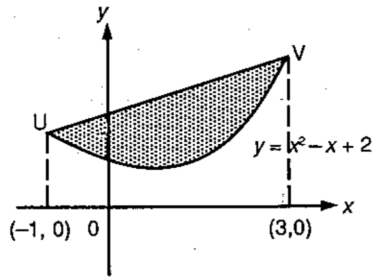


Fig.11.26

14 Fig.11.27 shows part of the curve $y = 5 - x - x^2$ and the line $y = 2x + 1$ which meet at A and B. Find

(a) the x-coordinate of A and of B, and

(b) the area of the shaded region.

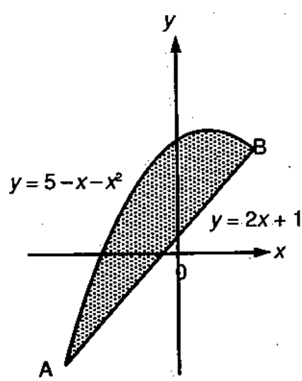


Fig.11.27

15 Fig.11.28 shows part of the curves $y = \frac{2}{x^2}$ and $y = x^2 - 4x$.

A is the point (1,2) and BC is part of the line $x = 3$. Find the area of the shaded region.

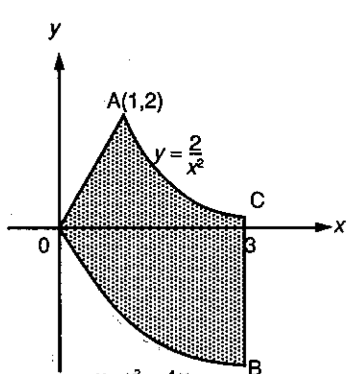


Fig.11.28

SOLIDS OF REVOLUTION

A portion of the curve $y = f(x)$ between the ordinates $x = a$ and $x = b$ is rotated about the x-axis through 360° (one revolution).

The outline will be that of a solid, called a **solid of revolution** (Fig.11.29). The x-axis is an axis of symmetry and any cross-section perpendicular to that axis will be a circle. Examples of such solids are a cylinder, a cone, a sphere etc (Fig.11.30).

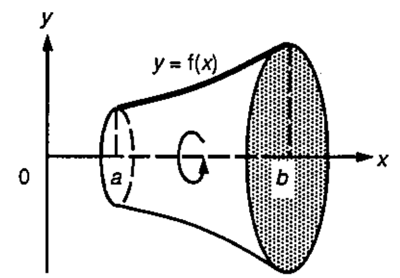


Fig.11.29

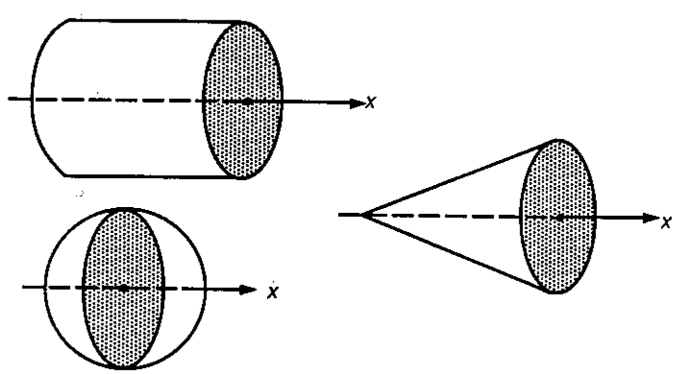


Fig.11.30

We can use calculus to find the volume of such a solid. Suppose V is the volume of the solid between $x = a$ and $x = b$ (Fig.11.31). Let x increase by δx. Then y will increase by δy and V by δV.

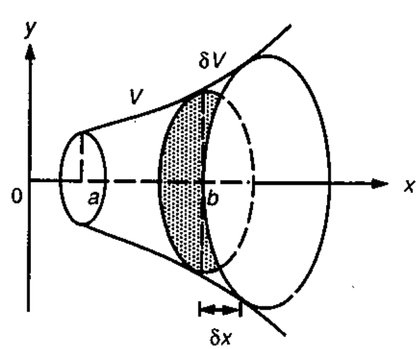


Fig.11.31

Fig.11.32 shows a section in the plane of the axes through the solid. The slice δV has two circular faces of radii y and $y + \delta y$.

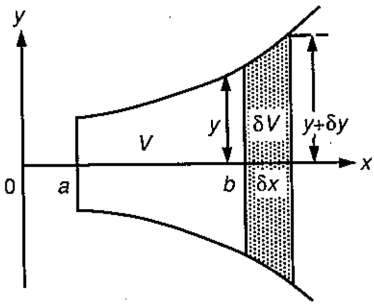


Fig.11.32

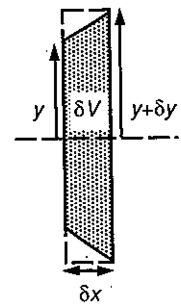


Fig.11.33

Hence its volume lies between the volumes of two cylinders of radii y and $y + \delta y$ and thickness δx (Fig.11.33).

Then $\pi y^2 \, \delta x < \delta V < \pi(y + \delta y)^2 \, \delta x$ which gives $\pi y^2 < \dfrac{\delta V}{\delta x} < \pi(y + \delta y)^2$

Now as $\delta x \to 0$, $\delta y \to 0$ and $y + \delta y \to y$ and the right hand term $\to \pi y^2$.

$\dfrac{\delta V}{\delta x} \to \dfrac{\mathrm{d}V}{\mathrm{d}x}$.

In the limit, $\dfrac{\mathrm{d}V}{\mathrm{d}x} = \pi y^2$ and $V = \displaystyle\int_a^b \pi y^2 \, \mathrm{d}x$

$$= \pi \int_a^b y^2 \, \mathrm{d}x \text{ as } \pi \text{ is a constant.}$$

The volume of revolution of $y = \mathrm{f}(x)$ between $x = a$ and $x = b$ is $\pi\int_a^b y^2 \, \mathrm{d}x$

Similarly, if a portion of the curve $y = \mathrm{f}(x)$ is rotated about the y-axis (Fig.11.34), the volume of revolution will be

$\pi \displaystyle\int_c^d x^2 \, \mathrm{d}y$ where c and d are the limits on the y-axis.

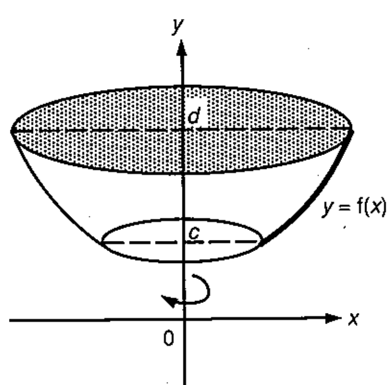


Fig.11.34

Example 24

The portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ is rotated through 360° about (i) the x-axis, (ii) the y-axis. Find the volumes created.

(i) In Fig.11.35, $V_x = \pi \int_1^2 y^2 \, dx = \pi \int_1^2 x^4 \, dx$

$$= \pi \left[\frac{x^5}{5}\right]_1^2$$

$$= \pi \left[\left(\frac{32}{5}\right) - \left(\frac{1}{5}\right)\right]$$

$$= \frac{\pi}{5}(32 - 1)$$

$$= \frac{31\pi}{5} \text{ units}^3$$

Such answers are usually left in terms of π.

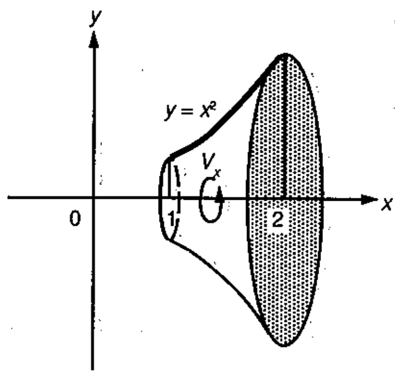


Fig.11.35

(ii) In Fig.11.36, the limits are now 1 and 4, corresponding to $x = 1$ and $x = 2$, and the function must be expressed as $x = \sqrt{y}$.

$V_y = \pi \int_1^4 x^2 \, dy = \pi \int_1^4 y \, dy$

$$= \pi \left[\frac{y^2}{2}\right]_1^4$$

$$= \pi \left(8 - \frac{1}{2}\right) = \frac{15\pi}{2} \text{ units}^3$$

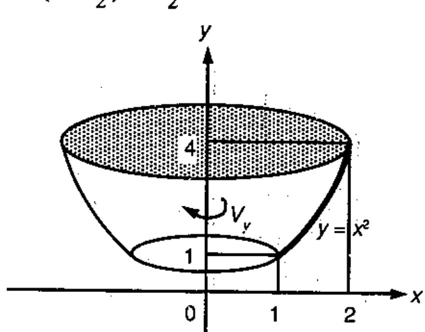


Fig.11.36

Solid of Revolution Created by a Region between Two Curves

In Fig.11.37, $y_1 = f(x)$ and $y_2 = g(x)$ are two curves intersecting at $x = a$ and $x = b$. If the shaded region is rotated about the x-axis, the volume created by y_1 is $\pi \int_a^b y_1^2\,dx$ and that by y_2 is $\pi \int_a^b y_2^2\,dx$. Hence the volume created by the region will be $\pi \int_a^b (y_1^2 - y_2^2)\,dx$.

The same principle will apply to rotation about the y-axis.

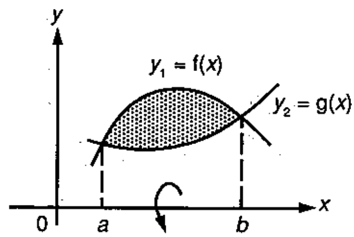


Fig.11.37

Example 25

Fig.11.38 shows the part AB of the curve $y^2 = x - 2$ where B is the point (3,1). CB is the tangent to the curve at B with gradient $\frac{1}{2}$. The curve meets the x-axis at A. Find
(a) the equation of CB,
(b) the coordinates of C,
(c) the volume swept out by the shaded region when revolved round the x-axis.

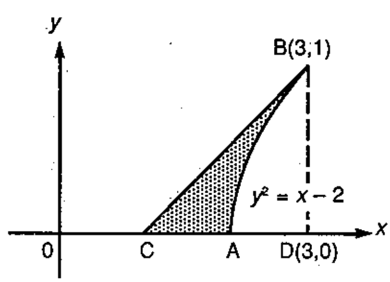


Fig.11.38

(a) Equation of CB is $y - 1 = \frac{1}{2}(x - 3)$ i.e. $2y = x - 1$ or $y = \frac{x-1}{2}$.

(b) The tangent meets the x-axis where $x = 1$ so C is (1,0).

(c) The x-coordinate of A is 2.

Between C and A the boundaries of the shaded region are the line and the x-axis, but between A and B the boundaries are the line and the curve. So we have to find the volume in two parts. The simplest method is to find the volume produced by CB and subtract the volume produced by AB.

$$\text{Volume produced by CB} = \pi \int_1^3 \left(\frac{x-1}{2}\right)^2 dx$$

$$= \frac{\pi}{4} \int_1^3 (x^2 - 2x + 1)\, dx$$

$$= \frac{\pi}{4} \left[\frac{x^3}{3} - x^2 + x\right]_1^3$$

$$= \frac{\pi}{4}(9 - 9 + 3) - \frac{\pi}{4}\left(\frac{1}{3} - 1 + 1\right)$$

$$= \frac{\pi}{4}\left(\frac{8}{3}\right) = \frac{2\pi}{3} \text{ units}^3$$

$$\text{Volume produced by AB} = \pi \int_2^3 (x - 2)\, dx = \pi \left[\frac{x^2}{2} - 2x\right]_2^3$$

$$= \pi\left(\frac{9}{2} - 6\right) - \pi(2 - 4)$$

$$= \frac{\pi}{2} \text{ units}^3$$

Hence the required volume $= \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$ units^3.

Exercise 11.5 *(Answers on page 631.)*

Unless otherwise stated, leave answers in terms of π.

1 Find the volumes created when the parts of the curves given below are rotated about the x-axis:

(a) $y = x^2$, $0 \le x \le 1$

(b) $y = x(1 - x)$, $0 \le x \le 2$

(c) $y = \frac{1}{x}$, $1 \le x \le 2$

(d) $y = x - \frac{1}{x}$, $1 \le x \le 2$

(e) $y = \sqrt{x}$, $1 \le x \le 4$

(f) $y = \sqrt{4 - x^2}$, $1 \le x \le 2$

2 Find the volumes made by rotating the parts of the following curves about the axis stated:

(a) $y = x^2 + 1$, $0 \le x \le 2$; x-axis

(b) $y = \sqrt{x}$, $0 \le x \le 9$; y-axis

(c) $y = x^2 - x$, $0 \le x \le 1$; x-axis

(d) $y = \frac{1}{x}$, $1 \le x \le 3$; x-axis

(e) $y = 1 - x^2$, $0 \le x \le 2$; y-axis

3 The part of the curve $y = x^2 + 1$ between $x = 1$ and $x = 2$ is rotated about the y-axis through 360°. Find the volume formed.

4 The negative part of the curve $y = x^2 - 2x$ is rotated completely about the x-axis to form a solid of revolution. Find its volume.

5 (a) Find the coordinates of the points of intersection of the line $y = 2x$ and the curve $y = x^2$.

(b) The region enclosed by the curve and the line between these points is rotated about the x-axis to form a solid. Find its volume.

6 The region between the curve $y = x^2 + 1$, the 2 axes and the line $x = 1$ is revolved round the x-axis. Find the volume generated.

7 Sketch the curve $y^2 = x + 4$. The area bounded by the curve and the y-axis is rotated about the y-axis through 360°. Calculate the volume created correct to 3 significant figures.

8 (a) A point P(x,y) moves so that it is always 2 units from the origin O. State the relation between x and y and the name of the curve this relation represents.

(b) The part of this curve above the x-axis is rotated about that axis to form a solid. Find the volume of the sphere created.

9 Fig.11.39 shows an ellipse whose equation is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

(a) State the coordinates of the points A and B.

(b) If the part above the x-axis is rotated about that axis through 360°, find the volume of the ellipsoid formed.

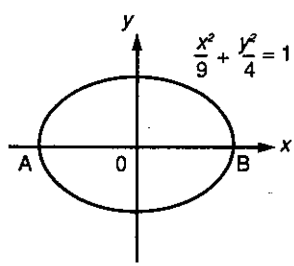


Fig. 11.39

10 Fig.11.40 shows part of the curve $y = 4x - x^2$ and a line OA where O is the origin. The x-coordinate of A is 2.

(a) Find the equation of OA.

(b) If the shaded region is rotated about the x-axis, find the volume formed.

(c) If the curve meets the x-axis at B, what is the volume created if the unshaded region OBA is rotated about the x-axis?

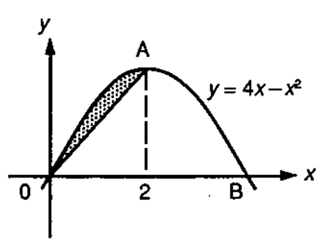


Fig.11.40

11 Show on a diagram the region enclosed by the curve $y = x^2 + 2$ and the line $y = 4x - 1$, stating the coordinates of the points of intersection. If this region is rotated completely round the x-axis, find the volume of the solid formed.

12 The curves $y = x^2$ and $y = 2 - x^2$ for $x > 0$, intersect at A. Find the coordinates of A. The region bounded by the curves and the y-axis is rotated about the x-axis through 360°. Find the volume of revolution.

13 In Fig.11.41, the curves $y^2 = 4x$ and $y = 2x^2$ intersect at O and A.

(a) Find the coordinates of A.

(b) The region bounded by the two curves is rotated about (i) the x-axis, (ii) the y-axis. Find the ratio of the two volumes created.

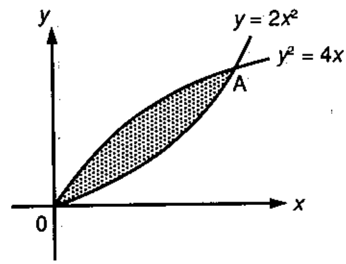


Fig.11.41

14 Fig.11.42 shows parts of the curves $y = \frac{1}{x}$ $(x > 0)$ and $y^2 = x$ which intersect at A.

(a) Find the coordinates of A.

(b) The shaded region between the curves, the x-axis and the line $x = 3$ is rotated about the x-axis. Find the volume of revolution.

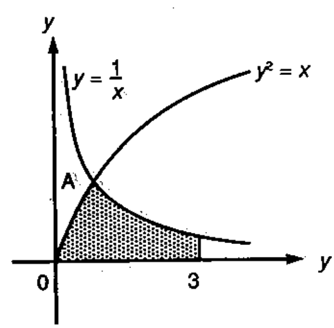


Fig.11.42

15 Copy Fig.11.43, which shows part of the curve $y = \frac{1}{x}$ and add the curve $y = x^2$ and the line $y = 2$. If the region bounded by these curves and the line is rotated about the x-axis, find the volume generated.

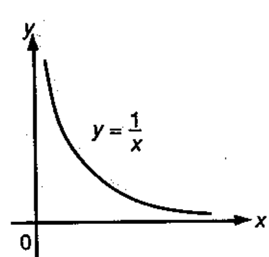


Fig.11.43

16 OA lies on the line $y = 3x$ ($x \geq 0$), where O is the origin, and AB is part of the curve $y = 4 - x^2$ ($x \geq 0$). B lies on the x-axis. Find (a) the coordinates of A and B, and (b) the volume of revolution made by rotating the region OAB about the x-axis.

17 Fig.11.44 shows part of the curve $y = x - \frac{1}{x}$ for $x > 0$. The curve meets the x-axis at A and the line $y = 1\frac{1}{2}$ at C. Find (a) the coordinates of A and C and (b) the volume made by rotating the shaded region about the x-axis.

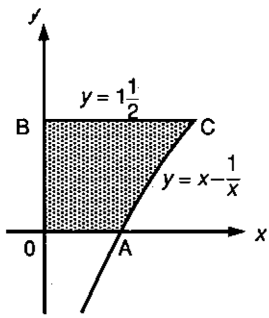


Fig.11.44

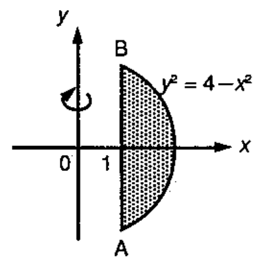


Fig.11.45

18 Fig.11.45 shows an arc of the circle $y^2 = 4 - x^2$ and a chord AB which lies on the line $x = 1$. Show that
(a) the coordinates of B are $(1, \sqrt{3})$,
(b) the volume created when the shaded region is rotated about the y-axis is $4\pi\sqrt{3}$ units^3.

19 B is the point $(h,0)$ and A the point (h,r).
(a) Find the equation of OA, where O is the origin, in terms of h and r.
(b) If OA is rotated through 360° about the x-axis, which solid is formed?
(c) Find the volume of this solid in terms of h and r.

20 Fig.11.46 shows the part of the curve $y = \frac{1}{x}$ for $x > 0$. A is the point (1,1) and the tangent to the curve at A meets the x-axis at B. C is the point (3,0). Find
(a) the equation of the tangent and the coordinates of B, and
(b) the volume made by revolving the shaded area round the x-axis.

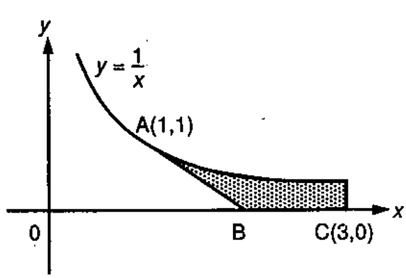


Fig.11.46

21 Fig.11.47 shows the part AB of the curve $y = x^2 + 2$ where B is the point (2,6). The tangent at B to the curve meets the x-axis at C. Find
   (a) the equation of the tangent,
   (b) the coordinates of C, and
   (c) the volume of the solid formed by rotating the shaded region completely about the x-axis.

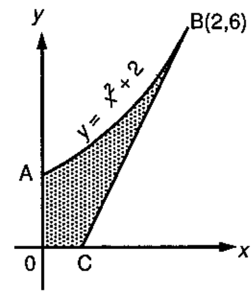
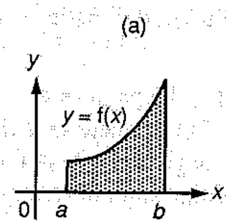


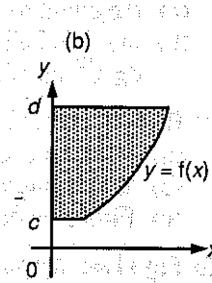
Fig. 11.47

SUMMARY

- If $\dfrac{dy}{dx} = f(x)$, then $y = \int f(x)\,dx + c$ (indefinite integral).
- If $\dfrac{dy}{dx} = ax^n$, $y = \dfrac{ax^{n+1}}{n+1} + c$, $(n \neq -1)$.
- Definite integral $\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$ where $g(x)$ is the indefinite integral of $f(x)$.
- Area between $y = f(x)$, the x-axis and $x = a$, $x = b$ (Fig. 11.48(a))

  $= \int_a^b y\,dx = \int_a^b f(x)\,dx.$





- If $f(x) < 0$ for $a \leq x \leq b$, the area

  $= -\int_a^b f(x)\,dx.$

- Area between $y = f(x)$ and the y-axis for $c \leq y \leq d$ (Fig. 11.48(b))

  $= \int_c^d x\,dy$ where x is found in terms of y.

- Area between $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ and $f(x) > g(x)$ Fig. 11.48(c))

  $= \int_a^b [f(x) - g(x)]\,dx.$

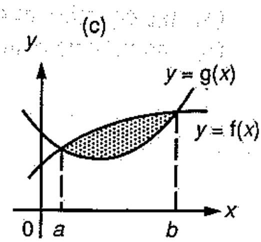


Fig. 11.48

- If $y = f(x)$ for $a \leq x \leq b$ is rotated completely round the x-axis, the volume of the solid of revolution produced $= \pi \int_a^b y^2\,dx$: if rotated about the y-axis, volume $= \pi \int_c^d x^2\,dy$, where c, d are the limits for y.

REVISION EXERCISE 11 *(Answers on page 632.)*

Answers for volumes may be left in terms of π.

A

1 On a curve for which $\frac{dy}{dx} = p + x$, where p is a constant, the tangent at the point (2,5) has a gradient of -2. Find the value of p and the equation of the curve.

2 Evaluate (a) $\int_{-2}^{-1} (x - \frac{1}{x^2})\,dx$ (b) $\int_{1}^{2} \frac{x+1}{x^3}\,dx$

3 Fig.11.49 shows part of the line $y = 2x$ and part of the curve $y = 4x - x^2$. Calculate the ratio of the areas of the regions P and Q. (C)

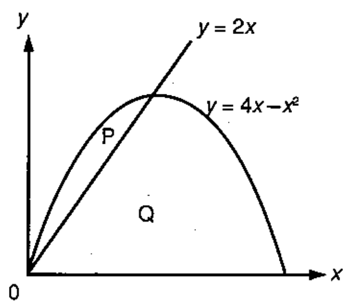


Fig. 11.49

4 A particle, moving in a straight line, passes through a fixed point O with a velocity of 8 m s^{-1}. Its acceleration, a m s^{-2}, t seconds after passing O is given by $a = 12 - 6t$. Find
(i) the velocity of the particle when $t = 2$,
(ii) the displacement of the particle from O when $t = 2$. (C)

5 Fig.11.50 shows part of the curve $y = x^2 - 2x + 2$ and a chord PQ. Find
(i) the coordinates of P and Q,
(ii) the ratio of the area of the shaded region A to the area of the shaded region B. (C)

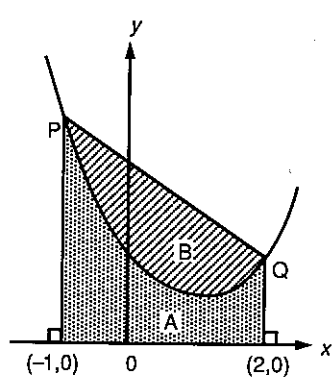


Fig.11.50

6 Fig.11.51 shows part of the curve $y = 5 + 4x - x^2$. A is the maximum point of the curve. Find

(a) the coordinates of A,

(b) the equation of OA

(c) the area of the shaded region.

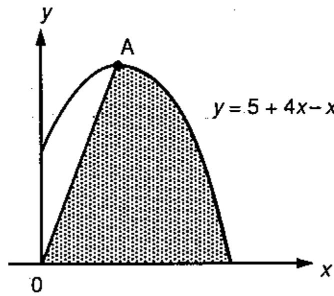


Fig.11.51

7 Fig.11.52 shows part of the curve $y = 6x - x^2$ and the line $y = 3x$. Show that the area enclosed by the curve and the x-axis is 36 units^2. Calculate the ratio of the areas of the regions marked A and B. (C)

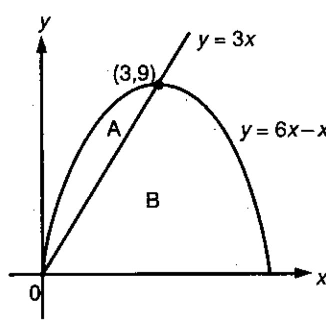


Fig.11.52

8 Fig. 11.53 shows part of the curve $y^2 = x$ and the line $2y = x$. If the shaded region is rotated through 360° about the y-axis and the x-axis, find the ratio of the volumes formed.

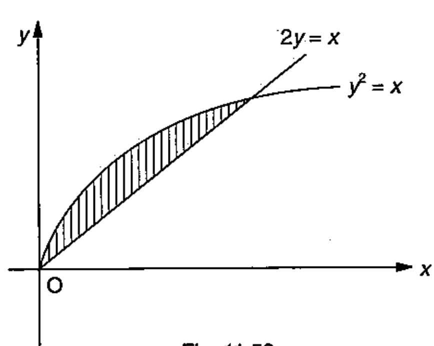


Fig. 11.53

9 In Fig. 11.54, the line $x + y = 5$ meets the curve $y = \frac{4}{x}$ at A and B.
Find (a) the coordinates of A and B, and (b) the volume obtained by rotating the shaded area round the x-axis through 360°.

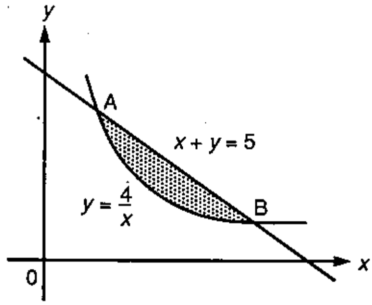


Fig. 11.54

10 A particle travelling in a straight line passes a fixed point O with a velocity of $1\frac{1}{2}$ m s^{-1}. It moves in such a manner that, t seconds after passing O, its acceleration a m s^{-2}, is given by $a = p + qt$, where p and q are constants.
Given that its velocity is $3\frac{1}{2}$ m s^{-1} when $t = 2$ and that it comes instantaneously to rest when $t = 3$, calculate the value of p and of q.
Find the distance travelled by the particle between $t = 1$ and $t = 2$. (C)

11 (a) Find $\int \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$

(b) In Fig. 11.55, the line $y = \frac{1}{2}x$ is the tangent to the curve $y^2 = x - 1$ at the point (2,1). Calculate the volume swept out when the shaded region shown is rotated through 360° about the x-axis. (C)

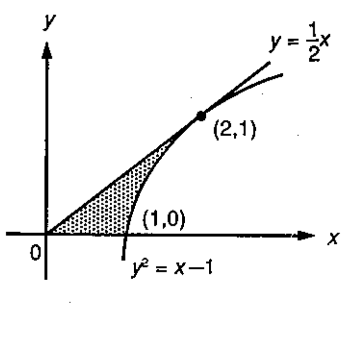


Fig. 11.55

12 A particle passes a fixed point O with a velocity of 3 and moves in a straight line with acceleration a given by $a = 3 - 2t$ where t is the time in seconds after passing O. Find the velocity and the distance of the particle from O when $t = 2$.

13 Calculate the volume generated when the shaded region in Fig. 11.56 is rotated through four right angles about the x-axis. (C)

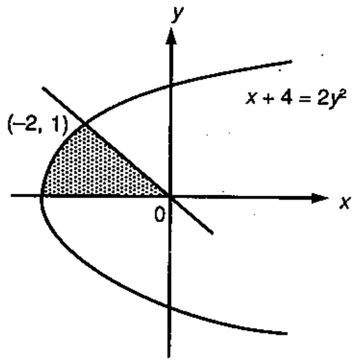


Fig. 11.56

14 Fig. 11.57 shows part of the curve $y^2 = x + 1$. AB, CD are parallel to the y-axis where A is the point (1,0) and C is the point $(k,0)$. If the volume produced by rotating the region Q about the x-axis is 3 times the volume produced by rotating the region P, find the value of k.

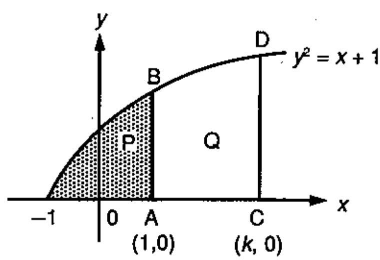


Fig. 11.57

15 The tangent at the point (2,4) on the curve $y = x^2$ meets the x-axis at A.
(a) Find the coordinates of A.
(b) If the region bounded by the curve, the x-axis and the tangent is rotated about the x-axis through 360°, find the volume created.

16 The part of the curve $y = 4 - x^2$ lying in the first quadrant meets the y-axis at A and the x-axis at B. C lies on the curve and the equation of OC is $y = 3x$, where O is the origin.
(a) Find the coordinates of A, B and C.
(b) The region OAC is rotated about the y-axis and the region OBC is rotated about the x-axis. Find the ratio of the volumes of revolution produced.

17 (a) Explain the geometrical meaning of the result $\int_0^6 \mathrm{f}(x)\,\mathrm{d}x = 0$ if $\mathrm{f}(x)$ is not everywhere zero.
(b) Given that $\int_0^4 \mathrm{p}(x)\,\mathrm{d}x = 6$ evaluate
(i) $\int_0^4 3\mathrm{p}(x)\,\mathrm{d}x$,
(ii) $\int_0^4 [2 - \mathrm{p}(x)]\,\mathrm{d}x$,

(iii) $\int_0^2 [p(x) - 2]\,dx + \int_2^4 [p(x) + 1]\,dx$

18 Fig. 11.58 shows part of the curve $(y - 2)^2 = x + 4$ and part of the line $y = \frac{1}{2}x$. Find (a) the coordinates of B, (b) the area of the shaded region.

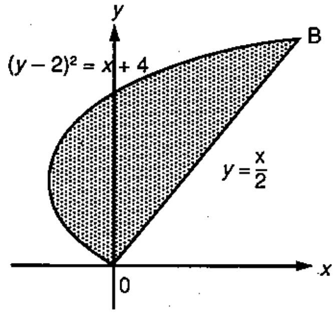


Fig. 11.58

19 Fig. 11.59 shows part of the curve $y = x - \frac{1}{x^2}$. Given that C is the point (2,0), find

(i) the equation of the tangent to the curve at the point A,
(ii) the coordinates of the point T where this tangent meets the x-axis,
(iii) the coordinates of the point B where the curve meets the x-axis,
(iv) the area of the region enclosed by the curve and the lines AT and BT,
(v) the ratio of the area found in part (iv) to the area of the triangle ATC. (C)

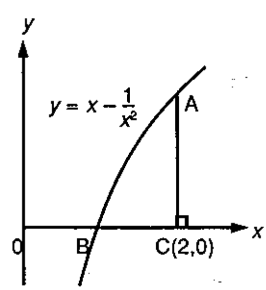


Fig. 11.59

20 Part of the curve $y = 2x^2$ is shown in Fig. 11.60 where A is the point $(a,0)$. AB is parallel to the y-axis and BC parallel to the x-axis.

(a) Show that the area of region P is twice that of region Q.
(b) If both regions are rotated about the y-axis, show the volumes produced are equal.

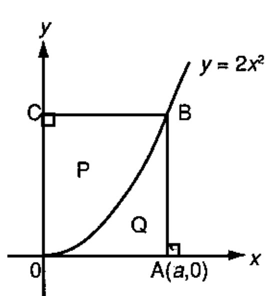


Fig. 11.60

21 $y = ax^2 + bx + c$ where a, b and c are constants. Given $\int_{-t}^{t} xy \, dx = 0$ find the value of b. If also $y = 2$ when $x = 1$ and $\int_{-1}^{1} y \, dx = 0$ find the value of a and of c.

B

22 (a) If $\int_0^4 f(x) \, dx = 12$, find the value of $\int_1^5 f(x-1) \, dx$.

(b) What is the value of a if $\int_0^3 f(x+a) \, dx = \int_2^5 f(x) \, dx$.

23 If $y = x^2 + 1$, evaluate $\dfrac{\int_0^2 xy \, dx}{\int_0^2 y \, dx}$

24 Sketch the graph of $y = |x^2 - 2x|$. Hence find $\int_0^3 |x^2 - 2x| \, dx$.

25 A solid bowl is formed by rotating the parts of the curves $y = x^2$ and $y = x^2 - 1$ for $x > 0$ and $0 \le y \le 1$ about the y-axis (Fig. 11.61). Calculate

(a) the capacity of the bowl, i.e. the amount of liquid it could hold,

(b) the volume of material in the bowl.

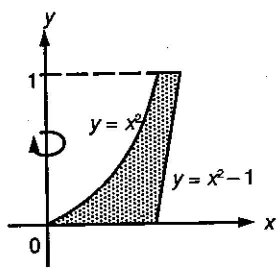


Fig. 11.61

26 Fig. 11.62 shows part of the curve $y^2 = 4x$. P is any point on the curve and PN is perpendicular to the x-axis.

Show that the volume generated by rotating the shaded region about the x-axis is equal to $\frac{\pi}{6} \times \text{ON} \times \text{PN}^2$.

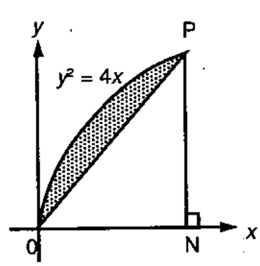


Fig. 11.62

27 The region below the curve $y = ax^2$ (where a is positive) for $0 \le x \le 2$ is rotated about the x-axis, while the region between the curve and the y-axis for $0 \le y \le 4a$ is rotated about the y-axis. Find the value of a if these volumes are equal.

28 Sketch for $0 \le x \le 4$ the graph of the positive function f, where

$$\mathrm{f} : x \longmapsto \frac{3x+2}{2} \text{ for } 0 \le x \le 1,$$

$$\mathrm{f} : x \longmapsto \frac{6-x}{2} \text{ for } 1 \le x \le 2 \text{ and}$$

$$\mathrm{f} : x \longmapsto \sqrt{8-2x} \text{ for } 2 \le x \le 4.$$

Show that the gradients of the last two parts are equal where $x = 2$. If the resulting figure is rotated about the x-axis through 360°, calculate (in terms of π) the volume produced.

Revision Papers 1 – 5

PAPER 1 *(Answers on page 632.)*

1 (a) Differentiate $\left(x - \frac{1}{2x}\right)^2$ with respect to x.

(b) Evaluate $\int_{-1}^{2} (x-2)^2 \, dx$

2 Solve the simultaneous equations $2x - 3y = 7$, $xy = x^2 - 6$.

3 The functions f and g are defined for $x > 0$ as $f : x \longmapsto 3 - x$, $g : x \longmapsto \frac{4}{x}$. Show that (a) f and g are each self-inverse functions, and (b) $(fg)^{-1} = gf$. (c) Prove that $fg(x) = gf(x)$ has no real solutions.

4 Given the points A(–1,2), B(3,1) and C(4,–2), find

(a) the equation of the line AB,

(b) the equation of the line through C perpendicular to AB.

5 On the same diagram, sketch the graphs of $y = |x - 1|$ and $y = |x + 2|$. Hence find the range of values of x for which $|x - 1| > |x + 2|$.

6 $\overrightarrow{AB} = 2\mathbf{i} + \mathbf{j}$ and A is the point (–3, 2).

(a) State the coordinates of B.

(b) If C is the point (0,–1), use a vector method to find $\angle ABC$.

7 (a) Prove the identity $(\cos\theta + \sin\theta)^2 \equiv 2 - (\cos\theta - \sin\theta)^2$.

(b) Solve the equation $\operatorname{cosec} 2\theta = -2.15$ for $0° \le \theta \le 360°$.

8 Calculate the area of the region lying between the curve $y = x^2 - 3x + 2$, the x-axis and the lines $x = -1$, $x = 2$.

9 The sector OAC of a circle centre O and radius 4 cm has an area of 12 cm². Find

(a) the angle of the sector in radians,

(b) the perimeter of the sector,

(c) the area of the segment cut off by the chord AC.

10 (a) For what values of k is the function $x^2 + 2kx + 5$ always positive?

(b) Given that $f(x)$ is a quadratic function and that $f(x)$ is only positive when x lies between –1 and 3, find $f(x)$ if $f(-2) = -10$.

PAPER 2 *(Answers on page 632.)*

1 The perimeter of a sector of a circle is 8 cm and its area is 4 cm².
Find (a) the angle of the sector, (b) the length of its arc.

2 Find the solutions of these equations for $0° \le \theta \le 360°$:
(a) $\sin 2\theta = -0.4$ (b) $\cos(\theta + 30°) = 0.25$

3 A particle starts from rest at a point O and moves in a straight line with velocity v m s^{-1} given by $v = 6t - 3t^2$ where t is the time in seconds after the start. Find
(a) its acceleration when the particle is next at instantaneous rest,
(b) the distance travelled to that position.

4 (a) Find the coefficient of x^5 in the expansion of $(2x - 3)^7$.
(b) If the first two terms in the expansion of $(ax + b)^5$ in descending powers of x are $32x^5 - 80x^4$, find the value of a and of b.

5 The sides AB and AC of a triangle lie on the lines $2x - y = 5$ and $x + 3y = 13$ respectively. Given that C is the point $(-2,5)$ and that $\angle ACB = 90°$, find the coordinates of A and of B.

6 Fig.R1 shows part of the curve $y = 2 - x - x^2$ and the lines $y = 2$ and $y = -4$. Find the area of the shaded region.

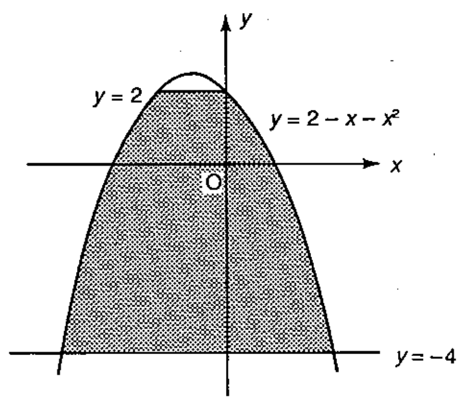


Fig. R1

7 The gradient of the curve $y = \frac{a}{x} + bx^2$ at the point $(1,-1)$ is -8. Find the values of a and b. Hence find the equation of the tangent to the curve where $x = 2$.

8 Given the function $y = x^2 - 3x + 2$, find the approximate percentage change in the value of y if x is increased by 2% when it is 3.

9 (a) The function f is defined by $f : x \longmapsto \frac{x+4}{x-2}$ $(x \ne 2)$.
Find (i) $f^{-1}(3)$ and (ii) the values of x for which $f(x) = x$.

(b) The function g is given by $g : x \longmapsto a - \frac{3}{x}$ $(x \ne 0)$ where a is a constant. If $g(2) - g^{-1}(-2) = 2$ find the values of a.

10 The position vectors of two points A and B referred to an origin O are $2\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + 4\mathbf{j}$ respectively. Find
(a) $\overrightarrow{AB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$,
(b) angle AOB.

PAPER 3 *(Answers on page 633.)*

1 (a) Find (i) the range of values of x for which the function $x^3 - 3x^2 - 9x + 1$ is increasing and (ii) the nature of the stationary points on the curve and the values of x where they occur.

(b) Find the maximum value of $(x + b)(a - x)$ where a and b are constants.

2 Solve the simultaneous equations $x + 2y = 8$, $x^2 - xy + y^2 = 7$.

3 A cone of radius 6 cm and height 24 cm is held vertex down with its axis vertical. Water is poured into the cone at the rate of 90 $cm^3\ s^{-1}$. At what rate is the water level rising when its greatest depth is 12 cm?

4 (a) A and B are points with position vectors $4\mathbf{i} + t\mathbf{j}$ and $\mathbf{i} - t\mathbf{j}$ respectively with reference to an origin O. If OA is perpendicular to OB, find the values of t.

(b) T has position vector $-2\mathbf{i} + 4\mathbf{j}$ and the line TP is parallel to the vector $\mathbf{i} + 3\mathbf{j}$. Show that the position vector of P is given by $\overrightarrow{OP} = (t-2)\mathbf{i} + (3t+4)\mathbf{j}$ where t is a scalar. Hence find the value of t for which OP is perpendicular to TP.

5 Find the volumes created when the shaded region in Fig.R2 is rotated about

(a) the x-axis,

(b) the y-axis,

each through 360°.

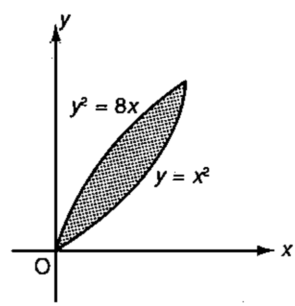


Fig. R2

6 (a) Find the equation of the normal to the curve $y = \frac{1}{x} - 2x$ at the point where $x = 1$.

(b) Evaluate $\int_1^3 \frac{2x^2 + 3}{x^2}\, dx$.

7 (a) Find and simplify the first three terms in the expansions, in ascending powers of x, of $(1 + 3x)^4$ and $(2 - x)^4$ and hence find the coefficient of x^2 in the expansion of $(2 + 5x - 3x^2)^4$.

(b) Find the range of values of y for which $|\ y^2 - 5y - 1\ | \leq 5$.

8 (a) Find the coordinates and type of turning points on the curve $y = x^3 + 2x^2 + x + 1$.

(b) If $P = \frac{1}{t} - \frac{t^3}{3}$ find $\frac{dP}{dt}$. Hence find the approximate percentage change in P, stating if P is increased or decreased, when t is decreased from 2 by 0.5%.

9 In $\triangle$OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the midpoint of AB. T lies on OB where $\overrightarrow{OT} = \frac{1}{3}\overrightarrow{OB}$. OM and TA intersect at N. Taking $\overrightarrow{TN} = k\overrightarrow{TA}$ and $\overrightarrow{ON} = m\overrightarrow{OM}$, find two vector expressions for $\overrightarrow{ON}$ in terms of k, m, $\mathbf{a}$ and $\mathbf{b}$ and hence find

(a) the values of k and m, and

(b) the ratios ON:NM and TN:NA.

10 (a) Solve the equation $3 \sin^2 \theta = 2 \cos \theta + 2$ for values of θ in the range $0° \leq \theta \leq 360°$.

(b) A particle moves along a straight line so that its velocity v m s^{-1} at time t seconds from the start is given by $v = t^2 - 4t$.

(i) Find the time after the start when the particle is at instantaneous rest.

(ii) What is the velocity of the particle when its acceleration is zero?

(iii) Find the displacement of the particle during the first 3 seconds of motion.

PAPER 4 *(Answers on page 633.)*

1 (a) Differentiate with respect to x (i) $(x^2 - x + 1)^2$ (ii) $x - \frac{2}{2-x}$.

(b) Show on a diagram a sketch of the curves $y = \cos 2x$ and $y = |2 \sin x|$ for $0 \leq x \leq 2\pi$. State how many values of x will satisfy the equation $\cos 2x = |2 \sin x|$ in that range.

2 (a) Given that $A = 3r + \frac{1}{r}$, find the rate of change of r with respect to t when $r = 2$ if the rate of change of A with respect to t is $\frac{11}{8}$.

(b) Solve the equation $\cos \frac{\theta}{2} = -0.35$ for $0° \leq \theta \leq 360°$.

3 A curve passes through the point (0,2) and its gradient at any point (x,y) on the curve is $1 - x - 2x^2$. Find

(a) the equation of the curve,

(b) the coordinates of the turning points on the curve, stating the nature of each one,

(c) the equation of the tangent to the curve at the point where $x = 1$.

4 In Fig. R3, the curve $y^2 = x + 1$ cuts the y-axis at A and meets the line $x = 3$ at B. Find the ratio of the volumes produced by rotation through 360° about the x-axis of the shaded regions P and Q.

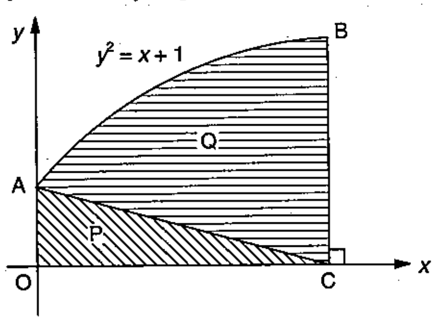


Fig. R3

5 (a) For each of the following functions, find the range of f(x) corresponding to the domain given:

(i) $f(x) = |3 - x|$ for $-1 \leq x \leq 4$

(ii) $f(x) = 4 - x^2$ for $-1 \leq x \leq 3$

(iii) $f(x) = x^2 - 4x + 3$ for $0 \leq x \leq 3$

(b) $\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j}$, $\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{r}_3 = \mathbf{i} + \mathbf{j}$ are three vectors. Find

(i) $|\mathbf{p}|$ where $\mathbf{p} = \mathbf{r}_1 - \mathbf{r}_2 + 2\mathbf{r}_3$,

(ii) the product $(\mathbf{r}_1 + \mathbf{r}_2).\mathbf{p}$,

(iii) the value of t if $\mathbf{r}_1 + t\mathbf{r}_2$ is perpendicular to $\mathbf{r}_2 + \mathbf{r}_3$.

6 (a) If $y = 6 - x - x^2$, find (i) the range of values of x for which y is positive and (ii) the maximum value of y.

(b) If $f : x \longmapsto \frac{x}{x+3}$ $(x \neq -3)$, find in a similar form f^2 and f^3.

(c) Given that $f : x \longmapsto x^2 + 2$ and $g : x \longmapsto x - 1$, find the value of x for which $fg(x) = gf(x)$.

7 OAB is a triangle with $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$. OP lies on OA where OP:PA = 2:1 and Q lies on AB where AQ:QB = 2:1. OQ and PB intersect at R.

(a) State $\vec{AB}$, $\vec{AQ}$, $\vec{OQ}$ and $\vec{PB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) By taking $\vec{PR} = p\vec{PB}$, show that $\vec{PR} = \frac{2}{3}(1 - p)\mathbf{a} + p\mathbf{b}$.

(c) If $\vec{OR} = q\vec{OQ}$ find the values of p and q.

(d) Hence find the ratios OR:RQ and PR:RB.

8 (a) Find the values of $\frac{d^2y}{dx^2}$ on the curve $y = 2x(x - 3)^2$ where $\frac{dy}{dx} = 0$.

(b) A, B, C and D are the points (–5,2), (2,3), (–1,–6) and (3,10) respectively. Find (i) the coordinates of the point M where AD and BC intersect, (ii) the equation of the line through M perpendicular to AB.

9 (a) Sketch the curve $y = |x^2 - x - 2|$ for $-2 \leq x \leq 1$ and hence find the finite area enclosed by the curve and the x-axis.

(b) OABC is a quadrilateral in the first quadrant where O is the origin. The equation of OA is $2y = x$ and the equation of OC is $y = 2x$. If the coordinates of B are (6,6), find the coordinates of A and C and show that OABC is a rhombus. Find the area of the rhombus.

10 In an acute-angled triangle ABC, the base BC = a and the height of the triangle = h. A rectangle PQRS is drawn inside the triangle with P and Q on BC, R on CA and S on AB. If PS = x, find an expression for the area of this rectangle in terms of a, h and x and hence find the maximum area of the rectangle.

PAPER 5 *(Answers on page 633.)*

1 A piece of wire 120 cm long is bent into the shape shown in Fig.R4. Show that the area A cm^2 is given by $A = 480x - 60x^2$. Hence find the value of x which gives the maximum area.

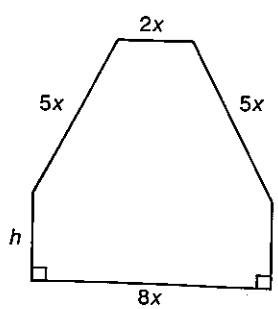


Fig. R4

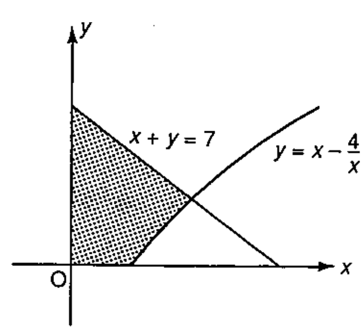


Fig. R5

2 Fig.R5 shows a part of the curve $y = x - \frac{4}{x}$ and the line $x + y = 7$.
Calculate the volume produced when the shaded region is rotated about the x-axis.

3 (a) Find, in ascending powers of x, the first three terms in the expansions of
(i) $(1 - 2x)^4$ and (ii) $(2 - x)^5$.
Hence find the coefficient of x^2 in the expansion of $(1 - 2x)^4(2 - x)^5$.

(b) The function f is defined as $f : x \longmapsto \frac{20}{ax + b}$. Given that $f(1) = 4$ and $f^{-1}(-4) = -4$, find (i) the value of a and of b, (ii) the values of x for which $f(x) = x$.

4 (a) The radius r of a sphere increases from 2 to 2.01. Find the approximate change in the surface area A. [$A = 4\pi r^2$].

(b) The radius r of a sphere is increasing at a constant rate of 0.02 cm s^{-1}.
Find the rate at which the volume is increasing when $r = 3$ cm.
[$V = \frac{4}{3}\pi r^3$].

(c) The volume V of a sphere decreases by approximately 6% when the radius decreases by p%. Find the value of p.

5 (a) Given that $\int_1^4 f(x)dx = 7$, find the values of

(i) $\int_1^2 [f(x) + 1]dx + \int_2^4 [f(x) - 2]dx$

(ii) $\int_1^4 2f(x)dx$

(iii) $\int_0^3 f(x + 1)\, dx$

(b) On one diagram, sketch the graphs of
(i) $y = \sin 2x$, (ii) $y = |\sin x|$ for $0 \le x \le 2\pi$.
State the number of solutions of the equation $\sin 2x = |\sin x|$ in this range.

(c) Prove the identity
$(\cos\theta - \sin\theta)(\operatorname{cosec}\theta - \sec\theta) \equiv \sec\theta \operatorname{cosec}\theta - 2$.

6 (a) (i) The equation $(k + 4)x^2 - 2(k + 1)x + k - 1 = 0$ has equal roots. Find the value of k.

(ii) If the equation has real roots, find the range of values of k.

(b) With respect to an origin O, the position vectors of the points A and B are $2\mathbf{i} + \mathbf{j}$ and $4\mathbf{i} - 2\mathbf{j}$ respectively.

(i) Show that $|\overrightarrow{OB}| = 2|\overrightarrow{OA}|$.

The position vectors of the points C and D are given by
$\overrightarrow{OC} = 2\overrightarrow{OB} - \overrightarrow{OA}$ and $\overrightarrow{OD} = \overrightarrow{BA}$.

(ii) Show the points A, B, C and D on a diagram and calculate the angle between $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

7 The mean value of function $f(x)$ for $a \leq x \leq b$ is defined as $\frac{1}{b-a}\int_a^b f(x)\,dx$.
A lies on the x-axis where OA = 2 and P is the point (1,1). If Q is any point on OA where OQ = x, find an expression for the length of PQ^2 in terms of x. Using the above definition, calculate the mean length of PQ^2 as Q moves from O to A. Hence state the mean length of PQ.

8 (a) Sketch on the same diagram the graphs of $y = |x|$ and $y = 2 - x^2$.
Hence find the finite area enclosed by $y = |x|$ and $y = 2 - x^2$.
(b) A point P (x,y) moves so that its distance from the point (1,0) is always twice its distance from the line $x + 2 = 0$. Find the equation of the curve on which P will lie and the coordinates of the points where this curve meets the x-axis.

9 On graph paper, taking scales of 2 cm for $\frac{\pi}{3}$ radians on the x-axis and 4 cm for 1 unit on the y-axis, draw the graphs of $y = 2\cos x$ and $2\pi y = x$ for $0 \leq x \leq 2\pi$. Hence find from your graph approximate solutions to the equation $x = 4\pi \cos x$.

10 (a) (i) Given that $f(x) = x - \frac{4}{x}$, sketch the graph of $y = f(x)$ for $1 \leq x \leq 4$. Hence sketch the graph of $y = f^{-1}(x)$ for $3 \leq x \leq 3$.
(ii) Calculate the volume created if $y = f(x)$ for $1 \leq x \leq 4$ is rotated about the x-axis.
(b) Find all the angles between 0° and 360° which satisfy the equations
(i) $3 \sec 2x = 4$,
(ii) $2 \cot y \cos y = 3$.

Remainder and Factor Theorems: Cubic Equations

12

THE REMAINDER THEOREM

Consider the polynomial $x^3 - 2x^2 + 4x - 3$. Divide this by $x - 3$. Using long division as in Arithmetic, the steps are as follows.

$$
\begin{array}{rlllll}
 & \textcircled{1}\; x^2 & + \;x & + \;7 & & \\
x - 3 & \overline{)\; x^3 - 2x^2} & + \;4x & - \;3 & & \\
 & \textcircled{2}\; \underline{x^3 - 3x^2} & & & & \\
 & \textcircled{3}\quad\; x^2 & + \;4x & \textcircled{4} & & \\
 & \textcircled{5}\quad\; \underline{x^2} & \underline{- \;3x} & & & \\
 & & \textcircled{6}\;\; 7x & - \;3 & \textcircled{7} & \\
 & & \textcircled{8}\;\; \underline{7x} & \underline{- \;21} & & \\
 & & \quad\textcircled{9} & \;\;18 & \text{remainder} &
\end{array}
$$

Step ① Divide x^3 by x to give x^2

Step ② Multiply $x - 3$ by x^2

Step ③ Subtract to get x^2 and bring down the next term, $4x$

Step ④ Divide x^2 by x to give x

Step ⑤ Multiply $x - 3$ by x

Step ⑥ Subtract to get $7x$ and bring down the next term, -3

Step ⑦ Divide $7x$ by x to give 7

Step ⑧ Multiply $x - 3$ by 7

Step ⑨ Subtract; this gives the remainder 18

In the following it is the remainder which is important.

Let $f(x) = x^3 - 2x^2 + 4x - 3$. Now find f(3). What do you notice?

This result is not a coincidence. In fact we shall prove that when a polynomial $f(x)$ is divided by a linear expression $x - a$ the remainder will be $f(a)$.

If we divide say 15 by 4, then the result (the quotient) is 3 and the remainder is 3. The connection between these is

$$15 = \underset{\text{divisor}}{\underset{\uparrow}{4}} \times \underset{\text{quotient}}{\underset{\uparrow}{3}} + \underset{\text{remainder}}{\underset{\uparrow}{3}}$$

So if we divide $f(x)$ by $(x - a)$ and the quotient is Q, and the remainder R, then

$$f(x) = (x - a) \times Q + R$$

This is true for all values of a.

Now put $x = a$.

Then $\quad f(a) = 0 \times Q + R$

i.e. $\quad R = f(a)$.

If we divide by the general linear expression $px + q$, then $f(x) = (px + q) \times Q + R$.

Put $x = -\frac{q}{p}$ and $f\left(-\frac{q}{p}\right) = R$.

This is the **remainder theorem** for a polynomial $f(x)$:

> If $f(x)$ is divided by $px + q$, the remainder is $f\left(-\frac{q}{p}\right)$

It is also worth remembering the simple form: when $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

The theorem only applies to *polynomials* and only to **linear** divisors. Note also that it tells us nothing about the quotient.

Example 1

What are the remainders when $x^3 - x^2 + 3x - 2$ is divided by (a) $x - 1$, (b) $x + 2$, (c) $2x - 1$?

(a) Here, $x - a$ is $x - 1$ so $a = 1$.
$f(1) = 1 - 1 + 3 - 2 = 1$
The remainder is 1.

(b) Here, $a = -2$.
$f(-2) = -8 - 4 - 6 - 2 = -20$
The remainder is -20.

(c) $px + q = 2x - 1$, so $-\frac{q}{p} = \frac{1}{2}$.
$f\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{1}{4} + \frac{3}{2} - 2 = -\frac{5}{8}$.
The remainder is $-\frac{5}{8}$.

Example 2

The polynomial $x^3 + ax^2 - 3x + 4$ is divided by $x - 2$ and the remainder is 14. What is the value of a?

Taking f(x) as the given expression,

the remainder = $f(2) = 8 + 4a - 6 + 4 = 6 + 4a$.

Then $6 + 4a = 14$ and $a = 2$.

Example 3

$f(x) = x^3 + ax^2 + bx - 3$. When $f(x)$ is divided by $x - 1$ and $x + 1$, the remainders are 1 and -9 respectively. Find the values of a and b.

Dividing by $x - 1$, the remainder = $f(1) = 1 + a + b - 3 = a + b - 2$.

Then $a + b - 2 = 1$, so $a + b = 3$.

Dividing by $x + 1$, the remainder is $f(-1) = -1 + a - b - 3 = a - b - 4$.

Then $a - b - 4 = -9$, so $a - b = -5$.

Solving the two equations for a and b, $a = -1$ and $b = 4$.

Example 4

The polynomial $f(x) = A(x - 1)^2 + B(x + 2)^2$ is divided by $x + 1$ and $x - 2$. The remainders are 3 and -15 respectively. Find the values of A and B.

Divisor $x + 1$: remainder = $f(-1) = A(-1 - 1)^2 + B(-1 + 2)^2 = 4A + B = 3$

Divisor $x - 2$: remainder = $f(2) = A(2 - 1)^2 + B(2 + 2)^2 = A + 16B = -15$

Solving the simultaneous equations, $A = 1$ and $B = -1$.

Example 5

(i) If the expression $x^3 + px^2 + qx + r$ gives the same remainder when divided by $x + 1$ or $x - 2$, show that $p + q = -3$.

(ii) If the remainder is 4 when the expression is divided by $x - 1$, find the value of r.

(iii) If also the remainder is -60 when the expression is divided by $x + 3$, find the values of p and q.

(i) Divisor $x + 1$: remainder = $f(-1) = -1 + p - q + r$

Divisor $x - 2$: remainder = $f(2) = 8 + 4p + 2q + r$

Then $-1 + p - q + r = 8 + 4p + 2q + r$

so $3p + 3q = -9$ or $p + q = -3$.

(ii) Divisor $x - 1$: remainder = $f(1) = 1 + p + q + r = 4$.
But $p + q = -3$
Then $1 - 3 + r = 4$ and $r = 6$.

(iii) Divisor $x + 3$: remainder = $f(-3) = -27 + 9p - 3q + 6 = -60$
Hence $9p - 3q = -39$ or $3p - q = -13$.
Solving the simultaneous equations in p and q, $p = -4$, $q = 1$.

Exercise 12.1 *(Answers on page 634.)*

1 Find the remainder when $x^3 + 2x^2 - x - 1$ is divided by

(a) $x - 1$	(b) $x + 1$	(c) $x - 3$	(d) $x + 4$
(e) $2x - 1$	(f) $3x + 2$	(g) $2x - 3$	(h) $x + t$

2 Find the remainder when
(a) $x^2 - 7x - 3$ is divided by $x + 2$
(b) $x^4 - 3x^2 - x + 3$ is divided by $x + 3$
(c) $3x^3 - x^2 - x - 1$ is divided by $x - 4$

3 Find the remainder when the following expressions are divided by the linear expression stated:

(a) $1 - 2x - 3x^2$ by $x - 2$	(b) $x^3 + 3x - 3$ by $3x - 2$
(c) $x^3 + x^2 - x - 3$ by $x + 3$	(d) $3x^3 - x^2 + 4$ by $3x + 1$
(e) $2x^3 - x^2 + 4x + 2$ by $2x + 1$	(f) $x^4 + x^3 - 2x^2 - 3$ by $x + 3$

4 If $2x^3 - x^2 - 1$ is divided by $x + 2$, what is the remainder?

5 If $x^3 - 2x + 1$ is divided by $3x + 2$, what is the remainder?

6 What is the remainder when $ax^3 + bx^2 + cx + d$ is divided by $x + 1$?

7 The expression $2x^3 + px^2 - x - 2$ is divided by $x + 3$. State the remainder in terms of p.

8 Given $f(x) = ax^3 + x^2 - 3x - 2$ and that the remainder on dividing $f(x)$ by $x + 2$ is 0, what is the value of a?

9 $x^3 + px - 4$ is divided by $x + 4$ and the remainder is -28. Find the value of p.

10 The polynomial $x^3 + ax^2 + bx - 1$ is divided by $x - 2$ and $x + 1$.
The remainders are 7 and 4 respectively. Find the value of a and of b.

11 When the expression $x^3 + px^2 + qx + 2$ is divided by $x + 2$, the remainder is double that obtained when it is divided by $x - 1$. Find a relation between p and q. If the remainder is also 6 when the expression is divided by $x + 1$, find the value of p and of q.

12 The remainders obtained when $px^3 + qx^2 + 4x - 2$ is divided by $x - 1$ and $x + 2$ are equal. Show that $3p - q = -4$. If also the remainder is -18 when the polynomial is divided by $x - 2$, find the value of p and of q.

13 The expression $x^2 - 4x - 2$ has the same remainder when it is divided by either $x - a$ or $x - b$ $(a \neq b)$. Show that $a + b = 4$.
Given also that the remainder is 10 when the expression is divided by $x - 2a$, find the values of a and b.

14 When $x^3 - x^2 + ax + b$ is divided by $x - 1$ and $x + 1$, the remainders are -5 and -1 respectively. Find the values of a and b.

15 The polynomial $x^4 + 3x^3 + ax^2 + bx - 1$ is divided by $x - 1$ and $x + 2$.
The remainders obtained are 4 and 19 respectively. Find the values of a and b.

16 When the polynomials $x^3 - 4x + 3$ and $x^3 - x^2 + x + 9$ are each divided by $x - a$, the remainders are equal. Find the possible values of a.

17 If $x^2 + (m - 2)x - m^2 - 3m + 5$ is divided by $x + m$, the remainder is -1. Find the values of m.

THE FACTOR THEOREM

If $(x - a)$ is a factor of $f(x)$, then there will be **no** remainder when $f(x)$ is divided by $(x - a)$. So $f(a) = 0$. Similarly, if $px + q$ is a factor of $f(x)$, $f\left(-\frac{q}{p}\right) = 0$. This is the **factor theorem** for a polynomial $f(x)$:

If $px + q$ is a factor of $f(x)$, $f\left(-\frac{q}{p}\right) = 0$.
If $f\left(-\frac{q}{p}\right) = 0$, $px + q$ is a factor of $f(x)$.

We use the factor theorem to factorize polynomials (if possible).

Example 6

Factorize $x^3 - 6x^2 - x + 6$.

Take $f(x)$ as $x^3 - 6x^2 - x + 6$. As $f(x)$ is of the third degree, it will have *at most* three linear factors of the form $px + a$, $qx + b$, $rx + c$.

Then $x^3 - 6x^2 - x + 6 = (px + a)(qx + b)(rx + c)$

As the first term is x^3, $p = q = r = 1$.
So $x^3 - 6x^2 - x + 6 = (x + a)(x + b)(x + c)$

The last term is $+6$ so $a \times b \times c = +6$.
Hence the possible factors come from $x \pm 1$, $x \pm 2$, $x \pm 3$, $x \pm 6$.
The first factor has to be found by trial.
Try $x - 1$. Then $f(1) = 1 - 6 - 1 + 6 = 0$ so $x - 1$ is a factor.

Now we could continue trying other possible factors. In simple cases, this would be quick and satisfactory but in general could be time consuming, especially if the polynomial had only one linear factor.

For a cubic polynomial the best method is to find one factor by trial and then deduce the remaining quadratic factor by inspection. The steps are shown in full here but in practice all the working is done mentally and only the results written down.

$x^3 - 6x^2 - x + 6 = (x - 1)(\qquad)$.

We now complete the blank bracket step by step.

$$x^3 - 6x^2 - x + 6 = (x - 1)(x^2 + tx - 6)$$

① ② ③

Step ① The first term in the second bracket must be x^2

Step ② The last term must be -6 as $-1 \times -6 = +6$

Step ③ Take the middle term as tx. To find t, equate the coefficients of x^2.

$$-6x^2 \qquad (x - 1)(x^2 + tx - 6)$$

So $-6x^2 = -x^2 + tx^2$ and $t = -5$.

Check the coefficient of x: $(x - 1)(x^2 - 5x - 6)$ i.e. $+5x - 6x = -x$ which is correct.

Hence $f(x) = (x - 1)(x^2 - 5x - 6) = (x - 1)(x - 6)(x + 1)$.

Example 7

Factorize $x^3 - 3x^2 - 2x + 8$.

The possible factors are $x \pm 1$, $x \pm 2$, $x \pm 4$, $x \pm 8$. The first factor is found by trial.

Try $x + 1$: remainder $= -1 - 3 + 2 + 8 \neq 0$. $x + 1$ is not a factor.

Try $x - 1$: remainder $= 1 - 3 - 2 + 8 \neq 0$. $x - 1$ is not a factor.

Try $x + 2$: remainder $= -8 - 12 + 4 + 8 \neq 0$. $x + 2$ is not a factor.

Try $x - 2$: remainder $= 8 - 12 - 4 + 8 = 0$. $x - 2$ is a factor.

Then $x^3 - 3x^2 - 2x + 8 = (x - 2)(x^2 - x - 4)$ $\quad = +8 \quad = -3x^2$

Hence the expression $= (x - 2)(x^2 - x - 4)$ as $x^2 - x - 4$ cannot be factorized.

Example 8

Factorize $4x^3 - 8x^2 - x + 2$.

The first term $4x^3$ makes the solution more complicated as one of the factors must be $2x + a$ or $4x + b$. However try $x - 1$ and $x + 1$ and confirm that they are not factors.

Now try $x - 2$: remainder $= 32 - 32 - 2 + 2 = 0$ so $x - 2$ is a factor.

Then $4x^3 - 8x^2 - x + 2 = (x - \underbrace{2)(4x^2}_{= -8x^2} \quad - 1)$ no middle term

Hence the expression $= (x - 2)(4x^2 - 1) = (x - 2)(2x - 1)(2x + 1)$

Example 9

The expression $2x^3 + ax^2 + bx - 2$ is exactly divisible by $x - 2$ and $2x + 1$. Find the values of a and b and hence find the third factor.

Divide by $x - 2$: remainder $= 16 + 4a + 2b - 2 = 0$ so $4a + 2b = -14$ or $2a + b = -7$.

Divide by $2x + 1$: remainder $= 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 2$

$= -\frac{1}{4} + \frac{a}{4} - \frac{b}{2} - 2 = 0$ so $a - 2b = 9$.

Solving the two equations for a and b, $a = -1$, $b = -5$.

Let $(px + q)$ be the third factor.

Hence $2x^3 - x^2 - 5x - 2 = (2x + 1)(x - 2)(px + q)$
$= (2x^2 - 3x - 2)(px + q)$

By inspection, $p = 1$ and by checking the last term, $q = +1$.

Hence the third factor is $x + 1$.

Exercise 12.2 *(Answers on page 634.)*

1 Factorize

(a) $x^3 + 1$
(b) $x^3 - 4x^2 + 5x - 2$
(c) $x^3 - 4x^2 + x + 6$
(d) $x^3 + 6x^2 + 11x + 6$
(e) $x^3 - x^2 + 2x - 2$
(f) $x^3 + 3x^2 - 6x - 8$
(g) $2x^3 + 7x^2 + 8x + 3$
(h) $3x^3 + 2x^2 - 3x - 2$
(i) $x^3 - 1$
(j) $x^3 - 2x^2 - 9x + 18$
(k) $2x^3 - 3x^2 - 8x + 12$
(l) $6x^3 - 13x^2 + 9x - 2$

2 If $x^3 + ax + 6$ is divided by $x + 1$, the remainder is 12. Find the value of a and factorize the expression.

3 Given the expression $ax^3 + bx^2 + cx + d$, show that $x - 1$ is a factor if $a + b + c + d = 0$. [This result is worth remembering: if the sum of the coefficients = 0, then $x - 1$ is a factor.]

4 The expression $2x^3 + ax^2 + bx + 1$ is exactly divisible by $2x - 1$ and $x + 1$. Find the value of a and of b and hence find the third factor of the expression.

5 Given that $f(x) = x^3 - 6x^2 + 11x + p$, find the value of p for which $x - 3$ is a factor of $f(x)$. With this value of p, find the other factors of $f(x)$.

6 $x^3 + ax + b$ and $x^2 + 2ax + 3b$ have a common factor $x + 1$. Find the value of a and of b and with these values factorize the cubic expression.

7 $f(x) = x^3 + ax^2 + bx + 4$ and $f(x)$ is exactly divisible by $x - 2$. If the remainder is -24 when $f(x)$ is divided by $x + 2$, find the value of a and of b and hence factorize $f(x)$.

8 $x^3 + ax^2 + x + b$ is exactly divisible by $x - 3$ and the remainder is -20 when it is divided by $x + 2$. Find the values of a and b and then factorize the expression.

9 The function $f(x) = 2x^3 + ax^2 - 2x + b$ has a factor $2x - 1$. When $f(x)$ is divided by $x + 2$, the remainder is -15. Find the value of a and of b and find the other two factors of $f(x)$.

10 Given that $f(x) = x^3 + ax^2 + bx + 8$ and that the remainders when $f(x)$ is divided by $x + 1$ and $x + 2$ are 6 and -8 respectively, find the value of a and of b and hence factorize $f(x)$.

11 If $x - 2$ is a common factor of the expressions $x^2 + (p + q)x - q$ and $2x^2 + (p - 1)x + (p + 2q)$, find the value of p and of q.

12 The remainder when $x(x + b)(x - 2b)$ is divided by $x - b$ is -16. Find the value of b.

13 Find the value of k ($\neq 0$) for which $x + k$ and $x - k$ are both factors of $x^3 - x^2 - 9x + 9$. Then find the third factor.

14 The expression $x^4 + 4x^3 + 6x^2 + 5x + 2$ has only two linear factors. Find these factors.

15 The expression $A(x - 1)^3 + B(x + 3)^2 + 20$ is exactly divisible by $x + 1$ and the remainder is 26 when it is divided by x. Find the value of A and of B. Using these values, rewrite the expression as a polynomial and factorize completely.

Solving a Cubic Equation

Example 10

Solve the equation $2x^3 + 3x^2 - 3x = 2$.

First we factorize the polynomial $2x^3 + 3x^2 - 3x - 2$.

The sum of the coefficients is 0 so $x - 1$ is a factor.

Then the polynomial is $(x - 1)(2x^2 + 5x + 2) = (x - 1)(2x + 1)(x + 2)$.

Hence the roots of the cubic equation $2x^3 + 3x^2 - 3x - 2 = 0$ are $x = 1$ or $-\frac{1}{2}$ or -2.

Example 11

If $f(x) = x^3 - 3x^2 + x + 2$, solve the equation $f(x) = 0$.

$x - 1$ is not a factor of $f(x)$.
Verify that $x + 1$ is also not a factor.
Try $x - 2$ and verify that this is a factor.
Then $f(x) = (x - 2)(x^2 - x - 1)$.
The roots of $f(x) = 0$ are $x = 2$ and the roots of $x^2 - x - 1 = 0$,
i.e. $x = \dfrac{1 \pm \sqrt{5}}{2} = 1.62$ or -0.62.

Example 12

Given that $f(x) = x^3 - 2x^2 + 2x$, solve the equation $f(x) = 4$.

$f(x) = 4$ gives $x^3 - 2x^2 + 2x - 4 = 0$. To solve this equation, we first factorise the polynomial $x^3 - 2x^2 + 2x - 4$.
Check that $x + 1$ and $x - 1$ are not factors. Now try $x - 2$.
Divide the polynomial by $x - 2$ to obtain the other factor.
The equation is $(x - 2)(x^2 + 2) = 0$ and the only root is $x = 2$ as $x^2 + 2 = 0$ has no real roots.

Example 13

Find the nature and x-coordinates of the turning points on the curve $y = 3x^4 + 4x^3 - 6x^2 - 12x + 1$.

$\dfrac{dy}{dx} = 12x^3 + 12x^2 - 12x - 12 = 12(x^3 + x^2 - x - 1)$

$\dfrac{dy}{dx} = 0$ when $x^3 + x^2 - x - 1 = 0$.

$x - 1$ is a factor of the left hand side of this equation.

Then $x^3 + x^2 - x - 1 = (x - 1)(x^2 + 2x + 1) = (x - 1)(x + 1)^2$ and so $\dfrac{dy}{dx} = 0$ when $x = 1$ or $x = -1$.

$\dfrac{d^2y}{dx^2} = 12(3x^2 + 2x - 1)$

When $x = 1$, $\dfrac{d^2y}{dx^2} > 0$ so this is a minimum point.

When $x = -1$, $\dfrac{d^2y}{dx^2} = 0$ so we use the sign test on $\dfrac{dy}{dx} = (x - 1)(x + 1)^2$.

x	slightly < -1	-1	slightly > -1
sign of $\frac{dy}{dx}$	$-$	0	$-$
sketch of tangent			

At $x = -1$, the curve has a point of inflexion.

IDENTICAL POLYNOMIALS

If we state that two cubic polynomials are **identical**, then corresponding coefficients must be equal.

So if $2x^3 - x^2 + x - 5 = ax^3 + bx^2 + cx + d$ then $a = 2$, $b = -1$, $c = 1$ and $d = -5$.

We can also say that the polynomials are equal for **all** values of x.

This enables us to convert a polynomial into a form which may be more suitable for further computations. The method is general and applies to any polynomials of the same degree.

Example 14

Given that $2x^3 - x^2 - 7x - 5 = (Ax + B)(x - 1)(x + 2) + C$ for all values of x, evaluate A, B and C.

First expand the right hand side, obtaining

$$\begin{aligned}(Ax + B)(x^2 + x - 2) + C &\equiv Ax^3 + (B + A)x^2 + (B - 2A)x - 2B + C \\ &\equiv 2x^3 - x^2 - 7x - 5\end{aligned}$$

Now compare coefficients:

The x^3 term gives $A = 2$.

The x^2 term gives $B + A = -1$ so $B = -3$.

Check that the x-coefficients are equal. $B - 2A = -3 - 4 = -7$ which is correct. finally $-2B + C = -5$ so $C = -11$.

An alternative method is to substitute suitable values of x into each polynomial, remembering that these are equal for all values of x.

Put $x = 1$. Then $2 - 1 - 7 - 5 = 0 + C$ so $C = -11$. Note why 1 was chosen. What other value of x could we have chosen instead?

Now put $x = 0$. Then $-5 = (B)(-1)(2) - 11$ so $B = -3$.

Put $x = -1$. Then $-2 - 1 + 7 - 5 = (-A - 3)(-2)(1) - 11$

i.e. $-1 = 2A + 6 - 11$ so $A = 2$.

Either method is simple to use, but the second method needs a careful choice of values for x.

Exercise 12.3 *(Answers on page 634.)*

1 Solve the equations

(a) $x^3 + x^2 - x = 1$

(b) $x^3 + 2x^2 - x = 2$

(c) $x^3 + 6x^2 + 11x + 6 = 0$

(d) $x^3 - 4x^2 + 5x - 2 = 0$

(e) $x^3 + 2x^2 - 2x + 3 = 0$

(f) $x^3 = 13x + 12$

(g) $x^3 - 9x^2 + 26x = 24$

(h) $x^3 = 6x + 5$

(i) $x^3 - 4x^2 + x + 6 = 0$

(j) $4x^3 - 12x^2 + 5x + 6 = 0$

(k) $x^2(2x + 1) = 13x - 6$

2 The expression $x^3 + ax^2 + bx + 12$ is exactly divisible by $x - 1$ and $x + 3$. Find the value of a and of b and the remaining factor of the expression. Hence solve the equation $x^3 + ax^2 + bx + 12 = 0$.

3 $(x - 2)$ is a factor of $2x^3 + ax^2 + bx - 2$ and when this expression is divided by $x + 3$, the remainder is -50. Find the value of a and of b and the other factors. Hence solve the equation $2x^3 + ax^2 + bx = 2$.

4 In each of the following, the polynomials are equal for all values of x. Evaluate A, B and C.

(a) $x^3 - x^2 - 2x - 7 = x(\mathrm{A}x + \mathrm{B})(x + 1) + \mathrm{C}$

(b) $3x^3 - 8x^2 + 4x - 5 = x(\mathrm{A}x + \mathrm{B})(x - 2) + \mathrm{C}$

(c) $3x^3 - 13x^2 + 18x - 10 = (\mathrm{A}x + \mathrm{B})(x - 1)(x - 2) + \mathrm{C}$

(d) $4x^3 - 7x^2 - 5x + 6 = (\mathrm{A}x + \mathrm{B})(x - 2)(x + 1) + \mathrm{C}$

(e) $2x^2 - x + 3 = \mathrm{A}(x + 1)^2 + \mathrm{B}(x + 1) + \mathrm{C}$

(f) $2x^3 - 7x^2 + 7x - 5 = \mathrm{A}(x - 1)^3 + \mathrm{B}x(x - 1) + \mathrm{C}$

5 (a) Solve the equation $x^3 - 7x + 6 = 0$. Hence state the solutions of the equation $(x - 2)^3 - 7(x - 2) + 6 = 0$.

(b) Solve the equation $2x^3 = 11x^2 - 17x + 6$.

6 If $f(x) = x^3 + ax^2 + bx + 6$ and the remainders when $f(x)$ is divided by $x + 1$ and $x - 2$ are 20 and 8 respectively, find the value of a and of b and hence solve the equation $f(x) = 0$.

7 (a) $f(x) = x^3 + ax^2 + bx + 12$. Given that the remainders when $f(x)$ is divided by $x + 1$ and $x + 3$ are 12 and -30 respectively, find the value of a and of b. With these values, solve the equation $f(x) = 0$.

(b) $f(x) = x^3 + kx^2 + 3x - 2$ is divided by $x + k$. If the remainder is 4, find the value divisible by $x - 4$. With this value for k, solve the equation $f(x) = 4(x - 1)$.

8 Find the x-coordinates of the points where the line $y = 5x - 1$ meets the curve $y = 2x^3 + x^2 + 1$.

9 Find the coordinates of the points of intersection of the curve $y = x^3$ and the line $y = 7x + 6$.

10 Show that $2x^3 - x^2 + 3x - 4$ cannot be equal to $x(\mathrm{A}x + \mathrm{B})(x - 2) + \mathrm{C}$ for all values of x. State the new coefficient of x in the first polynomial which will make the polynomials identical.

11 Find the x-coordinates of the points where the line $y = x + 6$ meets the curve $y = x^3 + 3x^2 + x + 2$ and show that the line is a tangent to the curve at one of these points.

12 Find the x-coordinates and the type of the turning points on the curve $y = x^4 - 8x^3 + 22x^2 - 24x + 4$.

13 Fig.12.1 shows part of the curve $y = x^3 + 1$. The tangent at A $(-1,0)$ meets the curve again at T. Find

(a) the equation of AT,

(b) the coordinates of T,

(c) the area of the shaded region in the figure.

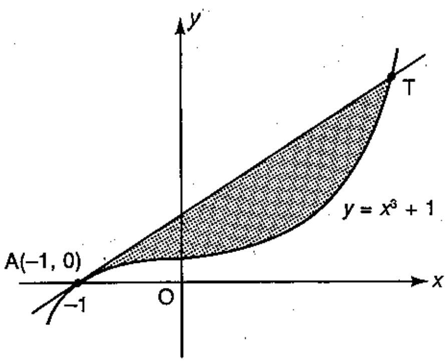


Fig.12.1

14 The tangent at P(1,1) on the curve $y = x^3$ meets the curve again at Q. Find

(a) the equation of PQ,

(b) the coordinates of Q,

(c) the area of the finite region enclosed by the tangent and the curve.

SUMMARY

- *Remainder theorem*: if a polynomial $f(x)$ is divided by $px + q$, the remainder is $f\left(-\frac{q}{p}\right)$; if divided by $x - a$, the remainder is $f(a)$.
- *Factor theorem*: if $px + q$ is a factor of $f(x)$, $f\left(-\frac{q}{p}\right) = 0$; if $x - a$ is a factor of $f(x)$, $f(a) = 0$.
- If $f\left(-\frac{q}{p}\right) = 0$, $px + q$ is a factor of $f(x)$; if $f(a) = 0$, $x - a$ is a factor of $f(x)$.
- If the sum of the coefficients of $f(x)$ is 0, then $x - 1$ is a factor of $f(x)$.

REVISION EXERCISE 12 *(Answers on page 635.)*

A

1 The expression $ax^3 - x^2 + bx - 1$ leaves remainders of -33 and 77 when divided by $x + 2$ and $x - 3$ respectively. Find the value of a and of b and the remainder when divided by $x - 2$.

2 (a) The expression $6x^2 + x + 7$ leaves the same remainder when divided by $x - a$ and $x + 2a$, where $a \neq 0$. Calculate the value of a.

(b) Given that $x^2 + px + q$ and $3x^2 + q$ have a common factor $x - b$, where p, q and b are non-zero, show that $3p^2 + 4q = 0$. (C)

3 (a) Find, in terms of p, the remainder when $3x^3 - 2x^2 + px - 6$ is divided by $x + 2$. Hence write down the value of p for which the expression is exactly divisible by $x + 2$.

(b) Solve the equation $x^3 - 12x + 16 = 0$.

(c) Given that the expression $x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x - 1$ or $x + 2$, prove that $a = b + 3$.
Given also that the remainder is 3 when the expression is divided by $x + 1$, calculate the value of c. (C)

4 Find the x-coordinates of the points where the curves $y = x^3 - 4x^2 - 5$ and $y = 2x^2 - 11x + 1$ intersect.

5 (a) The expression $2x^3 + ax^2 - 72x - 18$ leaves a remainder of 17 when divided by $x + 5$. Determine the value of a.

(b) Solve the equation $2x^3 = x^2 + 5x + 2$.

(c) Given that the expression $x^2 - 5x + 7$ leaves the same remainder whether divided by $x - b$ or $x - c$, where $b \neq c$, show that $b + c = 5$.
Given further that $4bc = 21$ and that $b > c$, find the value of b and of c. (C)

6 (a) Given that $x + 2$ is a factor of $f(x) = x^3 - 3x^2 - 4x + p$ find the value of p and hence factorise $f(x)$.

(b) Solve the equation $2x^3 + 3x^2 - 4x - 4$, giving the answers correct to 2 decimal places if necessary.

(c) If $4x^3 - 11x^2 - 6x + 7 = (Ax + B)(x + 1)(x - 3) + C$ for all values of x, evaluate A, B and C.

7 (a) The expressions $x^3 - 7x + 6$ and $x^3 - x^2 - 4x + 24$ have the same remainder when divided by $x + p$.

(i) Find the possible values of p.

(ii) Determine whether, for either or both of these values, $x + p$ is a factor of the expressions.

(b) Given that $E = x^4 - x^3 + 5x^2 + 4x - 36$, find (i) the remainder when E is divided by $x + 1$, (ii) the value of a ($a > 0$) for which $x + a$ and $x - a$ are both factors of E. (C)

8 If the polynomials (i) $3x^3 + x^2 + 2x + 4$ and (ii) $x^3 + 2x^2 + 6x - 10$ are each divided by $x - a$, the remainder from (i) is double the remainder from (ii). Find the possible values of a.

9 The gradient of a curve at the point (x,y) is given by $3x^2 - 12x + 12$ and the curve passes through the point $(0,-7)$. Find the equation of the curve and the coordinates of the point(s) where it meets the x-axis.

10 If the vectors $\mathbf{r}_1 = (t-1)\mathbf{i} - (t+2)\mathbf{j}$ and $\mathbf{r}_2 = t^2\mathbf{i} + (t-1)\mathbf{j}$ are perpendicular, find the possible values of t.

11 (a) Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.
(b) Find the value of a for which $(1 - 2a)x^2 + 5ax + (a-1)(a-8)$ is divisible by $x - 2$ but not by $x - 1$.
(c) Given that $16x^4 - 4x^3 - 4b^2x^2 + 7bx + 18$ is divisible by $2x + b$,
(i) show that $b^3 - 7b^2 + 36 = 0$,
(ii) find the possible values of b. (C)

12 The tangent at the point P(1,3) on the curve $y = x^3 - x + 3$ meets the curve again at T. Find (i) the equation of PT, (ii) the coordinates of T and (iii) the area of the finite region enclosed by the curve and the tangent.

13 Fig.12.2 (not drawn to scale) shows parts of the curves $y = x^3 + 2$ and $y = x(2x + 1)$. Find the coordinates of the points A, B and C where the curves intersect and the areas of the shaded regions P and Q.

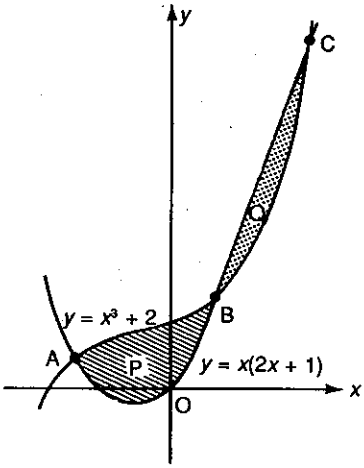


Fig. 12.2

14 By first solving the equation $x^3 - 3x + 2 = 0$ or otherwise, find the solutions of the equation $(x + 2)^3 = 3x + 4$.

B

15 If $f(x) = x^3 - 3x^2 + x - 3$, show that $x - 2$ is a factor of $f(x + 1)$.

16 Given that $2x + p$ is a factor of $2x^4 + px^3 + 2px^2 + 7x + 3$, show that $p^3 - 7p + 6 = 0$ and hence find the possible values of p.

17 Solve the equation $6\cos^3\theta = 7\cos^2\theta - 1$ for $0° \le \theta \le 360°$.

18 Factorize the polynomial $f(x) = x^3 - (2p + 1)x^2 + (2p - q)x + q$ where p and q are constants. If the equation $f(x) = 0$ has three real roots, show that $p^2 + q \ge 0$.

19 If the polynomial $ax^3 + bx^2 + cx - 4$ is divided by $x + 2$, the remainder is double that obtained when the polynomial is divided by $x + 1$. Show that c can have any value and find b in terms of a.

20 If the solutions of the equation $x^3 + px^2 + qx + r = 0$ are a, b and c it can be proved that $a^2 + b^2 + c^2 = p^2 - 2pq$ and that $a^3 + b^3 + c^3 = 3pq - p^3 - 3r$.
Verify these statements for the equation $x^3 + 2x^2 - 5x - 6 = 0$.
Make up a cubic equation $(x - a)(x - b)(x - c) = 0$ with values for a, b and c and verify the statements for your equation.

21 Find the ranges of values of x for which
$2x^3 - 3x^2 + x + 6 < (x + 2)(x + 1)(x - 1)$

Arithmetic and Geometric Progressions

13

ARITHMETIC PROGRESSIONS

Here are 3 sequences of numbers which follow a simple pattern. Can you say what the next two numbers should be?

(i) $1, 4, 7, 10, \ldots$
(ii) $-15, -11, -7, -3, \ldots$
(iii) $29, 27\frac{1}{2}, 26, 24\frac{1}{2}, \ldots$

You will have found that in (i) the numbers increase by 3 so the next two numbers are 13 and 16, in (ii) the numbers increase by 4 so the next two are 1 and 5, and in (iii) the numbers decrease by $1\frac{1}{2}$ so the next two are 23 and $21\frac{1}{2}$.

These are examples of an **arithmetic progression**, which we shall abbreviate as AP. An AP is a sequence of numbers which increase by a constant amount (+ or –). This amount is called the **common difference (*d*)** and the starting number is called the **first term (*a*)**. Hence the second term, or T_2 for short is $a + d$, the third term T_3 is $a + d + d = a + 2d$ and so on.

1st term	*2nd term*	*3rd term*	*4th term* ...	*nth term*
T_1	T_2	T_3	T_4 ...	T_n
a	$a + d$	$a + 2d$	$a + 3d$ ...	$a + (n - 1)d$

So the formula for the nth term (T_n) of an AP is

$$T_n = a + (n - 1)d$$

Note that the difference between consecutive terms is constant:

$$T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \ldots = d$$

Example 1

(a) What is the 15th term of the AP –5, –2, 1, ... ?
(b) Which term is 28?

In this AP, $a = -5$ and $d = 3$.

(a) $T_{15} = a + 14d = -5 + 42 = 37$
(b) $T_n = a + (n - 1)d$ so $28 = -5 + (n - 1) \times 3 = 3n - 8$

Hence $n = 12$ i.e. 28 is the 12th term.

Example 2

Find a formula in terms of n for the nth term of the AP 15, 9, 3, ... and hence find the 30th term.

$T_n = a + (n - 1)d = 15 + (n - 1) \times (-6) = 21 - 6n$

Then $T_{30} = 21 - 6 \times 30 = -159$.

Example 3

The nth term of an AP is given by $T_n = 2n + 9$. Find (a) the first term, (b) the common difference.

(a) $T_1 = 2 \times 1 + 9 = 11$
(b) $d = T_2 - T_1 = 2 \times 2 + 9 - 11 = 2$

Example 4

If the 5th term of an AP is –4 and the 10th term is 16, find the first term and the common difference.

$T_5 = a + 4d = -4$ and $T_{10} = a + 9d = 16$.

Solving these equations, $d = 4$ and $a = -20$.

Example 5

The first term of an AP is –4 and the 15th term is double the 5th term. Find the 12th term.

We must first find d.

$T_{15} = -4 + 14d$ and $T_5 = -4 + 4d$.

$T_{15} = 2T_5$ so $-4 + 14d = 2(-4 + 4d) = -8 + 8d$.

Hence $6d = -4$ and $d = -\frac{2}{3}$.

Then the 12th term $= a + 11d = -4 + 11 \times \left(-\frac{2}{3}\right) = -11\frac{1}{3}$.

Example 6

The sum of three consecutive terms of an AP is 18 and their product is 120. Find these terms.

We could take k, $k + d$ and $k + 2d$ as the three consecutive terms but it is simpler to take $k - d$, k and $k + d$ instead.

The sum of these is $3k = 18$ and so $k = 6$.

The product of the three terms is

$$(k - d)k(k + d) = (6 - d) \times 6 \times (6 + d) = 120$$

so $(6 - d)(6 + d) = 20$ i.e. $36 - d^2 = 20$ or $d^2 = 16$.

Hence $d = \pm 4$.

The numbers are either 2, 6, 10 or 10, 6, 2. The numbers are the same but they come from different APs.

Arithmetic Means

If a, b and c are three consecutive terms of an AP, b is called an **arithmetic mean** between a and c. Now $b - a = c - b$ so $2b = a + c$ and $b = \frac{1}{2}(a + c)$. For example, the arithmetic mean between -7 and 3 is $\frac{1}{2}(-7 + 3) = -2$.

Exercise 13.1 *(Answers on page 635.)*

1 State the first term and the common difference and then find the 7th and the 15th terms of the following APs:

(a) 2, 6, 10, ...　　(b) –3, 0, 3 ...

(c) $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$　　(d) 1.7, 1.4, 1.1, ...

2 For the AP $-2, 1, 4, \ldots$ state (a) the 8th term, (b) the 12th term. (c) Find a formula for the nth term.

3 Find (a) the 9th, (b) the 20th, terms of the AP $17, 13, 9, \ldots$

(c) Which term of this AP will be -35 ?

(d) Find a formula for the nth term.

4 (a) Find a formula for the nth term of the AP $-10, -7, -4, \ldots$

(b) If $2x - 3$ is the arithmetic mean of $x^2 - 4$ and $5x - 8$ find the values of x.

5 The 8th term of the AP $a, a + 4, a + 8, \ldots$ is 33. Find (a) the value of a, (b) the 15th term.

6 If the first term of an AP is 5 and the 9th term is 25, find the common difference.

7 Find (a) the 7th term and (b) a formula for the nth term of the AP $\frac{1}{2}, \frac{5}{6}, 1\frac{1}{6}, \ldots$

8 The 4th term of an AP is 13 and the 10th term is 31. Find the 20th term.

9 The 5th term of an AP is $5\frac{1}{2}$ and the 9th term is $8\frac{1}{2}$. Find the 17th term and a formula for the nth term.

10 The 4th term of an AP is 5 and the 11th term is 26. Find

(a) the first term and the common difference,

(b) a formula for the nth term.

11 A ball rolls down a slope so that it travels 4 cm in the lst second, 7 cm in the 2nd, 10 cm in the 3rd and so on. How far does it travel in

(a) the 7th,

(b) the nth, second?

12 The nth term (T_n) of an AP is given by $T_n = 3 - 5n$.

(a) State (i) the 8th term, (ii) the 21st term.

(b) What is the first term?

(c) What is the common difference?

13 The nth term (T_n) of an AP is given by $T_n = \frac{1}{2}(4n - 3)$.

(a) State (i) the 5th term, (ii) the 10th term.

(b) Find the common difference.

14 If the 5th term of an AP is 10 and the 10th term is 5, find the first term and the common difference.

15 If the 12th term of an AP is double the 5th term find the common difference, given that the first term is 7.

16 The 16th term of an AP is three times the 5th term. If the 12th term is 20 more than the 7th term, find the first term and the common difference.

17 In an AP the 9th term is -4 times the 4th term and the sum of the 5th and 7th terms is 9. Find the first term and the common difference.

18 In an AP the 22nd term is 4 times the 5th term while the 12th term is 12 more than the 8th term. Find the first term and the common difference.

19 Find a formula for the nth term of the AP $3, 2\frac{1}{2}, 2, \ldots$

20 The sum of 3 consecutive terms of an AP is 6 and their product is -42. Find these terms.

21 p, q and r are three consecutive terms of an AP. Express p and r in terms of q and d, where d is the common difference. If the sum of the terms is 21 and $p = 6r$, find p, q and r.

22 Which of the following sequences is an AP?

(a) $\frac{1}{8}, \frac{11}{24}, \frac{19}{24}, \ldots$ (b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

(c) $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \ldots$ (d) $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$

23 The 9th term of an AP is 3 times the 5th term.

(a) Find a relation between a and d.

(b) Prove that the 8th term is 5 times the 4th term.

24 If $x + 1$, $2x - 1$ and $x + 5$ are three consecutive terms of an AP, find the value of x.

25 $x + 2$, $x + 3$ and $2x^2 + 1$ are three consecutive terms of an AP, find the possible values of x.

Sum of an Arithmetic Progression

Suppose you were asked in a quiz: 'What is the sum of the first 20 numbers?'. Could you give the answer quickly?

It would be too slow (and difficult!) to add them all up, so we think of a quicker way. The sequence is an AP. Call the sum S_{20}.

$$S_{20} = 1 + 2 + 3 + \ldots + 18 + 19 + 20$$
$$S_{20} = 20 + 19 + 18 + \ldots + 3 + 2 + 1$$

Now add:

$$2S_{20} = 21 + 21 + 21 + \ldots + 21 + 21 + 21$$

which is $20 \times$ (first number + last number)

$$= 20 \times 21 = 420$$

so $S_{20} = 210$

Generalizing this for any AP with n terms of which the first is a and the last is L, $2S_n = n(a + L)$ and therefore $S_n = \frac{n}{2}(a + L)$.

Hence a formula for the sum S_n of n terms of an AP, with first term a and last term L is

$$S_n = \frac{n}{2}(a + L)$$

However we may not know the last term. So we convert this formula into a more suitable one. L is the nth term and so $L = a + (n - 1)d$.

Then, $S_n = \frac{n}{2}(a + L) = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$ giving this alternative formula for S_n:

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

We also say that adding the terms of a sequence gives a **series.** Thus $1 + 2 + 3 + 4$ is an arithmetic series of 4 terms and the sum of this series is 10. The formula of S_n above gives the sum of an arithmetic series of n terms.

S_n is the sum of all the terms $T_1, T_2, \ldots T_n$.
S_{n-1} is the sum of all the terms $T_1, T_2, \ldots T_{n-1}$.
Hence $T_n = S_n - S_{n-1}$.
For example the 9th term $= S_9 - S_8$.

Example 7

The integers from 12 to 55 are added. What is their sum?

There are 44 integers.

Hence $S_n = \frac{n}{2}(a + L) = \frac{44}{2}(12 + 55) = 22 \times 67 = 1474$.

Example 8

A spot of light is made to travel across a screen in a straight line and the total distance travelled is 115.5 cm. The distances travelled in successive seconds are in AP. It travels 1.5 cm in the first second and 9.5 cm in the last. How long did it take to cover the total distance?

In this AP, $a = 1.5$, $L = 9.5$ and $S = 115.5$.
Then $115.5 = \frac{n}{2}(1.5 + 9.5) = \frac{11n}{2}$ so $11n = 231$ and $n = 21$.
It took 21 seconds.

Example 9

The sum of the first 8 terms of an AP is 12 and the sum of the first 16 terms is 56. Find the AP.

$S_8 = \frac{8}{2}(2a + 7d) = 12$ and $S_{16} = \frac{16}{2}(2a + 15d) = 56$.

Then $2a + 7d = 3$ and $2a + 15d = 7$.

Solving these equations, we obtain $a = -\frac{1}{4}$ and $d = \frac{1}{2}$.

The AP is $-\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \ldots$

Example 10

The sum of the first 10 terms of an AP is 80 and the sum of the next 12 terms is 624. What is the AP?

$S_{10} = \frac{10}{2}(2a + 9d) = 80$ so $2a + 9d = 16$ (i)

We cannot use the formula for the next 12 terms as they do not start from a. But we know that the sum of the first 22 terms is $80 + 624 = 704$.

Hence $S_{22} = \frac{22}{2}(2a + 21d) = 704$ so $2a + 21d = 64$ (ii)

Now solving equations (i) and (ii), $a = -10$ and $d = 4$.

The AP is $-10, -6, -2, \ldots$

Example 11

An AP contains 30 terms. Given that the 10th term is 21 and that the sum of the last 10 terms is 675, find the sum of the first 10 terms.

$T_{10} = a + 9d = 21$ (i)

$$\begin{aligned}\text{The sum of the last 10 terms} &= S_{30} - S_{20}\\ &= \tfrac{30}{2}(2a + 29d) - \tfrac{20}{2}(2a + 19d)\\ &= 15(2a + 29d) - 10(2a + 19d)\\ &= 10a + 245d = 675\end{aligned}$$

Hence $2a + 49d = 135$ (ii)

Solving (i) and (ii), $a = -6$ and $d = 3$.

Then $S_{10} = \frac{10}{2}(-12 + 9 \times 3) = 5 \times 15 = 75$.

Example 12

The sum of the first 10 terms of an AP is $3\frac{1}{2}$ times the sum of the first 4 terms.

(a) Find the ratio of the 10th term to the 4th term.

(b) Given that the 5th term is 2, find the sum of the first 10 terms.

(a) $S_{10} = 5(2a + 9d)$ and $S_4 = 2(2a + 3d)$

Then $10a + 45d = \frac{7}{2}(4a + 6d) = 14a + 21d$.

Hence $4a = 24d$ or $a = 6d$.

Then $T_{10} = a + 9d = 15d$ and $T_4 = a + 3d = 9d$

and the ratio $T_{10}{:}T_4 = 15d{:}9d = 5{:}3$.

(b) $T_5 = a + 4d = 2$ so $10d = 2$ and $d = 0.2$. Then $a = 1.2$.

$S_{10} = 5(2a + 9d) = 5(2.4 + 1.8) = 21$

13 126 logs of wood are piled up as shown in Fig. 13.1. There are 21 logs in the bottom row. How many rows are there and how many logs are there in the top row?

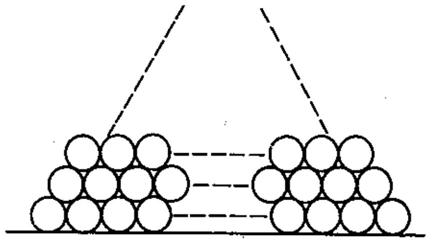


Fig. 13.1

14 In an AP, $T_{18} = 2 \times T_9$. Find the ratio $S_{18}:S_9$.

15 What is the least number of terms of the AP 5, 11, 17, ... that must be added to give a total sum greater than 100?

16 A line which is 140 cm long is divided into 20 pieces whose lengths form an AP. If the shortest piece is 3 cm long, find the length of the longest piece.

17 (a) 20 pieces of wood have lengths in AP. The shortest is 5 cm long and the total length of the pieces is 480 cm. What is the length of the longest piece?

(b) In an AP the sum of the first 11 terms equals the sum of the first 20 terms. Given that the 5th term is 33 find the sum of the first 31 terms.

18 The 6th term of an AP with 12 terms is 14 and the sum of the last 6 terms is 126. Calculate the sum of the first 6 terms.

19 The first term of an AP is 3. Given that the sum of the first 6 terms is 48 and that the sum of all the terms is 168, calculate

(a) the common difference,

(b) the number of terms in the AP,

(c) the last term.

20 (a) What is the least number of terms of the AP 4, 4.5, 5, ... to be added to obtain a total greater than 50?

(b) Strips of wood whose lengths form the AP 10, 12, 14 ... cm are laid end to end in a straight line. How many must be laid to give a total length L cm where $160 < L < 190$?

21 Given that the ratio of the 18th term to the 6th term in an AP is 3:1, calculate the ratio of the sum of the first 18 terms to the sum of the first 6 terms.

GEOMETRIC PROGRESSIONS

Can you see the difference in the pattern of these sequences of numbers?

(i) 2, 4, 6, 8, 10, ... (ii) 2, 4, 8, 16, 32, 64 ...

The first is an AP with common difference 2. In the second however, each term is *multiplied* by 2 to form the next term. This is the feature of a **geometric progression** or GP for short. In a GP, each term is **multiplied** by a constant, called the **common ratio (r)**, to give the next term.

Taking a as the first term, the pattern will be as follows:

first term	*second term*	*third term*	*fourth term*	...	*nth term*
T_1	T_2	T_3	T_4	...	T_n
a	$a \times r$	$ar \times r$	$ar^2 \times r$		
a	ar	ar^2	ar^3	...	ar^{n-1}

The formula for the nth term of a GP is

$$T_n = ar^{n-1}$$

Note carefully – the index for T_n is $\boldsymbol{n-1}$.

In a GP, the *ratio* of a term to the preceding one is always equal to r.

$$T_2{:}T_1 = T_3{:}T_2 = T_4{:}T_3 = \dots = r$$

Example 14

State the common ratio of

(a) $12, -4, 1\frac{1}{3}, \dots$ *(b)* $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \dots$ *(c)* $10\frac{2}{3}, 16, 24, \dots$

To find r, divide a term by the preceding one.

(a) $r = -4 \div 12 = -\frac{1}{3}$

(b) $r = \frac{1}{3} \div \frac{1}{2} = \frac{2}{3}$

(c) $r = 16 \div 10\frac{2}{3} = 16 \times \frac{3}{32} = \frac{3}{2}$

Example 15

Find (a) the 12th term of the GP 128, 64, 32, ... (b) a formula for the nth term.

(a) $a = 128$, $r = \frac{1}{2}$.

Then $T_{12} = ar^{11} = 128 \times \left(\frac{1}{2}\right)^{11} = \frac{2^7}{2^{11}} = \frac{1}{2^4} = \frac{1}{16}$.

(b) $T_n = ar^{n-1} = 128 \times \left(\frac{1}{2}\right)^{n-1} = \frac{2^7}{2^{n-1}} = 2^{7-(n-1)} = 2^{8-n}$

Example 16

If $x + 1$, $x + 3$ and $x + 8$ are the first three terms of a GP, find (a) the value of x, (b) the common ratio r.

(a) $r = \frac{x+3}{x+1} = \frac{x+8}{x+3}$

Then $(x + 3)(x + 3) = (x + 1)(x + 8)$

i.e. $x^2 + 6x + 9 = x^2 + 9x + 8$ which gives $x = \frac{1}{3}$.

(b) $r = \frac{x+3}{x+1} = \frac{10}{3} \div \frac{4}{3} = \frac{5}{2}$

Example 17

The 4th and 8th terms of a GP are 3 and $\frac{1}{27}$ respectively. Find the possible values of a and of r.

$T_4 = ar^3 = 3$ and $T_8 = ar^7 = \frac{1}{27}$.

Divide to eliminate a: $\frac{ar^7}{ar^3} = r^4 = \frac{1}{27} \div 3 = \frac{1}{81}$.

Hence $r = \pm \frac{1}{3}$.

If $r = +\frac{1}{3}$, $a\left(\frac{1}{3}\right)^3 = 3$ and $a = 3 \times 27 = 81$.

If $r = -\frac{1}{3}$, $a\left(-\frac{1}{3}\right)^3 = 3$ and $a = -81$.

This gives two possible GPs: 81, 27, 9, ... or -81, 27, -9, ...

Example 18

The first term of a GP exceeds the second term by 4 and the sum of the 2nd and 3rd terms is $2\frac{2}{3}$. Find the first three terms.

$a - ar = 4$ (i)

and $ar + ar^2 = \frac{8}{3}$ (ii)

Divide (i) by (ii) to eliminate a: $\dfrac{a - ar}{ar + ar^2} = 4 \div \frac{8}{3} = \frac{3}{2}$

Then $\dfrac{1 - r}{r + r^2} = \frac{3}{2}$

which gives $2 - 2r = 3r + 3r^2$ or $3r^2 + 5r - 2 = 0$.

Hence $(3r - 1)(r + 2) = 0$ giving $r = \frac{1}{3}$ or -2.

From (i), when $r = \frac{1}{3}$, $a = 6$ and when $r = -2$, $a = \frac{4}{3}$.

The first three terms are therefore either 6, 2, $\frac{2}{3}$ or $\frac{4}{3}, -\frac{8}{3}, \frac{16}{3}$.

Example 19

A store finds that it is selling 10% less of an article each week. In the first week it sold 500. In which week will it be first selling less than 200?

The number of articles sold forms a GP with $a = 500$ and $r = 0.9$.
[If $a = 500$ then T_2 is 10% less i.e. $T_2 = 0.9a$]
$T_n = 500(0.9)^{n-1}$ and we require the least value of n for which $500(0.9)^{n-1} < 200$ i.e. $0.9^{n-1} < 0.4$. Then $0.9^n < 0.4 \times 0.9 = 0.36$.
This can be found quickly using the x^y key of a calculator and testing values of 0.9^n for say $n = 5, 6, \ldots$ and stopping when the result is first < 0.36. This will be for $n = 10$. In the 10th week less than 200 are sold. (An alternative method using logarithms is shown in Chapter 15).

Geometric Means

If a, b and c are consecutive terms of a GP, then b is the **geometric mean** of a and c. $\frac{b}{a} = \frac{c}{b}$ so $b^2 = ac$ or $b = \sqrt{ac}$. For example, the geometric mean of 2 and 32 is 8 as $\sqrt{2 \times 32} = 8$.

Exercise 13.3 *(Answers on page 635.)*

1 Find the term stated for the following GPs:
(a) 5th term of 80, 20, 5, ...
(b) 6th term of the GP 60, 40, $26\frac{2}{3}$, ...
(c) 8th term of $\frac{729}{4}$, $\frac{243}{2}$, 81, ...

2 Find a formula in terms of n for the nth term of the GP $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$,...

3 The 2nd and 7th terms of a GP are $40\frac{1}{2}$ and $\frac{1}{6}$ respectively.
Find the first term and the common ratio.

4 The sum of the 1st and 3rd terms of a GP is $\frac{5}{8}$ and the sum of the 2nd and 4th terms is $1\frac{1}{4}$. Find the first term and the common ratio.

5 The 3rd and 6th terms of a GP are $2\frac{1}{4}$ and $-\frac{2}{3}$ respectively. Find the 2nd term.

6 Which of these sequences of numbers is a GP?
(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ (b) $\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \ldots$ (c) $\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \ldots$
(d) $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \ldots$ (e) $\frac{3}{8}, \frac{1}{4}, \frac{1}{6}, \ldots$ (f) 8, 12, 18, ...
(g) 9, 12, 15, ... (h) $\frac{1}{9}, \frac{1}{6}, \frac{1}{4}, \ldots$

7 The 2nd and 6th terms of a GP (with common ratio $r > 0$) are $\frac{8}{9}$ and $4\frac{1}{2}$ respectively. Find
(a) r,
(b) the first term,
(c) the 3rd term.

8 The 2nd and 5th terms of a GP are $\frac{1}{18}$ and $\frac{4}{243}$ respectively. Find
(a) the common ratio,
(b) the first term,
(c) the 3rd term.

9 The nth term T_n of a GP is given by $T_n = ak^{2n-1}$. State the first two terms. If the 5th term is 16 times the 3rd term, find the common ratio.

10 If 8, x, y and 27 are four consecutive terms of a GP, find the value of x and y.

11 The arithmetic mean of two numbers is 15 and their geometric mean is 9. Find the two numbers.

12 Given that $x + 2$, $x + 3$ and $x + 6$ are the first three terms of a GP, find (a) the value of x and (b) the 5th term of the GP.

13 Show that $x + 1$, $x + 3$ and $x + 5$ cannot be three consecutive terms of a GP, whatever the value of x.

14 If $x - 2$, $x - 1$ and $3x - 5$ are the first three terms of a GP, find
(a) the possible values of x,
(b) the common ratio for each of the possible GPs.

15 Each year the value of an article decreases by 8% of its value at the beginning of the year. If the original value was \$4000, after how many years will its value first be less than \$1000?

16 Which term of the GP $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \ldots$ is $\frac{4}{27}$?

17 The first term of a GP is 4 and the sum of the first three terms is 7. Find
(a) the possible values of r and
(b) the 5th term of each possible GP.

18 In a GP the sum of the 2nd and 3rd terms is $1\frac{5}{9}$ and the 4th term is $\frac{32}{27}$.
Find (a) the possible values of r and (b) the first term of the GP.

19 In a GP where $r > 0$, the sum of the 2nd and 3rd terms is $7\frac{1}{32}$ and the 4th term is $1\frac{7}{8}$.
Calculate the value of r.

20 For the GP 4, 6, 9, ... find the value of n if the nth term is the first term greater than 100.

21 When a ball is dropped onto a floor, it always rebounds a distance equal to $\frac{2}{3}$ of the height from which it fell. If it is dropped from a height of 300 cm, calculate, correct to the nearest cm, the height to which it will rise after the 6th bounce.

22 The sum of the first 3 terms of a GP is 7 times the first term. Find the possible values of the common ratio r if $r \neq 1$.

Sum of a Geometric Progression

If S_n denotes the sum of the first n terms of a GP then

$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

Now multiply throughout by r:

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$ (ii)

Subtract (ii) from (i):

$S_n - rS_n = a - ar^n$

or $S_n(1 - r) = a(1 - r^n)$ which gives $S_n = \dfrac{a(1-r^n)}{1-r}$

Hence the sum S_n of the first n terms of a GP, i.e. the sum of the first n terms of a geometric series, is given by:

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

This formula can also be written as $S_n = \frac{a(r^n-1)}{r-1}$. This form is more suitable when $r > 1$.
If $r = 1$, the formula cannot be used but the GP is then $a + a + a + \ldots + a$ and $S_n = na$.

Example 20

Find the sum of the first 10 terms of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$

$a = \frac{1}{8}, r = 2, n = 10$

$$S_{10} = \frac{\frac{1}{8}(2^{10} - 1)}{2 - 1} = \frac{1}{8}(1024 - 1) = \frac{1023}{8} = 127\tfrac{7}{8}$$

Example 21

Calculate, correct to 3 significant figures, the sum of the first 8 terms of the GP 12, 8, $5\frac{1}{3}, \ldots$.

$a = 12, r = \frac{2}{3}, n = 8$

$$S_8 = \frac{12[1 - (\frac{2}{3})^8]}{1 - \frac{2}{3}} = \frac{12[1 - (\frac{2}{3})^8]}{\frac{1}{3}} = 36[1 - \left(\tfrac{2}{3}\right)^8] \approx 36 \times 0.961 \approx 34.6$$

Example 22

What is the least number of terms of the GP 2, 3, $\frac{9}{2}$*, ... to be added to give a total greater than 30?*

Here $a = 2$ and $r = \frac{3}{2}$.

$$S_n = \frac{2[(\frac{3}{2})^n - 1]}{\frac{3}{2} - 1} = 4(1.5^n - 1) \text{ and this must be } >30.$$

Hence $1.5^n - 1 > 7.5$ or $1.5^n > 8.5$.
By calculator the least value of n (which must be an integer) satisfying this inequality is 6. So 6 terms must be added.

Exercise 13.4 *(Answers on page 635.)*

1 Find the sum of the first 6 terms of 64, 16, 4, ...

2 Calculate, correct to 3 significant figures, the sum of the first 7 terms of the GP 2, 1, $\frac{1}{2}, \ldots$

3 The 3rd and 6th terms of a GP are 108 and -32 respectively. Find
   (a) the common ratio,
   (b) the first term,
   (c) the sum of the first 6 terms.

4 In a GP the product of the 2nd and 4th terms is double the 5th term.
(a) Find the first term.
(b) If the sum of the first four terms is 80, show that $r^3 + r^2 + r + 1 = 40$.
(c) Hence find the value of r.
(d) Show that the sum of the first n terms is $3^n - 1$.

5 In a GP the product of the 1st and 7th terms is equal to the 4th term. Given that the sum of the 1st and 4th terms is 9, find
(a) the first term,
(b) the common ratio,
(c) the sum of the first six terms.

6 Three consecutive terms of a GP are $x + 1$, $x - 3$ and $x - 6$. Find
(a) the value of x,
(b) the common ratio
If $x + 1$ is the 4th term of the GP, find
(c) the first term and
(d) the sum of the first 5 terms correct to 3 significant figures.

7 The numbers $p - 1$, $2p - 2$ and $3p - 1$ are the first three terms of a GP, where $p > 0$.
(a) Find the value of p.
(b) Using this value of p, calculate the sum of the first 8 terms of the GP.

8 In a GP the 3rd and 5th terms are 9 and 1 respectively. Find the possible values of
(a) the common ratio,
(b) the first term,
(c) the sum of the first 6 terms.

9 A particle moves along a straight line starting from a point O on the line. It covers a distance of 24 cm in the 1st second, 16 cm in the 2nd, $10\frac{2}{3}$ cm in the 3rd and so on. What is its distance from O after 10 seconds, correct to the nearest cm?

10 The first term of a GP is 64 and the common ratio is $\frac{1}{2}$. How many terms must be added to obtain a total of $127\frac{1}{2}$?

11 The 1st, 3rd and 7th terms of an AP with common difference 2 are the first three terms of a GP. Find
(a) the common ratio of the GP and
(b) the sum of the first 6 terms of the GP.

12 Find 3 geometrical means between $40\frac{1}{2}$ and 8. Hence find the sum of these 5 terms of the GP.

13 A ball is thrown vertically upwards to a height of 81 cm above a floor. After each bounce it rises to a height of $\frac{2}{3}$ of the distance it dropped. Show that the total distance it travels until it reaches the floor for the nth bounce is given by $486[1 - \left(\frac{2}{3}\right)^n]$ cm. Calculate this distance correct to the nearest cm if $n = 8$.

14 The sum of n terms of a GP is given by $10 - \frac{10}{2^n}$. Find
(a) the sum of the first 4 terms,
(b) the 4th term,
(c) the first term,
(d) the common ratio.

15 The nth term (T_n) of a GP is given by $T_n = 2^{9-n}$. Find
(a) the first term,
(b) the common ratio,
(c) the sum of the first 9 terms.

16 What is the least number of terms of the GP 5, 6, 7.2, ... to be added for the total to exceed 100?

17 The annual output from an oil well diminishes each year by 5%. 200 000 barrels were extracted in the first year. It will become uneconomic to use the well when the output is less than 80 000 barrels per year. For how many years can the well be used? How much oil (in barrels correct to 2 significant figures) will have been extracted by then?

Sum to Infinity of a Geometric Progression

If we take more and more terms of a GP so that there is never a last term, we have an *infinite* GP. We look at what happens to the sum of such a GP.

Consider these two GPs:

(i) 1, 2, 4, 8, ... (ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$

In (i) $a = 1$ and $r = 2$. Then the sum of n terms (S_n) is given by $S_n = \dfrac{1(2^n - 1)}{2 - 1} = 2^n - 1$.

Now as n increases, 2^n will also increase, in fact doubling for each successive value of n. $2^n - 1$ will increase beyond any bound. We say $S_n \to \infty$ (tends to infinity) as $n \to \infty$. Note carefully that ∞ is NOT a number but a symbol to indicate that n and S_n continually increase without limit.

Now in (ii), $a = 1$, $r = \frac{1}{2}$ and $S_n = \dfrac{1\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$.

As n gets larger and larger (i.e. $n \to \infty$), $\frac{1}{2^n}$ will become smaller and smaller, i.e. $\to 0$. So $1 - \frac{1}{2^n}$ will get closer and closer to the value 1.

Then $S_n = 2\left(1 - \frac{1}{2^n}\right)$ will get closer and closer to 2 .

Here are some calculated values of S_n to show this:

n	S_n
5	1.9375
10	1.998 046 88
15	1.999 938 96
20	1.999 998 09
25	1.999 999 94

We write $\lim_{n \to \infty} 2\left(1 - \frac{1}{2^n}\right) = 2$ i.e. the limiting value of S_n is 2 as $n \to \infty$.

So in GP (ii), S_n continually increases as n increases and its limiting value is 2. This is illustrated in Fig.13.2, where the dots show values of S_n for increasing values of n.

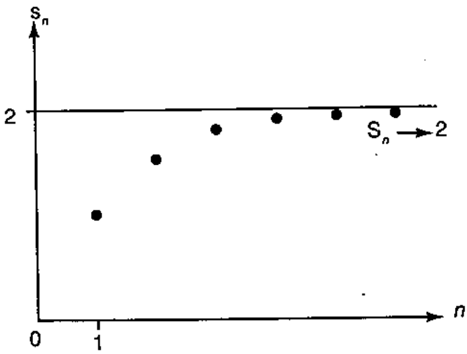


Fig. 13.2

We call this limiting value of S_n the **sum to infinity** (S_∞) of the GP. S_∞ is not the sum of an 'infinite' number of terms as there is no such number but is the limit of S_n when $n \to \infty$.

For the general GP $a, ar, ar^2, \ldots, S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$.

Now if r is numerically larger than 1, i.e. $r > 1$ or $r < -1$, $r^n \to \infty$ as $n \to \infty$ as in (i) above. So $S_n \to \infty$ as $n \to \infty$ and the GP has no sum to infinity.

However if r is numerically less than 1, i.e. $-1 < r < 1$, then r^n will get smaller and smaller as $n \to \infty$ and $S_n \to \frac{a}{1-r}$. The GP will have a sum to infinity and $S_\infty = \frac{a}{1-r}$.

The sum to infinity of the GP $a, ar, ar^2, \ldots$ is
$S_\infty = \frac{a}{1-r}$ provided $-1 < r < 1$.

Example 23

Find the sum to infinity of the GPs

(a) $20, 12, \frac{36}{5}, \ldots$

(b) $12, -3, \frac{3}{4}, \ldots$

(a) Here $a = 20$, $r = \frac{3}{5} < 1$. So $S_\infty = \frac{20}{1 - \frac{3}{5}} = 50$

(b) $|\,r\,| = |-\frac{1}{4}| < 1$ so $S_\infty = \frac{12}{1-(-\frac{1}{4})} = \frac{48}{5} = 9.6$

Example 24

What is the least number of terms of the GP 3, 1, $\frac{1}{3}$, ... for which their sum differs from the sum to infinity by less than 0.001?

$$S_\infty = \frac{30}{1-\frac{1}{3}} = \frac{9}{2}. \qquad S_n = \frac{3\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}} = \frac{9}{2}\left(1-\frac{1}{3^n}\right)$$

$S_\infty - S_n = \frac{9}{2} \times \frac{1}{3^n} < 0.001$. Then $\frac{9}{2} \times 1000 < 3^n$ or $3^n > 4500$.

By calculator the least value of n to satisfy this inequality is 8.

Example 25

The sum to infinity of a certain GP is 27. If the first term is 36, find r.

$S_\infty = \frac{a}{1-r} = \frac{36}{1-r} = 27$

so $36 = 27 - 27r$ giving $r = -\frac{1}{3}$.

This GP is 36, –12, 4, $-\frac{4}{3}$, $\frac{4}{9}$, $-\frac{4}{27}$ ...

If we add the terms one by one we get 36, 24, 28, $26\frac{2}{3}$, $27\frac{1}{9}$, $26\frac{26}{27}$, ... which are alternately greater and smaller than 27 but tend to the limit 27.

Example 26

In its first year a tin mine produced 100 tonnes but thereafter output fell by $12\frac{1}{2}$% per year. (Assume that production could continue in this way indefinitely).

(a) What is the maximum amount that could be extracted from the mine?

(b) It is decided to close the mine when the annual production falls below 40 tons. After how many years will this be done?

(c) How many tonnes of tin (correct to 2 significant figures) had been taken up to that time?

(d) What percentage of the maximum amount was extracted in that time?

This is a GP with $a = 100$ and $r = 1 - 0.125 = 0.875$

(a) Maximum amount = sum to infinity = $\frac{100}{1-0.875}$ = 800 tonnes.

(b) We must find n where $100(0.875)^{n-1} < 40$ i.e. $0.875^{n-1} < 0.4$ so $0.875^n < 0.35$.

BY calculator, $n = 8$, so the mine will be closed after 8 years.

(c) Amount produced in 8 years $= \frac{100[1-0.875^8]}{1-0.875}$

$$= \frac{100[1-0.344]}{0.125} \approx 530 \text{ tonnes}$$

(d) Percentage produced = $\frac{530}{800} \times 100\% \approx 66\%$.

Exercise 13.5 *(Answers on page 636.)*

1 Find the sum to infinity (if possible) of the following:

(a) $2 + \frac{1}{2} + \frac{1}{8} + \ldots$ (b) $24 + 8 + 2\frac{2}{3} + \ldots$

(c) $1 - 2 + 4 - 8 + \ldots$ (d) $36 + 30 + 25 + \ldots$

(e) $1 + 3 + 9 + \ldots$ (f) $10 - 5 + 2\frac{1}{2} + \ldots$

(g) $12 + 8 + 5\frac{1}{3} + \ldots$ (h) $16 - 4 + 1 + \ldots$

(i) $\frac{16}{27} + \frac{8}{9} + \frac{4}{3} \ldots$ (j) $1 + \cos 60° + \cos^2 60° + \ldots$

2 The sum to infinity of a GP is $\frac{2}{3}$ and the first term is $\frac{1}{12}$. Find the common ratio.

3 In Question 13 of Exercise 13.4, the distance the ball travels up to the nth bounce was given as $486[1 - \left(\frac{2}{3}\right)^n]$ cm. What would be the maximum distance the ball will travel before coming to a stop?

4 What is the sum to infinity of the GP 2, 0.2, 0.02, ...?

5 The sum to infinity of a GP with common ratio $\frac{5}{6}$ is 12. Find the first term.

6 The sum to infinity of a GP is 75 and the common ratio is $\frac{1}{5}$. Find the first term.

7 The sum to infinity of a GP is 9 and its second term is 2. Find the possible values of (a) the first term and (b) the common ratio.

8 If the sum to infinity of a GP is 4 times the first term, find the common ratio.

9 Find the sum of the first 8 terms of the GP $1, \frac{1}{2}, \frac{1}{4}, \ldots$

(a) What is the difference between this value and the sum to infinity?

(b) Express this difference as a percentage of the sum to infinity.

(c) Find the smallest value of n for which the sum of n terms differs from the sum to infinity by less than 0.001.

10 If the sum to infinity of a GP is twice the first term, find the common ratio.

11 If the sum to infinity of the GP $e + e^3 + e^5 + \ldots$ is $\frac{3}{8}$, find the value of e.

12 If the sum to infinity of a GP is 18 and the second term is 4, find the first term and the common ratio of the possible GPs.

13 The sum to infinity of a GP whose first term is a is a^2. Find the common ratio in terms of a.

14 If $1 + p$, $1 + 3p$ and $1 + 4p$ ($p \neq 0$) are the first three terms of a GP, find

(a) the value of p,

(b) the common ratio,

(c) the sum to infinity of the GP.

15 The first three terms of a GP are $2x - 1$, $x + 1$ and $x - 1$ ($x \neq 0$). Find the value of x and the sum to infinity of the GP.

16 A pendulum is released from the position OP (Fig.13.3) and swings freely. It swings through angles of 60°, 50°, $41\frac{2}{3}$° and so on. Find the total angle it swings through before coming to a halt.

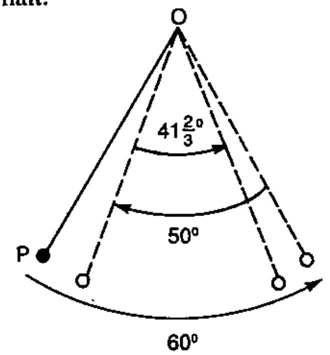


Fig. 13.3

17 The 3rd and 6th terms of a GP are $2\frac{2}{3}$ and $\frac{8}{81}$ respectively. Find
(a) the common ratio,
(b) the first term,
(c) the sum to infinity of this GP.

18 The 5th term of a GP is $\frac{2}{81}$ and the sum of the 3rd and 4th terms is $\frac{8}{27}$. Find the possible values of (a) r and (b) the sum to infinity of this GP.

19 What is the common ratio of a GP if the ratio of the sum to infinity to the sum of the first 7 terms is 128:127?

20 1200 people visited an exhibition on its opening day. Thereafter the attendance fell each day by 4% of the number on the previous day.
(a) The exhibition closed after 10 days. How many people visited it?
(b) If it had been kept open indefinitely, what would be the maximum number of visitors?

21 An oil well produced 100 000 barrels of oil in its first year but output fell by $7\frac{1}{2}$% each year thereafter. (Give answers correct to 2 significant figures).
(a) What is the maximum amount of oil that could be extracted?
(b) If the well is closed down after 15 years, what was the total amount of oil extracted in that time?
(c) What percentage of the maximum amount is left in the well?

22 Find the sum to infinity of the GP 0.9, 0.09, 0.009, ...

23 In Fig.13.4, ABC is an isosceles triangle with AB = AC. B_1, C_1 are the midpoints of AB, AC respectively. B_2, C_2 are the midpoints of AB_1, AC_1 respectively and so on.

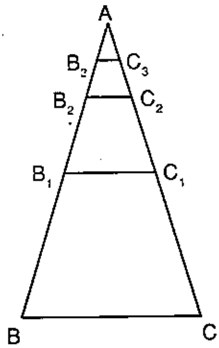


Fig. 13.4

(a) Show that the areas of the triangles ABC, AB_1C_1, AB_2C_2, AB_3C_3... form a GP and state the common ratio of this GP.

(b) If this sequence of triangles is continued indefinitely, find the sum of the areas AB_1C_1, AB_2C_2, AB_3C_3, ... as a fraction of the area of triangle ABC.

24 The first three terms of a GP are 1, a and b while the first three terms of an AP are 1, b and a where $a \neq b \neq 1$. Find

(a) the value of a and of b,

(b) the sum to infinity of the GP.

25 (a) If each term of the GP $a, ar, ar^2, \ldots$ $(r \neq 1)$ is squared, show that the new sequence is also a GP and find its sum to infinity.

(b) If this sum is equal to the sum to infinity of the original GP, find a relation between a and r.

SUMMARY

- Arithmetic progression (AP): a sequence $a, a + d, a + 2d, \ldots$ in which the terms increase by a fixed amount (the common difference d). a is the first term
- The nth term $T_n = a + (n - 1)d$.
- The sum of n terms $S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + L)$ where L is the last term being added.
- If a, b, c are consecutive terms of an AP, b is the arithmetic mean of a and c. $b = \frac{1}{2}(a + c)$.
- Geometric progression (GP): a sequence $a, ar, ar^2, \ldots$ in which each term is multiplied by a constant (the common ratio r). a is the first term.
- The nth term $T_n = ar^{n-1}$.
- The sum of n terms $S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}$. (Either form can be used).
- If a, b, c are consecutive terms of a GP, b is the geometric mean of a and c. $b = \sqrt{ac}$.
- Provided $-1 < r < 1$, a GP will have a sum to infinity S_∞ which is the limit of S_n when $n \to \infty$. $S_\infty = \frac{a}{1 - r}$.

REVISION EXERCISE 13 *(Answers on page 636.)*

A

1 The 5th and 12th terms of an AP are 2 and 23 respectively. Find
(a) the 9th term, (b) the sum of the first 20 terms.

2 There are 20 terms in an AP. The sum of the first 10 terms is 55 and the sum of the last 10 terms is 355. Find the first term and the common difference.

3 If the 9th term of an AP is 3 times the 3rd term and the sum of the first 10 terms is 110, find the first term and the common difference.

4 What is the greatest number of terms of the AP 40, 45, 50, ... which can be added if the total is not to exceed 500?

5 Find the sum of all the multiples of 3 between 5 and 91.

6 In an AP the sum of the first three terms is 12 and their product is 28. Find the possible values of the first term and the common difference.

7 The numbers $x + 3$, $5x + 3$ and $11x + 3$ ($x \neq 0$) are three consecutive terms of a GP. Find the value of x and the common ratio.

8 (a) The sum of the first 8 terms of an arithmetic progression is 24 and the sum of the first 18 terms is 90. Calculate the value of the seventh term.

(b) A geometric progression with a positive common ratio is such that the sum of the first 2 terms is $17\frac{1}{2}$ and the third term is $4\frac{2}{3}$. Calculate the value of the common ratio. (C)

9 (a) An arithmetic progression contains 20 terms. Given that the 8th term is 25 and that the sum of the last 8 terms is 404, calculate the sum of the first 8 terms.

(b) The first term of a geometric progression exceeds the second term by 2, and the sum of the second and third terms is $\frac{4}{3}$.
Calculate the possible values of the first term and of the common ratio of the progression. Given further that all the terms of the progression are positive, calculate the sum to infinity. (C)

10 (a) The first term of an arithmetic progression is 2. The sum of the first 8 terms is 58 and the sum of the whole series is 325.
Calculate (i) the common difference, (ii) the number of terms (iii) the last term.

(b) The first three terms of a geometric progression are $x + 5$, $x + 1$, x.
Calculate (i) the value of x, (ii) the common ratio, (iii) the sum to infinity. (C)

11 (a) A length of 200 cm is divided into 25 sections whose lengths are in arithmetic progression. Given that the sum of the lengths of the 3 smallest sections is 4.2 cm, find the length of the largest section.

(b) An infinite geometric progression has a finite sum. Given that the first term is 18 and that the sum of the first 3 terms is 38, calculate the value of (i) the common ratio, (ii) the sum to infinity. (C)

12 In an AP, the 15th term is double the 9th term. If also the sum of the first 15 and the sum of the first 9 terms added together is 279, find the first term and the common difference.

13 Deliveries from a warehouse are reduced each week by 10%. In the first week 2000 loads were taken. Assuming that this operation can continue indefinitely, find (i) the maximum number of loads in the warehouse. If it is decided to stop the operation as soon as the number of loads taken falls below 400, find (ii) the number of weeks the operation was in action, (iii) the total number of loads taken and (iv) the percentage of the maximum left in the warehouse.

14 What percentage of the sum to infinity is the sum of 10 terms of the GP $1 + 0.8 + 0.64 + \dots$? Find the least value of n for which the sum of n terms differs from the sum to infinity by less than 0.1.

15 r is the common ratio of a GP ($r \neq 1$) and the sum of the first 4 terms is 5 times the sum of the first 2 terms. Find the possible values of r.

16 The positive numbers p, 6 and q are three consecutive terms of a GP. p and q are also the 3rd and 5th terms respectively of an AP whose common difference is $2\frac{1}{2}$. Find the possible values of p and q and the common ratio of the GP.

B

17 Given that the first term of an AP is 1 and that the sum of first n terms is n^2, find the AP.

18 Three successive terms of a GP are 1, $\sin\theta$ and $\cos\theta$ where $0 < \theta < \pi$. Find

(a) the value of θ in radians and

(b) the common ratio of the GP.

19 Given that the $T_7 = 23$, $T_n = 43$ and $T_{2n} = 91$ in an AP, find the values of a, d and n.

20 In Fig.13.5, the outer square is divided into 4 equal squares and one is shaded. One of the three remaining squares is again divided into 4 smaller squares with 1 square shaded. This process is then repeated indefinitely. What is the ultimate value of the ratio of the total shaded area to the area of the original square?

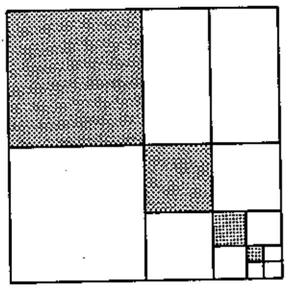


Fig. 13.5

21 If the common ratio of a GP with a sum to infinity is $x^2 - x - 1$, within what limits must x lie?

22 Three squares are placed side by side on the line PQ as shown in Fig.13.6. The lengths of their sides form a geometric progression.

(a) Show that the vertices A, B and C lie in a straight line.

(b) If the length of a side of the largest square is double the length of a side of the smallest square, what angle does ABC make with PQ?

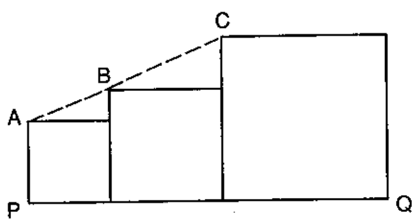


Fig. 13.6

23 If the value of an article is assumed to increase annually by 5% of its value at the beginning of the year, after how many years will its value have doubled?

24 What is the least number of terms of the GP $1 + \frac{1}{4} + \frac{1}{16} + \dots$ which should be added so that their sum is less than the sum to infinity by 10^{-6}?

Further Trigonometry: Compound and Multiple Angles: $a \cos \theta + b \sin \theta$

14

In this Chapter, we study the trigonometric functions for **compound angles** such as $A \pm B$ where A and B are any two angles.

Let us consider the values of sin A, sin B and sin(A + B) where A = 60° and B = 30°. Is sin(A + B) = sin A + sin B? Is sin(A − B) = sin A − sin B? The answers are 'No' in both cases. We now derive suitable formulae to give the correct results.

ADDITION FORMULAE

I Sum of two angles A + B

Fig.14.1 shows two angles, ∠UOP = A, ∠TOU = B. Then ∠TOP = A + B.

For simplicity we take OT as 1 unit in length.

TP is perpendicular to OP, TU is perpendicular to OU and UR is perpendicular to TP.

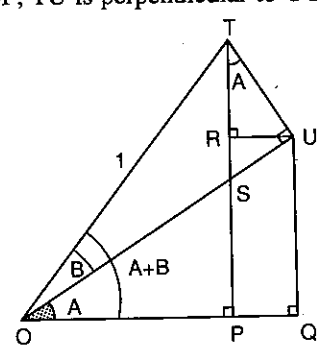


Fig. 14.1

Now ∠OSP = 90° − A = ∠TSU. So ∠RTU = A.
PT = PR + RT = QU + RT

Now PT = 1 × sin(A + B), QU = OU sin A and RT = UT cos A.

So sin(A + B) = OU sin A + UT cos A.

But OU = 1 × cos B and UT = 1 × sin B.

Hence **sin(A + B) = sin A cos B + cos A sin B** (i)

This gives us a formula for sin(A + B) in terms of A and B. It is rather more complicated than you may have expected.

Now we find a formula for cos(A + B).

OP = OQ – PQ = OQ – RU
OP = 1 × cos(A + B), OQ = OU cos A and RU = UT sin A.

So cos(A + B) = OU cos A – UT sin A.

But OU = cos B and UT = sin B.

Hence **cos(A + B) = cos A cos B – sin A sin B** (ii)

II Difference of two angles, A – B

In each of (i) and (ii), change B to –B.
Remember that sin(–B) = – sin B and that cos(–B) = cos B.
Then **sin(A – B) = sin A cos B – cos A sin B** (iii)
and **cos(A – B) = cos A cos B + sin A sin B** (iv)

These four formulae are very important. Note the pattern of sin and cos in each pair to help you remember them. Also note carefully the reversal of the sign in the two cos formulae.

Addition formulae

sin(A + B) = sin A cos B + cos A sin B
sin(A – B) = sin A cos B – cos A sin B
cos(A + B) = cos A cos B – sin A sin B
cos(A – B) = cos A cos B + sin A sin B

These formulae are *identities* and are true for *any* angles A and B.

Example 1

Show that cos (90° + θ) = –sin θ.

$\cos(90° + \theta) = \cos 90° \cos\theta - \sin 90° \sin\theta$
$\qquad = 0 - \sin\theta$ as $\cos 90° = 0$ and $\sin 90° = 1$.

Example 2

If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, calculate (without using tables or calculators) the values of $\sin(A + B)$ and $\cos(A + B)$ when
(a) A and B are both acute angles,
(b) A is obtuse and B is acute.

If $\sin A = \frac{4}{5}$, then $\cos^2 A = 1 - \sin^2 A = \frac{9}{25}$

and $\cos A = \pm \frac{3}{5}$ (+ if A is acute, – if A is obtuse).

If $\cos B = \frac{12}{13}$, then $\sin^2 B = 1 - \cos^2 B = \frac{25}{169}$

and $\sin B = \frac{5}{13}$ (+ for B acute and obtuse).

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13} = \frac{63}{65}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = \frac{16}{65}$

(b) $\sin(A + B) = \frac{4}{5} \times \frac{12}{13} + \left(-\frac{3}{5}\right) \times \frac{5}{13} = \frac{33}{65}$ (as A is obtuse)

$\cos(A + B) = \left(-\frac{3}{5}\right) \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = -\frac{56}{65}$

Example 3

In a triangle ABC, $\cos A = \frac{12}{13}$ and $\cos B = -\frac{4}{5}$.
Calculate, without using tables or calculators, (a) $\cos(A + B)$, (b) $\cos C$.

A is acute but B is obtuse. Hence C must be acute.

$\sin^2 A = 1 - \cos^2 A = 1 - \frac{144}{169} = \frac{25}{169}$ so $\sin A = \frac{5}{13}$ (+ as A is acute).

$\sin^2 B = 1 - \cos^2 B = 1 - \frac{16}{25} = \frac{9}{25}$ so $\sin B = \frac{3}{5}$ (+ as B is obtuse).

(a) $\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{12}{13} \times \left(-\frac{4}{5}\right) - \frac{5}{13} \times \frac{3}{5} = -\frac{63}{65}$

(b) $A + B + C = 180°$ so $C = 180° - (A + B)$.

Then $\cos C = \cos [180° - (A + B)] = -\cos(A + B) = \frac{63}{65}$.

Example 4

Given that $\sin \theta = \frac{3}{5}$ where $90° < \theta < 180°$ and that $\cos \phi = -\frac{5}{13}$ where $180° < \phi < 270°$, find $\cos(\theta - \phi)$.

$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$ so $\cos \theta = -\frac{4}{5}$ as $90° < \theta < 180°$.

$\sin^2 \phi = 1 - \cos^2 \phi = 1 - \frac{25}{169} = \frac{144}{169}$ so $\sin \phi = -\frac{12}{13}$ as $180° < \phi < 270°$.

Hence $\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$

$= \left(-\frac{4}{5}\right) \times \left(-\frac{5}{13}\right) + \left(\frac{3}{5}\right) \times \left(-\frac{12}{13}\right) = -\frac{16}{65}$

Example 5

Given that $\sin A = 0.6$ where $90° < A < 180°$ and that $\cos B = \frac{8}{17}$ where $270° < B < 360°$, calculate the value of $\cos(A - B)$ without using tables or calculators.

If $\sin A = 0.6 = \frac{3}{5}$, then $\cos^2 A = 1 - \frac{9}{25} = \frac{16}{25}$

and $\cos A = -\frac{4}{5}$ (as $90° < A < 180°$).

If $\cos B = \frac{8}{17}$, then $\sin^2 B = 1 - \frac{64}{289} = \frac{225}{289}$

and $\sin B = -\frac{15}{17}$ (as $270° < B < 360°$).

Then $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$= \left(-\frac{4}{5}\right) \times \left(\frac{8}{17}\right) + \left(\frac{3}{5}\right) \times \left(-\frac{15}{17}\right) = -\frac{77}{85}$$

Example 6

Simplify $\cos\theta \cos 60° - \sin\theta \sin 60°$ and hence solve the equation $\cos\theta \cos 60° - \sin\theta \sin 60° = -0.25$ for $0° < \theta < 360°$.

We must recognize that the given expression is the right hand side of the formula for $\cos(A + B)$ where $A = \theta$, $B = 60°$.

Hence $\cos\theta \cos 60° - \sin\theta \sin 60° = \cos(\theta + 60°)$.

Then $\cos(\theta + 60°) = -0.25$ and $\theta + 60° = 104.5°$ or $255.5°$ giving $\theta = 44.5°$ or $195.5°$.

Example 7

Solve the equation $\sin(\theta + 30°) = 3\cos\theta$ for $0° < \theta < 360°$.

Expand the left hand side: $\sin\theta \cos 30° + \cos\theta \sin 30° = \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = 3\cos\theta$. This reduces to $\sqrt{3}\sin\theta = 5\cos\theta$ so $\tan\theta = \frac{5}{\sqrt{3}}$ and $\theta = 70.9°$ or $250.9°$.

TANGENTS OF COMPOUND ANGLES A ± B

We can derive the formulae for tan(A ± B) directly from the four formulae derived earlier.

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} \quad \text{dividing each term by } \cos A \cos B$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Similarly, $\tan(A - B) = \dfrac{\sin(A - B)}{\cos(A - B)} = \dfrac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{after dividing each term by } \cos A \cos B.$$

So the two formulae for tan(A ± B) are

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 8

Angles A and B are both obtuse angles. Given that $\sin A = \frac{5}{13}$ *and that* $\cos B = -\frac{3}{5}$, *find* $\tan(A - B)$.

$\cos^2 A = 1 - \frac{25}{169} = \frac{144}{169}$ so $\cos A = -\frac{12}{13}$

$\tan A = \frac{\sin A}{\cos A} = \frac{5}{13} \div \left(-\frac{12}{13}\right) = -\frac{5}{12}$

$\sin^2 B = 1 - \frac{9}{25} = \frac{16}{25}$ so $\sin B = \frac{4}{5}$

$\tan B = \frac{\sin B}{\cos B} = \frac{4}{5} \div \left(-\frac{3}{5}\right) = -\frac{4}{3}$

Then $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{-\frac{5}{12} - \left(-\frac{4}{3}\right)}{1 + \left(-\frac{5}{12}\right)\left(-\frac{4}{3}\right)} = \dfrac{\frac{11}{12}}{1 + \frac{5}{9}} = \dfrac{33}{56}$

Exercise 14.1 *(Answers on page 636.)*

1 Simplify

(a) cos(A – B) – cos(A + B),

(b) sin(A + B) – sin(A – B).

2 Prove that
(a) $\sin(90° + \theta) = \cos\theta$,
(b) $\cos(180° + \theta) = -\cos\theta$.

3 If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ where A and B are both acute angles, find, without using tables or a calculator,
(a) $\sin(A + B)$,
(b) $\cos(A - B)$,
(c) $\tan(A + B)$.

4 Simplify $\cos\theta\cos\left(\frac{\pi}{2} + \theta\right) + \sin\theta\sin\left(\frac{\pi}{2} + \theta\right)$.

5 Simplify $\cos 40°\cos x - \sin 40°\sin x$ and hence solve the equation $\cos 40°\cos x - \sin 40°\sin x = 0.7$ for $0° < x < 360°$.

6 Given that $\sin A = -\frac{8}{17}$ $(180° < A < 270°)$ and that $\cos B = -\frac{5}{13}$ $(90° < B < 180°)$, calculate (without using tables or a calculator) the value of (a) $\cos(A - B)$, (b) $\sin(A + B)$, (c) $\tan(A - B)$.

7 Prove that $\tan(\theta + 45°) = \frac{1 + \tan\theta}{1 - \tan\theta}$.
Given that $\tan(\theta + 45°) = \frac{3}{2}$, find the value of $\tan\theta$.

8 Given that $\frac{\cos(A + B)}{\cos(A - B)} = \frac{1}{3}$, show that $2\sin A\sin B = \cos A\cos B$ and that $2\tan A = \cot B$. Given also that $\tan A = \frac{2}{5}$, find $\tan B$ and hence find $\tan(A - B)$.

9 If A and B are acute angles such that $\tan A = \frac{4}{5}$ and $\tan B = \frac{1}{9}$, show that $A + B = 45°$.

10 Find the value of $\sin\left(\frac{\pi}{4} + \theta\right)\cos\theta - \cos\left(\frac{\pi}{4} + \theta\right)\sin\theta$.

11 Express $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$. Given that $\tan\alpha = a$ and $\tan(\alpha + \beta) = b$, find $\tan\beta$ in terms of a and b. Hence express $\tan(\alpha - \beta)$ in terms of a and b.

12 Solve, in the domain $0° \le \theta \le 360°$, the equations
(a) $\cos(\theta + 30°) = \sin\theta$
(b) $\sqrt{2}\cos(\theta - 45°) = 4\sin\theta$
(c) $\cos(\theta - 30°) = \sqrt{2}\sin(\theta + 45°)$
(d) $\cos(\theta + 60°) = 2\sin(\theta + 30°)$
(e) $2\sin(\theta + 120°) = \cos(\theta + 150°)$.

13 Solve the equations (a) $\sin 50°\cos x + \cos 50°\sin x = 1$ and (b) $\tan x - \tan 35° = 1 + \tan x\tan 35°$ for $0° < x < 360°$.

14 Given that $\cos A = 0.6$ and $\cos B = 0.8$ (A and B both acute angles), find, without using tables or a calculator, the value of
(a) $\sin(A + B)$,
(b) $\cos(A - B)$,
(c) $\tan(A - B)$.

15 θ and ϕ are acute angles where $\cos\theta = \frac{3}{5}$ and $\tan\phi = \frac{8}{15}$. Without using tables or a calculator, find the value of
(a) $\sin(\theta + \phi)$, (b) $\tan(\theta - \phi)$, (c) $\sec(\theta - \phi)$, (d) $\cot(\theta + \phi)$.

16 A and B are both acute angles such that $\cos A = \frac{12}{13}$ and $\tan B = \frac{20}{21}$. Without using tables or calculators, find the value of
(a) $\sin(A - B)$, (b) $\cos(A + B)$, (c) $\tan(A - B)$, (d) $\cot(A + B)$.

17 If $\cos\theta + \cos\phi = p$ and $\sin\theta + \sin\phi = q$, by squaring each one and adding, prove that $\cos(\theta - \phi) = \frac{1}{2}(p^2 + q^2 - 2)$.

18 If $\sin A \sin B = \frac{1}{2}$ and $\cos A \cos B = \frac{1}{3}$ (A and B both acute), find the values of $\cos(A + B)$ and $\cos(A - B)$. Using tables or a calculator, find the values of the angles $A + B$ and $A - B$ and hence find the values of A and B.

19 If $\sin\theta = 2\sin(A - \theta)$, prove that $\tan\theta = \frac{2\sin A}{1 + 2\cos A}$.
Given that $\cos A = \frac{1}{3}$, where A is an acute angle, solve the equation
$\sin\theta = 2\sin(A - \theta)$ for $0° \leq \theta \leq 360°$.

20 Simplify $\sin(A + B) - \sin(A - B)$ and $\cos(A + B) - \cos(A - B)$.
By taking $X = A + B$ and $Y = A - B$, show that
$\sin X - \sin Y = 2\sin\frac{1}{2}(X - Y)\cos\frac{1}{2}(X + Y)$ and
$\cos X + \cos Y = 2\cos\frac{1}{2}(X + Y)\cos\frac{1}{2}(X - Y)$.

21 The position vectors with respect to an origin O of the points A and B are $\mathbf{a} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{b} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ respectively, where $\theta > \phi$.
(a) Show by means of a diagram that the angle between OA and OB is $\theta - \phi$.
(b) Find the scalar product $\mathbf{a}.\mathbf{b}$ and hence derive the formula
$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$.

MULTIPLE ANGLES

The addition formulae can be used to find expressions for trigonometric functions of **multiple angles** such as 2A, $\frac{A}{2}$, etc.

In the formula for $\sin(A + B)$, put $B = A$.

Then $\mathbf{\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A.}$ (i)

Again, in the formula for $\cos(A + B)$, put $B = A$.

Then $\mathbf{\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A}$ (ii)

This can also be expressed in two other useful forms.

$\mathbf{\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1}$ (iii)

or $\mathbf{\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A}$ (iv)

Also putting $B = A$ in the formula for $\tan(A + B)$,

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A\tan A} = \frac{2\tan A}{1 - \tan^2 A} \qquad \text{(v)}$$

These five formulae are known as the **double-angle formulae.**

$$\sin 2A \equiv 2\sin A\cos A$$

$$\left.\begin{aligned}\cos 2A &\equiv \cos^2 A - \sin^2 A\\ &\equiv 2\cos^2 A - 1\\ &\equiv 1 - 2\sin^2 A\end{aligned}\right\}\ \text{Note that in these, the cos always comes first, sin second.}$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

Note that in each one, the angle on the left is **double** the angle appearing on the right. So, for example, we could have

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ or $\cos 4A = 1 - 2\sin^2 2A$, etc.

If we start with angle A on the left hand side, then the formulae become

$$\begin{aligned}\sin A &\equiv 2\sin\tfrac{A}{2}\cos\tfrac{A}{2}\\ \cos A &\equiv \cos^2\tfrac{A}{2} - \sin^2\tfrac{A}{2}\\ &\equiv 2\cos^2\tfrac{A}{2} - 1\\ &\equiv 1 - 2\sin^2\tfrac{A}{2}\\ \tan A &\equiv \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}\end{aligned}$$

These are sometimes called the **half-angle formulae.**

Also, for future use, note that

$$\begin{aligned}\sin^2 A &\equiv \tfrac{1}{2}(1-\cos 2A)\\ \cos^2 A &\equiv \tfrac{1}{2}(1+\cos 2A)\end{aligned}$$

Example 9

Given that $\sin A = \frac{4}{5}$ where $0° < A < 90°$, find the values of
(a) $\sin 2A$, (b) $\cos 2A$, (c) $\cos 4A$, (d) $\tan 2A$, (e) $\tan 4A$.

If $\sin A = \frac{4}{5}$, then $\cos^2 A = 1 - \frac{16}{25}$ and $\cos A = \frac{3}{5}$ (+ as A is acute).

(a) $\sin 2A = 2\sin A\cos A = 2 \times \frac{4}{5} \times \frac{3}{5} = \frac{24}{25}$

(b) $\cos 2A = 1 - 2\sin^2 A = 1 - 2 \times \frac{16}{25} = -\frac{7}{25}$

(Either of the other two formulae for cos 2A could have been used instead.)

(c) $\cos 4A = 2\cos^2 2A - 1 = 2 \times \frac{49}{625} - 1 = -\frac{527}{625}$

(d) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{24}{25} \div \left(-\frac{7}{25}\right) = -\frac{24}{7}$

(e) $\tan 4A = \frac{2 \tan 2A}{1 - \tan^2 2A} = \frac{2 \times (-\frac{24}{7})}{1 - (-\frac{24}{7})^2} = -\frac{48}{7} \times \left(-\frac{49}{527}\right) = \frac{336}{527}$

Example 10

If $\sin \frac{\theta}{2} = \frac{5}{13}$, find the value of $\sin \theta$ where $0° < \theta < 90°$.

$\cos^2 \frac{\theta}{2} = 1 - \frac{25}{169}$ so $\cos \frac{\theta}{2} = \frac{12}{13}$ (+ as $0° < \frac{\theta}{2} < 45°$).

$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ and

hence $\sin \theta = 2 \times \frac{5}{13} \times \frac{12}{13} = \frac{120}{169}$

Example 10

If $\cos \theta = -\frac{7}{9}$ $(90° < \theta < 180°)$, find the values of (a) $\cos \frac{\theta}{2}$, (b) $\tan \frac{\theta}{2}$.

(a) We can obtain $\cos \frac{\theta}{2}$ by using $\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1$.

Then $-\frac{7}{9} = 2 \cos^2 \frac{\theta}{2} - 1$ so $\cos^2 \frac{\theta}{2} = \frac{1}{9}$ and

$\cos \frac{\theta}{2} = \frac{1}{3}$ (+ because $45° < \frac{\theta}{2} < 90°$).

(b) To find $\tan \frac{\theta}{2}$ we use the identity $\sec^2 \frac{\theta}{2} = 1 + \tan^2 \frac{\theta}{2}$.

$\sec^2 \frac{\theta}{2} = \frac{1}{\cos^2 \frac{\theta}{2}} = 9$

Hence $\tan^2 \frac{\theta}{2} = 9 - 1 = 8$

and $\tan \frac{\theta}{2} = \sqrt{8}$ (+ as $45° < \frac{\theta}{2} < 90°$).

Example 11

Solve the equations (a) $3 \sin x \cos x + 1 = 0$, (b) $\tan 2x = 3 \cot x$ for $0° \le x \le 360°$.

(a) $\sin x \cos x = \frac{1}{2} \sin 2x$

Then $\frac{3}{2} \sin 2x = -1$ or $\sin 2x = -0.6667$.

This gives $2x = 221.8°$ or $318.2°$ or $581.8°$ or $678.2°$ $(0° \le 2x \le 720°)$.
Then $x = 110.9°, 159.1°, 290.9°$ or $339.1°$.

(b) $\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{3}{\tan x}$

Then $2\tan^2 x = 3 - 3\tan^2 x$ or $5\tan^2 x = 3$

and $\tan x = \pm\sqrt{0.6} = \pm 0.7746$.

Hence $x = 37.8°, 142.2°, 217.8°$ or $322.2°$.

Example 12

Solve the equation $2\sin 2\theta = \tan\theta$ for $0° \leq \theta \leq 360°$ and show the solutions on a sketch graph.

We rewrite the equation as $2 \times 2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}$.

Then $4\sin\theta\cos^2\theta - \sin\theta = 0$ or $\sin\theta(4\cos^2\theta - 1) = 0$.

(*Note:* We must not divide through by $\sin\theta$ as $\sin\theta = 0$ contains possible solutions.)

Then $\sin\theta = 0$ or $\cos^2\theta = \frac{1}{4}$ i.e. $\cos\theta = \pm\frac{1}{2}$.

Verify that the first gives $\theta = 0°, 180°, 360°$

and that the second gives $\theta = 60°, 120°, 240°, 300°$.

So the complete solutions are $0°, 60°, 120°, 180°, 240°, 300°, 360°$.

The graphs of $y = 2\sin 2\theta$ and $y = \tan\theta$ are shown in Fig.14.2 with these solutions marked.

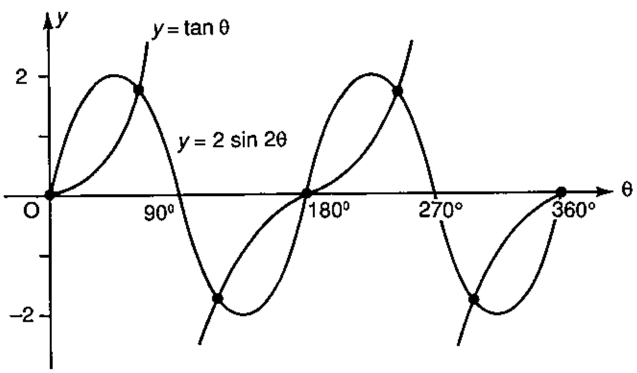


Fig.14.2

Exercise 14.2 *(Answers on page 637.)*

1 Show that $\sin^2 \frac{\theta}{4} = \frac{1}{2}\left(1 - \cos \frac{\theta}{2}\right)$ and that $\cos^2 \frac{3\theta}{2} = \frac{1}{2}(1 + \cos 3\theta)$.

2 Given that $\cos A = \frac{5}{13}$ where A is an acute angle, calculate without using tables or a calculator the value of (a) $\sin 2A$, (b) $\cos 2A$, (c) $\tan 2A$, (d) $\cos \frac{A}{2}$.

3 If $\cos \theta = \frac{4}{5}$ $(0° < \theta < 90°)$, find, without using tables or a calculator, the value of

(a) $\sin \theta$ (b) $\sin 2\theta$ (c) $\cos 2\theta$

(d) $\tan 2\theta$ (e) $\sin \frac{\theta}{2}$ (f) $\cos \frac{\theta}{2}$

4 If θ is an acute angle and $\cos 2\theta = \frac{119}{169}$, find, without using tables or a calculator, the value of (a) $\sin \theta$, (b) $\cos \theta$, (c) $\tan 2\theta$.

5 Given that $\sin \frac{\theta}{2} = \frac{1}{3}$, find, without using tables or a calculator, the value of (a) $\cos \frac{\theta}{2}$, (b) $\sin \theta$, (c) $\cos \theta$.

6 Prove that

(a) $(\sin \theta + \cos \theta)^2 \equiv 1 + \sin 2\theta$ (b) $\tan \theta \equiv \dfrac{\sin 2\theta}{1 + \cos 2\theta}$

(c) $\dfrac{1 - \cos \theta}{1 + \cos \theta} \equiv \tan^2 \dfrac{\theta}{2}$ (d) $\dfrac{1 - \cos \theta}{\sin \theta} \equiv \tan \dfrac{\theta}{2}$

(e) $(2 \cos \theta + 1)(2 \cos \theta - 1) \equiv 2 \cos 2\theta + 1$

(f) $\cos^4 \theta - \sin^4 \theta \equiv \cos 2\theta$

(g) $\cot \theta - \tan \theta \equiv 2 \cot 2\theta$

(h) $\cos^2\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1 + \sin 2\theta}{2}$

7 Find, without using tables or a calculator, the value of $\tan 2\theta$ if $\sin \theta = \frac{1}{2}$.

8 If $\tan 2\theta = \frac{4}{3}$, find, without using tables or a calculator, the value of (a) $\tan \theta$, (b) $\sin \theta$.

9 If $\tan \frac{A}{2} = \frac{1}{2}$, find, without using tables or a calculator, the value of $\cot A$.

10 If $\cos \theta = p$, find in terms of p, the value of (a) $\sin \theta$, (b) $\sin \frac{\theta}{2}$, (c) $\cos \frac{\theta}{2}$, (d) $\cos 2\theta$, (e) $\sin 2\theta$, (f) $\cos 4\theta$.

11 Express each of the following as a single trigonometrical function:

(a) $\sin A \cos A$ (b) $1 - 2 \sin^2 A$

(c) $\dfrac{2 \tan A}{1 - \tan^2 A}$ (d) $\cos^2 A - \sin^2 A$.

Given that $\sin A = s$, find the values of each one in terms of s.

12 State each of the following as a single trigonometrical function

(a) $\sin 35° \cos 35°$

(b) $\dfrac{\tan 40°}{1 - \tan^2 40°}$

(c) $\sin 25° \cos 35° - \cos 25° \sin 35°$

(d) $\dfrac{\tan 50° - \tan 40°}{1 + \tan 50° \tan 40°}$

(e) $0.5 - \sin^2 30°$

(b) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x} = \dfrac{3}{\tan x}$

Then $2 \tan^2 x = 3 - 3 \tan^2 x$ or $5 \tan^2 x = 3$

and $\tan x = \pm \sqrt{0.6} = \pm 0.7746$.

Hence $x = 37.8°, 142.2°, 217.8°$ or $322.2°$.

Example 12

Solve the equation $2 \sin 2\theta = \tan \theta$ for $0° \le \theta \le 360°$ and show the solutions on a sketch graph.

We rewrite the equation as $2 \times 2 \sin \theta \cos \theta = \dfrac{\sin \theta}{\cos \theta}$.

Then $4 \sin \theta \cos^2 \theta - \sin \theta = 0$ or $\sin \theta (4 \cos^2 \theta - 1) = 0$.

(*Note:* We must not divide through by $\sin \theta$ as $\sin \theta = 0$ contains possible solutions.)

Then $\sin \theta = 0$ or $\cos^2 \theta = \frac{1}{4}$ i.e. $\cos \theta = \pm \frac{1}{2}$.

Verify that the first gives $\theta = 0°, 180°, 360°$

and that the second gives $\theta = 60°, 120°, 240°, 300°$.

So the complete solutions are $0°, 60°, 120°, 180°, 240°, 300°, 360°$.

The graphs of $y = 2 \sin 2\theta$ and $y = \tan \theta$ are shown in Fig.14.2 with these solutions marked.

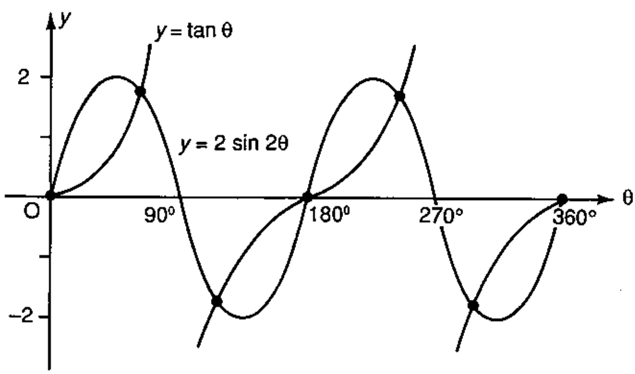


Fig.14.2

Exercise 14.2 *(Answers on page 637.)*

1 Show that $\sin^2 \frac{\theta}{4} = \frac{1}{2}\left(1 - \cos\frac{\theta}{2}\right)$ and that $\cos^2 \frac{3\theta}{2} = \frac{1}{2}(1 + \cos 3\theta)$.

2 Given that $\cos A = \frac{5}{13}$ where A is an acute angle, calculate without using tables or a calculator the value of (a) $\sin 2A$, (b) $\cos 2A$, (c) $\tan 2A$, (d) $\cos \frac{A}{2}$.

3 If $\cos\theta = \frac{4}{5}$ $(0° < \theta < 90°)$, find, without using tables or a calculator, the value of

(a) $\sin\theta$	(b) $\sin 2\theta$	(c) $\cos 2\theta$
(d) $\tan 2\theta$	(e) $\sin\frac{\theta}{2}$	(f) $\cos\frac{\theta}{2}$

4 If θ is an acute angle and $\cos 2\theta = \frac{119}{169}$, find, without using tables or a calculator, the value of (a) $\sin\theta$, (b) $\cos\theta$, (c) $\tan 2\theta$.

5 Given that $\sin\frac{\theta}{2} = \frac{1}{3}$, find, without using tables or a calculator, the value of (a) $\cos\frac{\theta}{2}$, (b) $\sin\theta$, (c) $\cos\theta$.

6 Prove that

(a) $(\sin\theta + \cos\theta)^2 \equiv 1 + \sin 2\theta$

(b) $\tan\theta \equiv \dfrac{\sin 2\theta}{1 + \cos 2\theta}$

(c) $\dfrac{1 - \cos\theta}{1 + \cos\theta} \equiv \tan^2\dfrac{\theta}{2}$

(d) $\dfrac{1 - \cos\theta}{\sin\theta} \equiv \tan\dfrac{\theta}{2}$

(e) $(2\cos\theta + 1)(2\cos\theta - 1) \equiv 2\cos 2\theta + 1$

(f) $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$

(g) $\cot\theta - \tan\theta \equiv 2\cot 2\theta$

(h) $\cos^2\left(\frac{\pi}{4} - \theta\right) = \dfrac{1 + \sin 2\theta}{2}$

7 Find, without using tables or a calculator, the value of $\tan 2\theta$ if $\sin\theta = \frac{1}{2}$.

8 If $\tan 2\theta = \frac{4}{3}$, find, without using tables or a calculator, the value of (a) $\tan\theta$, (b) $\sin\theta$.

9 If $\tan\frac{A}{2} = \frac{1}{2}$, find, without using tables or a calculator, the value of $\cot A$.

10 If $\cos\theta = p$, find in terms of p, the value of (a) $\sin\theta$, (b) $\sin\frac{\theta}{2}$, (c) $\cos\frac{\theta}{2}$, (d) $\cos 2\theta$, (e) $\sin 2\theta$, (f) $\cos 4\theta$.

11 Express each of the following as a single trigonometrical function:

(a) $\sin A \cos A$

(b) $1 - 2\sin^2 A$

(c) $\dfrac{2\tan A}{1 - \tan^2 A}$

(d) $\cos^2 A - \sin^2 A$.

Given that $\sin A = s$, find the values of each one in terms of s.

12 State each of the following as a single trigonometrical function

(a) $\sin 35° \cos 35°$

(b) $\dfrac{\tan 40°}{1 - \tan^2 40°}$

(c) $\sin 25° \cos 35° - \cos 25° \sin 35°$

(d) $\dfrac{\tan 50° - \tan 40°}{1 + \tan 50° \tan 40°}$

(e) $0.5 - \sin^2 30°$

13 Solve the equations (a) $\cos 2\theta = 3\cos\theta - 2$, (b) $\cos(2\theta + 30°) = \sin 2\theta$, (c) $\sin\left(\frac{\theta}{2} - 45°\right) = \sqrt{2}\cos\frac{\theta}{2}$ for $0° \le \theta \le 360°$.

14 Solve the equations (a) $\sin 2x = \cos x$, (b) $\sin\frac{x}{2} = \cos x$, for $0° < x < 360°$.

15 Convert the equation $3\cos 2\theta = \cos\theta - 1$ into a quadratic equation in $\cos\theta$ and hence find the solutions of the equation for $\theta° < \theta < 360°$.

16 By writing 3A as 2A + A, show that
(a) $\sin 3A = 3\sin A - 4\sin^3 A$, and
(b) $\cos 3A = 4\cos^3 A - 3\cos A$.
Using (b), solve the equation $\cos 3A = \cos 2A$ for $0° < A < 360°$.

17 Show that $\cos^2 A + 2\sin^2 A = \frac{1}{2}(3 - \cos 2A)$
Given that $\tan B = 2\tan A$, show that $\tan(B - A) = \frac{\sin 2A}{3 - \cos 2A}$.
Given also that $\sin 2A = \frac{3}{5}$, find the size of the angle $B - A$.

18 Show that $\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{\sin^2 A}{1 + \cos^2 A}$.

Given that $\frac{1 - \cos 2A}{3 + \cos 2A} = \frac{1}{7}$, find the values of A for $0° \le A \le 180°$.

THE FUNCTION $a\cos\theta + b\sin\theta$

This function occurs frequently and we now show how it can be converted to one of the forms $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$ where the constant R and the angle α have to be found.

Example 13

Convert $3\cos\theta + 4\sin\theta$ into (a) $R\sin(\theta + \alpha)$, (b) $R\cos(\theta - \alpha)$ by finding the necessary values of R and α.

(a) $R\sin(\theta + \alpha) = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$
and this must be identical to $4\sin\theta + 3\cos\theta$ matching the terms with $\sin\theta$ and $\cos\theta$.

Hence $R\cos\alpha = 4$
and $R\sin\alpha = 3$.

Squaring each one and then adding,

$R^2(\cos^2\alpha + \sin^2\alpha) = R^2 = 3^2 + 4^2 = 25$.

Therefore $R = 5$ (the + value is always taken for R).

Also $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{3}{4}$ so $\alpha = 36.87°$ (acute value taken as $\sin\alpha$ and $\cos\alpha$ are both +).

Hence $3\cos\theta + 4\sin\theta \equiv 5\sin(\theta + 36.87°)$.

(b) This is converted in a similar manner.

$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

which is identical to $3\cos\theta + 4\sin\theta$

so $R\cos\alpha = 3$ and $R\sin\alpha = 4$.

Now verify that $R = 5$ and $\tan\alpha = \frac{4}{3}$ giving $\alpha = 53.13°$.

Hence $3\cos\theta + 4\sin\theta = 5\cos(\theta - 53.13°)$.

(a) and (b) are alternative forms for the original expression. The other forms $R\sin(\theta - \alpha)$ and $R\cos(\theta + \alpha)$ are not suitable if α is to be acute as they contain a minus sign in the middle when expanded.

Example 14

Find the maximum and minimum values of $E = 3\cos\theta + 4\sin\theta$ and the values of θ when they occur.

From Example 13(a), $E = 5\sin(\theta + 36.87°)$.

The maximum value of a sine is 1 when the angle is 90° and the minimum value is –1 when the angle is 270°.

Hence the maximum value of $5\sin(\theta + 36.87°)$ is 5 when $\theta + 36.87° = 90°$ i.e. when $\theta = 53.13°$.

The minimum value is –5 when $\theta + 36.87° = 270°$ i.e. when $\theta = 233.13°$.

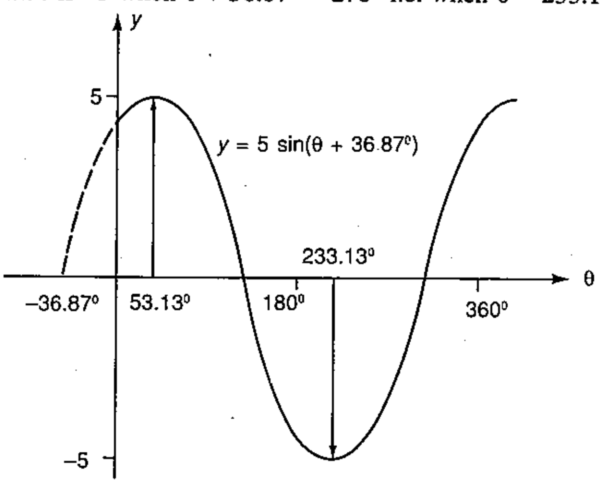


Fig. 14.3

Fig. 14.3 shows the graph of $y = 5\sin(\theta + 36.87°)$ which is the graph of $5\sin\theta$ shifted through 36.87° to the left. It is also the graph of $5\cos\theta$ shifted through 53.13° to the right.

Note: All functions of the type $a\cos\theta + b\sin\theta$ will have a similar graph. Those converting to $R\sin(\theta + \alpha)$ will show a sine curve shifted $\alpha°$ to the left and so on. The maximum and minimum values will be $+R$ and $-R$ respectively and will occur at values of θ differing by 180°.

Example 15

Express 5 sin θ – 12 cos θ in the form R sin(θ – α).
Hence find the values of θ ($0° < \theta < 360°$) for which the expression has a maximum or minimum value.

$$\begin{aligned} R\sin(\theta-\alpha) &= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha \\ &= 5\sin\theta - 12\cos\theta \end{aligned}$$

Hence $R\cos\alpha = 5$ and $R\sin\alpha = 12$.

Verify that $R = 13$ and $\alpha = 67.38°$

Hence $5\sin\theta - 12\cos\theta = 13\sin(\theta - 67.38°)$.

The maximum value is 13 when $\theta - 67.38° = 90°$ i.e. $\theta = 157.38°$.
The minimum value is –13 when $\theta - 67.38° = 270°$ i.e. $\theta = 337.38°$.

Exercise 14.3 *(Answers on page 637.)*

1 Express the following functions of θ in the form stated:
- (a) $4\cos\theta + 3\sin\theta$: $R\cos(\theta - \alpha)$
- (b) $5\sin\theta + 12\cos\theta$: $R\sin(\theta + \alpha)$
- (c) $2\cos\theta - 3\sin\theta$: $R\cos(\theta + \alpha)$
- (d) $2\sqrt{2}\sin\theta - \cos\theta$: $R\sin(\theta - \alpha)$
- (e) $2\cos\theta + \sqrt{5}\sin\theta$: $R\sin(\theta + \alpha)$

2 Using the results in Question 1, find the maximum and minimum values of the functions and the values of θ where these occur for $0° \le \theta \le 360°$.

3 Express $3\cos 2\theta - 4\sin 2\theta$ in the form $R\cos(2\theta + \alpha)$. Hence find the values of θ ($0° \le \theta \le 360°$) for which the expression has a maximum or minimum value.

4 By converting the expression $\sqrt{7}\sin\frac{\theta}{2} + \sqrt{2}\cos\frac{\theta}{2}$ into the form $R\sin\left(\frac{\theta}{2} + \alpha\right)$, find the values of θ between 0° and 360° for which the expression has a maximum or a minimum value.

5 Express $\sqrt{5}\sin\theta + 2\cos\theta$ in the form $R\sin(\theta + \alpha)$. Hence sketch the graph of $y = \sqrt{5}\sin\theta + 2\cos\theta$ for $0° \le \theta \le 360°$. State the value of α for which $3\cos(\theta - \alpha)$ will have the same graph.

THE EQUATION $a \cos\theta + b \sin\theta = c$

Example 17

Solve the equation $3\cos\theta + 4\sin\theta = 2$ for $0° \le \theta \le 360°$.

From Example 13, $3\cos\theta + 4\sin\theta = 5\sin(\theta + 36.87°)$.

Then $\sin(\theta + 36.87°) = \frac{2}{5} = 0.4$.

This gives $\theta + 36.87° = 23.58° + 360°$ or $156.42°$ $(36.87° \le \theta + 36.87° \le 396.87°)$.

Note that 360° is added to 23.58° as 23.58° is not in the domain for $\theta + 36.87°$.

Then $\theta = 346.71°$, or $119.55°$.

If we use the alternative form $5\cos(\theta - 53.13°)$, then $\cos(\theta - 53.13°) = 0.4$ and $\theta - 53.13° = 66.42°$ or $293.58°$.
Verify that the values for θ are the same.

The solutions are shown on the graph of $y = 3\cos\theta + 4\sin\theta$ in Fig.14.4.

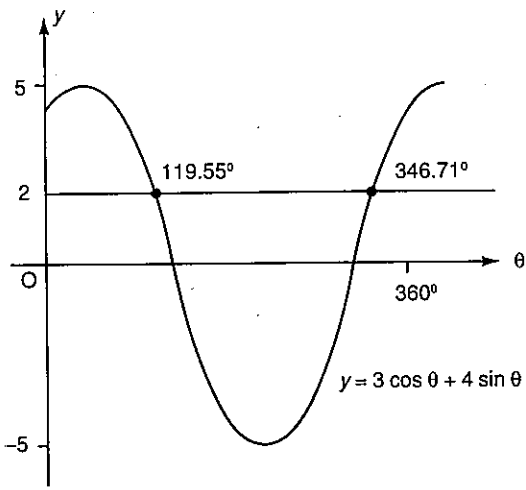


Fig. 14.4

Example 18

Solve the equation $\sqrt{3}\cos 2\theta - \sin 2\theta = \sqrt{2}$ for $0° \le \theta \le 360°$.

We convert the left hand side to $R\cos(2\theta + \alpha)$.

Verify that $R = 2$ and $\alpha = 30°$.

Then $\cos(2\theta + 30°) = \frac{\sqrt{2}}{2} = 0.7071$.

Hence $2\theta + 30° = 45°$ or $315°$ or $405°$ or $675°$ $(0° \le 2\theta \le 720°)$
and $2\theta = 15°$ or $285°$ or $375°$ or $645°$.

So $\theta = 7.5°, 142.5°, 187.5°, 322.5°$.

Exercise 14.4 *(Answers on page 637.)*

1 Using the results from Question 1 of Exercise 14.3, solve the following equations for $0° \le \theta \le 360°$:

(a) $4 \cos \theta + 3 \sin \theta = 2$

(b) $5 \sin \theta + 12 \cos \theta = 6.5$

(c) $2 \cos \theta - 3 \sin \theta = \frac{\sqrt{13}}{4}$

(d) $2\sqrt{2} \sin \theta - \cos \theta = 2$

(e) $2 \cos \theta + \sqrt{5} \sin \theta = 1$

2 Express $\sqrt{3} \sin \frac{\theta}{2} + \cos \frac{\theta}{2}$ in the form $R \sin\left(\frac{\theta}{2} + \alpha\right)$.

Hence find the solutions of the equation $\sqrt{3} \sin \frac{\theta}{2} + \cos \frac{\theta}{2} = 1$ in the range $0° \le \theta \le 360°$.

3 Solve the equation $\cos \theta + 2 \sin \theta = 2$ if $0° \le \theta \le 360°$.

4 Find all the angles between 0° and 360° which satisfy the equation $3 \cos x - \sqrt{7} \sin x = 2$.

5 Express the function $\cos 2x - 3 \sin 2x$ in the form $R \cos(2x + \alpha)$.
Using this form, find the solutions of the equation $\cos 2x = 3(1 + \sin 2x)$ in the interval $0° \le x \le 360°$.

6 Find the values of x in the range $0° < x < 360°$ satisfying the equation $2 \cos(2x - 60°) + \sin(2x - 60°) = \frac{1}{\sqrt{5}}$.

SUMMARY

- Addition formulae

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

- Double angle formulae

$$\sin 2A \equiv 2 \sin A \cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2 \cos^2 A - 1$$
$$\equiv 1 - 2 \sin^2 A$$
$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin^2 A = \tfrac{1}{2}(1 - \cos 2A), \quad \cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

- The function $a \cos\theta + b \sin\theta$ can be converted into one of the forms $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$, where $R > 0$.
  [If a and b are both positive, use either $R\sin(\theta + \alpha)$ or $R\cos(\theta - \alpha)$.
  If one of a or b is negative, use either $R\sin(\theta - \alpha)$ or $R\cos(\theta + \alpha)$.]

  $R^2 = a^2 + b^2$; $\tan\alpha$ (α acute) is found from a and b.
- To solve the equation $a\cos\theta + b\sin\theta = c$, use one of the above conversions.
  Then $\cos(\theta \pm \alpha)$ or $\sin(\theta \pm \alpha) = \frac{c}{R}$.

REVISION EXERCISE 14 *(Answers on page 637.)*

A

1 Solve the equation $\tan 2\theta = 3\tan\theta$ for $0° \leq \theta \leq 360°$.

2 Prove that $\dfrac{\sin(A + B) - \sin(A - B)}{\cos(A + B) - \cos(A - B)} = \tan B$.

3 $\tan A$ and $\tan B$ ($\tan A > \tan B$) are the roots of the equation $2t^2 - t = 3$. Calculate, without using tables or a calculator, the values of $\tan(A + B)$, $\tan(A - B)$ and $\tan 2A$.

4 Solve the equation $3\cos 2x + 8\sin x + 5 = 0$ for $0° \leq x \leq 360°$.

5 Find all the angles between 0° and 360° which satisfy the equation $3\sin x = \sec x$.

6 Given that $\dfrac{\cos(A - B)}{\cos(A + B)} = \frac{7}{3}$, show that $5\tan A = 2\cot B$.
Given further that A is acute and that $\tan B = 2$ find, without using tables or a calculator, the value of (i) $\tan(A + B)$, (ii) $\sin A$, (iii) $\cos 2A$. (C)

7 Given that $\cos A = \frac{1}{2}$ where A is an acute angle, find, without using tables or a calculator, the values of $\cos 2A$, $\sin 2A$ and $\sin\frac{A}{2}$.

8 (a) Given that $\sin\alpha = \frac{4}{5}$. Where $90° < \alpha < 180°$ and that $\cos\beta = -\frac{5}{13}$ where $180° < \beta < 270°$, calculate, without using tables or a calculator, (i) $\sin(\alpha - \beta)$, (ii) $\cos 2\alpha$, (iii) $\sin 2\beta$.
(b) Given that $3\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$, where R is positive and α is acute, evaluate R and α. Hence solve the equation $3\cos\theta + \sin\theta = 2$ for $0° < \theta < 360°$. (C)

9 Prove that (a) $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$ and (b) $\dfrac{\sin 2A + \sin A}{1 + \cos 2A + \cos A} = \tan A$.

10 (a) Find all the angles between 0° and 360° which satisfy the equation $4\cos x - 3\sin x = 1$.
(b) Given that $\cos\theta = c$ and that θ is acute, express in terms of c, (i) $\operatorname{cosec}\theta$, (ii) $\cot\theta$, (iii) $\sin 2\theta$, (iv) $\tan(\theta + 45°)$. (C)

11 Solve the equation $\cos 4\theta\cos\theta + \sin 4\theta\sin\theta = \frac{1}{2}$ for $0° \leq \theta \leq 180°$.

12 If $\sin x = -\frac{15}{17}$ and $270° < x < 360°$, calculate without using tables or a calculator, the values of (a) $\cos 2x$, (b) $\sin 2x$.

13 (a) Find all the angles between 0° and 360° which satisfy the equation $4 \cos \theta + 2 \sin \theta = 1$.

(b) Given that $\frac{\cos(A + B)}{\cos(A - B)} = \frac{3}{4}$, prove that $\cos A \cos B = 7 \sin A \sin B$ and deduce a relationship between tan A and tan B. Given further that $A + B = 45°$, calculate the value of $\tan A + \tan B$. (C)

14 (a) Find the maximum and minimum values of $\sqrt{5} \cos \theta + 2 \sin \theta$ and the values of θ where they occur.

(b) Solve the equation $\sin 2\alpha + \cos 2\alpha = \frac{1}{\sqrt{2}}$ for $0° < \alpha < 360°$.

15 (a) Convert the expression $\sqrt{2} \sin x + \sqrt{7} \cos x$ to the form $R \sin(x + \alpha)$ where $R > 0$ and α is acute. Hence sketch the graph of $y = \sqrt{2} \sin x + \sqrt{7} \cos x$ for $0° \leq x \leq 360°$.

(b) Solve the equation $\sqrt{7} \cos x + \sqrt{2} \sin x = 2$ for that domain.

16 Find all the values of x between 0° and 360° which are solutions of the equation $3 \tan 2x - 2 \cot x = 0$.

17 Given that $\tan 30° = \frac{1}{\sqrt{3}}$ and that $\tan 45° = 1$, calculate, without using tables or a calculator, the values of (a) tan 150°, (b) tan 15°, (c) tan 75°.

18 If $\sin(\theta + \phi) = 2 \sin(\theta - \phi)$, prove that $\tan \theta = 3 \tan \phi$. Given that $\tan \phi = \frac{2}{3}$, find, without using tables or a calculator, the values of (a) $\tan \theta$, (b) $\operatorname{cosec} \theta$, (c) $\cos \phi$, (d) $\tan(\theta + \phi)$, where θ and ϕ are both acute angles.

19 If A, B and C are the angles of a triangle and tan A = 2, tan B = 3, find the value of tan C without using tables or a calculator.

20 If $t = \tan \theta$ $(t \neq \pm 1)$, show that $\sin 2\theta = \frac{2t}{1 + t^2}$ and $\cos 2\theta = \frac{1 - t^2}{1 + t^2}$.
Given that $\sec 2\theta + \tan 2\theta = k$, prove that $t = \frac{k - 1}{k + 1}$.

B

21 A and B are both acute angles. Given that $\tan A + \tan B = 3x$ and that $\tan A \tan B = 2x^2$ find, in terms of x

(a) $\tan A - \tan B$,
(b) $\tan(A + B)$,
(c) $\tan(A - B)$.
(d) Hence solve the simultaneous equations $\tan A + \tan B = 3$, $\tan A \tan B = 2$.

22 Solve for $0° \leq \theta \leq 360°$, the equation $\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta} = 6$.

23 Given that $A - B = \frac{\pi}{2}$, show that $1 + \tan A \tan B = 0$.

24 (a) If $\sin x$, $\tan x$ and $\sin 2x$ are three consecutive terms of a GP, show that $\cos^3 x = \frac{1}{2}$ and find the common ratio of the GP.

(b) $2 \sin 2x$, $\frac{1}{2}$ and $3 \cos 2x$ are three consecutive terms of an AP. Find the possible values of x (to the nearest degree) if $0° \leq x \leq 360°$.

25 In Fig.14.5, P is a point on the circle centre O. PQ is a tangent and QS is perpendicular to PQ. PT is a diameter. If $\angle QPR = \theta$, show that $\angle PTR = \theta$ and deduce that $\angle PSR = \theta$.
Prove that $PQ = QS \tan \theta$ and that $QR = PQ \tan \theta$. Hence show that $RS = 2PQ \cot 2\theta$ and find the value of θ if $RS = 2PQ$.

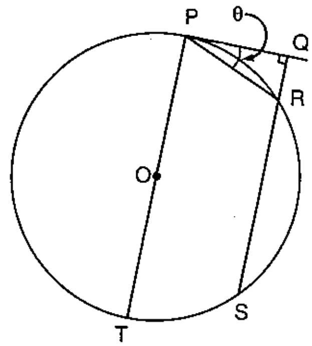


Fig. 14.5

26 In Fig. 14.6, ABCD is a square of side a and AFE is a sector of a circle centre A and angle 2θ. Show that

(a) $AE = \dfrac{a}{\cos(\frac{\pi}{4} - \theta)}$, (b) $AE^2 = \dfrac{2a^2}{1 + \sin 2\theta}$.

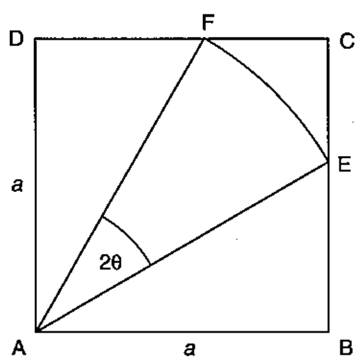


Fig. 14.6

Given that the area of the sector is half the area of the square, deduce that $1 + \sin 2\theta = 4\theta$. Writing this equation in the form $1 + \sin x = 2x$, draw the graphs of $y = 1 + \sin x$ and $y = 2x$ for $0 \leq x \leq \frac{\pi}{2}$ to find an approximate value for x and hence for θ. Use a calculator to confirm your result.

Exponential and Logarithmic Functions

15

In this chapter, we study functions such as 3^x where x is an index or **exponent.** Hence 3^x is called an **exponential function** and 3 is the **base** of the function.

First, here is a revision check on the rules for working with indices.

RULES FOR INDICES

The three basic rules are:

I $x^m \times x^n = x^{m+n}$ When multiplying exponential functions with the same base, ADD the indices.

II $x^m \div x^n = x^{m-n}$ When dividing exponential functions with the same base, SUBTRACT the indices.

III $(x^m)^n = x^{mn}$ If an exponential function is raised to another power, MULTIPLY the indices.

Using these rules we can deduce that $x^0 = 1$ $(x \neq 0)$ and that $x^{-m} = \frac{1}{x^m}$ $(x \neq 0)$. A negative index gives the reciprocal of the function.

Fractional indices

We can also find a meaning for fractional indices. For example

$(9^{\frac{1}{2}})^2 = 9^{\frac{1}{2} \times 2} = 9^1 = 9$. So $9^{\frac{1}{2}} = \sqrt{9} = 3$.

In general, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

Further, $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = (3)^2 = 9$.

In general, $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$.

Example 1

Find the values of (a) $100^{\frac{3}{2}}$ (b) $32^{-\frac{2}{5}}$.

(a) $100^{\frac{3}{2}} = (\sqrt{100})^3 = (10)^3 = 1000$

(b) $32^{-\frac{2}{5}} = \frac{1}{32^{\frac{2}{5}}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{(2)^2} = \frac{1}{4}$.

Example 2

Show that (a) $3^{2x-1} = \frac{9^x}{3}$, (b) $2^{x-1} \times 8^{x+1} = 4^{2x+1}$

(a) $3^{2x-1} = 3^{2x} \times 3^{-1} = (3^2)^x \times \frac{1}{3} = \frac{9^x}{3}$

(b) $2^{x-1} \times 8^{x+1} = 2^{x-1} \times (2^3)^{x+1}$

$= 2^{x-1} \times 2^{3x+3}$

$= 2^{4x+2} = (2^2)^{2x+1} = 4^{2x+1}$

Exercise 15.1 *(Answers on page 638.)*

1 Find the value of:

(a) 8^0 (b) 4^{-2} (c) $(-3)^{-3}$

(d) $8^{\frac{1}{3}}$ (e) $81^{\frac{3}{4}}$ (f) $32^{\frac{2}{5}}$

(g) $16^{-\frac{3}{4}}$ (h) $(x^3)^{-\frac{2}{3}}$ (i) $\left(-\frac{8}{27}\right)^{-\frac{1}{3}}$

2 Show that

(a) $2^{x+2} = 4 \times 2^x$ (b) $16^x = 4^{2x}$ (c) $2^{2x+3} = 8 \times 4^x$

(d) $2^{3-x} = \frac{8}{2^x}$ (e) $4^{2-x} = \frac{16}{2^{2x}}$ (f) $5^x \times 25^y = 5^{x+2y}$

3 Simplify (a) $2^{2-2x} \times 4^{x-1}$, (b) $27^{\frac{x}{3}} \div 9^{\frac{x}{2}}$, (c) $8^{\frac{2x}{3}} \times 4^{2-x}$.

EXPONENTIAL EQUATIONS

An equation such as $3^x = 81$ is an **exponential equation.** The unknown (x) is the exponent. We can solve such equations by expressing both sides in terms of the *same base*. Sometimes this can be done directly. If not, a more general method using logarithms can be used. This will be shown later.

Example 3

Solve the equations (a) $3^x = 81$, (b) $8^x = 0.25$.

(a) We have to see that 81 is a power of 3 (the base of the left hand side). $81 = 3^4$ so the equation is $3^x = 3^4$. Both sides are now expressed to the same base, so the exponents must be equal. Hence $x = 4$.

(b) Here we must see that 8^x and 0.25 can both be expressed to base 2.

$8^x = (2^3)^x = 2^{3x}$

$0.25 = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

Then $3x = -2$ or $x = -\frac{2}{3}$.

Example 4

Solve the equation $2^{2x+3} + 1 = 9 \times 2^x$.

$2^{2x+3} = 2^{2x} \times 2^3 = (2^x)^2 \times 8$

Now if we put $p = 2^x$, then $2^{2x+3} = 8p^2$ and $9 \times 2^x = 9p$.

The equation then becomes $8p^2 - 9p + 1 = 0$ i.e. $(8p - 1)(p - 1) = 0$

giving $p = \frac{1}{8}$ or $p = 1$.

Then $2^x = \frac{1}{8} = 2^{-3}$ and $x = -3$

or $2^x = 1 = 2^0$ and $x = 0$.

The solutions of the equation are $x = -3$ or $x = 0$.

However, note that not all such equations will have two solutions or even any solutions.

Example 5

Solve the equation $2^{2x+1} + 15 \times 2^x - 8 = 0$.

$2^{2x+1} = 2^{2x} \times 2^1 = (2^x)^2 \times 2$

As before, put $p = 2^x$.

The equation becomes $2p^2 + 15p - 8 = 0$ i.e. $(2p - 1)(p + 8) = 0$ giving $p = \frac{1}{2}$ or -8.

So $2^x = \frac{1}{2} = 2^{-1}$ and $x = -1$

or $2^x = -8$. As 2^x is never negative, this part has no solution.

The equation has therefore only one solution, $x = -1$.

Example 6

Solve the equation $2^x + 2^{1-x} = 3$.

The equation is $2^x + 2^1 \times 2^{-x} = 3$ i.e. $2^x + \frac{2}{2^x} = 3$.

Take $p = 2^x$.

The equation becomes $p + \frac{2}{p} = 3$ i.e. $p^2 - 3p + 2 = 0$.

Then $(p - 2)(p - 1) = 0$ and $p = 2$ or 1.

Then $2^x = 2 = 2^1$ and $x = 1$ or $2^x = 1 = 2^0$ and $x = 0$.

The solutions are $x = 1$ or 0.

Example 7

Solve the simultaneous equations

$$3^x \times 9^y = 1 \qquad \text{(i)}$$

and

$$2^{2x} \times 4^y = \frac{1}{8} \qquad \text{(ii)}$$

In equation (i), we see that each term can be expressed as a power of 3.

Then $3^x \times (3^2)^y = 3^0$

so $x + 2y = 0$ (iii)

Similarly, each term of equation (ii) can be expressed as a power of 2.

Then $2^{2x} \times (2^2)^y = 2^{-3}$

so $2x + 2y = -3$ (iv)

Solving equations (iii) and (iv), we obtain $x = -3$, $y = 1\frac{1}{2}$.

Exercise 15.2 *(Answers on page 638.)*

1 Solve the following equations:

(a) $2^x = 64$

(b) $5^x = 125$

(c) $5^x = 1$

(d) $9^x = 81$

(e) $16^x = 0.125$

(f) $4^x = 0.5$

(g) $9^x = \frac{1}{729}$

(h) $2^{2x} - 9(2^x) + 8 = 0$

(i) $3^{2x} - 12(3^x) + 27 = 0$

(j) $5^{2x} + 1 = 26(5^{x-1})$

(k) $2^{2x+1} - 129(2^x) + 64 = 0$

(l) $3^{2x+1} + 26(3^x) - 9 = 0$

(m) $2^{2x+1} - 15(2^x) = 8$

(n) $2^x + 2^{2-x} = 5$

(o) $2^{x+3} = 2^{1-x} + 15$

(p) $3^{2x+1} - 28(3^{x-1}) + 1 = 0$

2 Show that the equation $2^{2x} + 3(2^{x+1}) + 8 = 0$ has no solutions.

3 Solve the simultaneous equations $5^x \times 25^{2y} = 5$, $3^{2x} \times 9^{y-1} = \frac{1}{27}$.

4 Find x and y given that $2^{2x} \times 8^y = 4$, $3^x \times 9^{y+1} = \frac{1}{81}$.

The Graph of the Exponential Function

Complete the following table for values of 2^x and 3^x using a calculator where necessary.

x	–2	–1	0	0.5	1	1.5	2	2.5	3
2^x	0.25		1						8
3^x	0.11					5.2			–

(The value 3^3 is omitted as it is out of scale.)

Now taking scales of 2 cm for 1 unit on the x-axis and 1 cm for 1 unit on the y-axis, plot the points given by these values and draw the two curves (Fig.15.1).

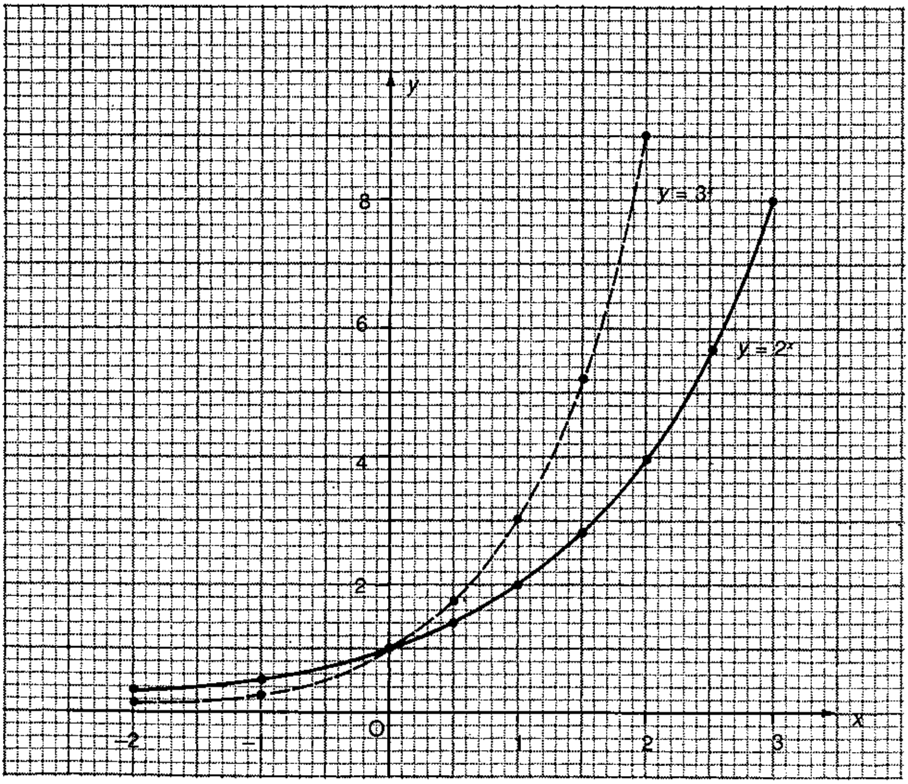


Fig.15.1

These are typical graphs of the exponential function a^x ($a > 1$). They all have the same shape and pass through the point (0,1) as $a^0 = 1$ for all values of $a \neq 0$.

Fig. 15.2 shows the graphs of $y = a^x$ and $y = a^{-x}$ ($a > 1$). These are reflections of each other in the y-axis. Note that a^x is never negative but $a^x \to 0$ as $x \to -\infty$.

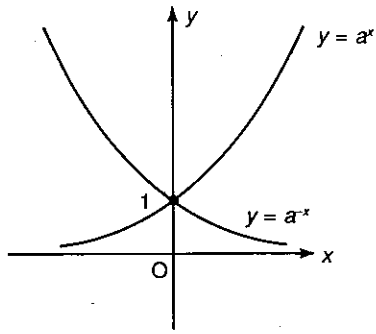


Fig. 15.2

Exercise 15.3 *(Answers on page 638.)*

1 From Fig. 15.1, estimate the values of (a) $2^{1.2}$, (b) $3^{0.8}$ and find approximate values of x if (c) $2^x = 5$, (d) $3^x = 7$.

2 Copy the curve $y = 3^x$ from Fig. 15.1 and add the curve $y = 3^{-x}$ for the same domain. Now draw the graph of the curve $y = \frac{1}{2}(3^x + 3^{-x})$ by taking the values of y halfway between the two curves $y = 3^x$ and $y = 3^{-x}$. [This curve is called a **catenary** and is the curve made by a flexible chain when suspended from two points on the same level.]

3 Using a calculator, make a table of values for the function $y = e^x$ where $e = 2.72$. Take values of x from -1 to 2.5 at intervals of 0.5. With scales of 2 cm for 1 unit on the x-axis and 1 cm for 1 unit on the y-axis, plot the points and draw the curve.

By drawing tangents to the curve at the points where $x = 0$, 1 and 2, estimate the gradient of the curve at these points. Compare your results with the value of y at these points. [The significance of the exponential function e^x will be seen later in Chapter 18. Your results suggest the unique property of this function that $\frac{dy}{dx} = y$.]

THE LOGARITHMIC FUNCTION

The inverse of the exponential function is called the **logarithmic function.** If $y = a^x$, we define x as the **logarithm** of y to the base a ($a > 0$). This is written as $\log_a y$.

	exponential form		logarithmic form
	$y = a^x$	$\longleftrightarrow$	$\log_a y = x$
		index becomes logarithm	
For example,	$8 = 2^3$		$\log_2 8 = 3$
		base	
	$100 = 10^2$		$\log_{10} 100 = 2$
	$r = a^3$		$\log_a r = 3$

Conversely, if $\log_x 10 = 4$, then $10 = x^4$
if $\log_3 x = 5$, then $x = 3^5$.

Example 8

Write in logarithmic form: (a) $3^2 = 9$, (b) $x^3 = 10$. (c) $2^{-2} = \frac{1}{4}$.

(a) If $3^2 = 9$, then $\log_3 9 = 2$.
(b) If $x^3 = 10$, then $\log_x 10 = 3$.
(c) If $2^{-2} = \frac{1}{4}$, then $\log_2 \left(\frac{1}{4}\right) = -2$. (Logarithms can be negative)

Example 9

Write in exponential form: (a) $4 = \log_3 x$, (b) $x = \log_5 7$, (c) $2 = \log_x 5$.

(a) $4 = \log_3 x$ becomes $3^4 = x$
(b) $x = \log_5 7$ becomes $5^x = 7$
(c) $2 = \log_x 5$ becomes $x^2 = 5$

Example 10

Find the value of x if (a) $x = \log_2 64$, (b) $\log_x 25 = 2$, (c) $x = \log_3 \left(\frac{1}{3}\right)$, (d) $\log_3 x = 4$.

(a) If $x = \log_2 64$, then $2^x = 64$ and $x = 6$ (as $64 = 2^6$).
(b) If $\log_x 25 = 2$, then $x^2 = 25$ and $x = 5$ (+ as base must be positive).
(c) If $x = \log_3 \left(\frac{1}{3}\right)$, then $3^x = \frac{1}{3} = 3^{-1}$ and $x = -1$.
(d) If $\log_3 x = 4$, then $3^4 = x = 81$.

Exercise 15.4 *(Answers on page 638.)*

1 Write in logarithmic form:

(a) $4^2 = 16$ (b) $3^3 = 81$ (c) $10^3 = 1000$
(d) $10^{-3} = 0.001$ (e) $4^{\frac{1}{2}} = 2$ (f) $x^3 = 2$
(g) $7^x = 21$ (h) $x^{-4} = 16$ (i) $10^{-1} = 0.1$
(j) $8^2 = 64$ (k) $4^x = 9$ (l) $x^{-3} = 0.3$

2 Write the following in exponential form and hence find the value of x:

(a) $x = \log_2 16$ (b) $x = \log_3 27$
(c) $x = \log_4 64$ (d) $x = \log_2 \left(\frac{1}{8}\right)$
(e) $x = \log_{10} 0.001$ (f) $x = \log_{64} 4$
(g) $x = \log_7 \left(\frac{1}{49}\right)$ (h) $\log_5 625 = x$
(i) $x = \log_3 \left(\frac{1}{27}\right)$ (j) $x = \log_{13} 169$
(k) $x = \log_{169} 13$

3 Find the value of:

(a) $\log_5 5$ (b) $\log_8 64$ (c) $\log_3 1$
(d) $\log_a 1$ (e) $\log_4 16$ (f) $\log_3 243$
(g) $\log_9 3$ (h) $\log_p p^2$ (i) $\log_5 \frac{1}{25}$
(j) $\log_4 8$ (k) $\log_{16} 8$ (l) $\log_4 16$
(m) $\log_x x$ (n) $\log_x \left(\frac{1}{x}\right)$ (o) $\log_2 \left(\frac{1}{4}\right)$

4 Find the value of x if:

(a) $\log_x 9 = 2$ (b) $\log_2 x = -3$
(c) $\log_x 125 = 3$ (d) $\log_{x+1} 27 = 3$
(e) $\log_4 (x-2) = 3$ (f) $\log_{2x} 36 = 2$
(g) $\log_x 81 = 4$ (h) $\log_3 x = -1$
(i) $\log_{x-2} 3 = 1$ (j) $\log_3 (x-2) = 4$
(k) $\log_{2x} 64 = 3$

The Graph of the Logarithmic Function

As the logarithmic function is the inverse of the exponential function $y = a^x$, we can obtain its graph by reflecting $y = a^x$ in the line $y = x$ (Fig.15.3).

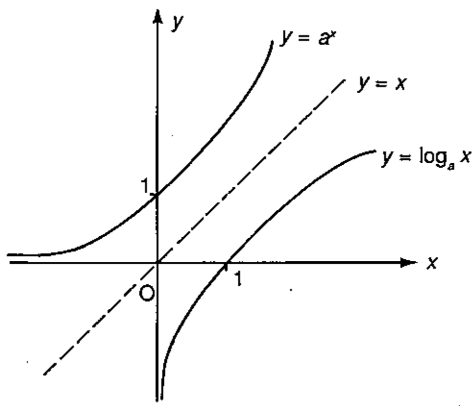


Fig.15.3

Note the following:

(1) $\log_a 1 = 0$. This follows because $a^0 = 1$.
(2) $\log_a x$ is not defined if $x < 0$. The logarithm of a negative number does not exist.
(3) If $0 < x < 1$, $\log_a x < 0$. The logarithm of a positive number < 1 is always negative.
(4) $\log_a 0$ is undefined.
(5) As x increases, $\log_a x$ increases.

Rules for Logarithms

These are similar to the rules for indices.

Let $P = a^m$. Then $m = \log_a P$.
Let $Q = a^n$. Then $n = \log_a Q$.

I $PQ = a^m \times a^n = a^{m+n}$

Then $\log_a PQ = m + n = \log_a P + \log_a Q$.

$$\log_a PQ = \log_a P + \log_a Q$$

For example, $\log_a 12 = \log_a (4 \times 3) = \log_a 4 + \log_a 3$.

Note: Do NOT write $\log_a (P + Q) = \log_a P + \log_a Q$. This is not true.

II $\frac{P}{Q} = \frac{a^m}{a^n} = a^{m-n}$.

Then $\log_a \frac{P}{Q} = m - n = \log_a P - \log_a Q$.

$$\log_a \frac{P}{Q} = \log_a P - \log_a Q$$

For example, $\log_a 3 = \log_a 12 - \log_a 4$.

Note: This rule does NOT apply to $\frac{\log_a P}{\log_a Q}$ which is the division of two logarithms.

III $P^n = (a^m)^n = a^{mn}$

Then $\log_a P^n = mn = n \log_a P$

$$\log_a P^n = n \log_a P$$

For example, $\log_a 2^3 = 3 \log_a 2$ and $\log_a \sqrt{3} = \log_a 3^{\frac{1}{2}} = \frac{1}{2}\log_a 3$.

Two Special Logarithms

1 For any base, $a^0 = 1$. Hence

$$\log_a 1 = 0$$

The logarithm of 1 is always 0.

2 $a^1 = a$. Hence

$$\log_a a = 1$$

The logarithm of the base is always 1.

Example 11

Simplify (a) $\log_7 49$, *(b)* $\log_3 \left(\frac{1}{9}\right)$.

(a) $\log_7 49 = \log_7 7^2 = 2 \log_7 7 = 2$ as $\log_7 7 = 1$

(b) $\log_3 \left(\frac{1}{9}\right) = \log_3 1 - \log_3 9 = 0 - \log_3 3^2$
$= -2 \log_3 3$
$= -2$

Example 12

Simplify $\log_4 9 + \log_4 21 - \log_4 7$.

$\log_4 9 + \log_4 21 - \log_4 7 = \log_4 (9 \times 21 \div 7) = \log_4 27 = \log_4 3^3 = 3 \log_4 3$

Example 13

Given that $\log_5 2 = 0.431$ *and* $\log_5 3 = 0.683$, *find the value of*
(a) $\log_5 6$, *(b)* $\log_5 1.5$, *(c)* $\log_5 8$, *(d)* $\log_5 12$, *(e)* $\log_5 \frac{1}{18}$.

We must express each in terms of powers of 2 and 3.

(a) $\log_5 6 = \log_5 (3 \times 2) = \log_5 3 + \log_5 2 = 1.114$

(b) $\log_5 1.5 = \log_5 \left(\frac{3}{2}\right) = \log_5 3 - \log_5 2 = 0.252$

(c) $\log_5 8 = \log_5 2^3 = 3 \log_5 2 = 1.293$

(d) $\log_5 12 = \log_5 (4 \times 3) = \log_5 4 + \log_5 3 = \log_5 2^2 + \log_5 3$
$= 2 \log_5 2 + \log_5 3 = 1.545$

(e) $\log_5 \frac{1}{18} = \log_5 1 - \log_5 18 = 0 - \log_5 (9 \times 2)$
$= -\log_5 9 - \log_5 2$
$= -\log_5 3^2 - \log_5 2$
$= -2 \log_5 3 - \log_5 2 = -1.797$

(With practice, some of these steps could be omitted).

Example 14

Given that $\log_a x = p$ *and* $\log_a y = q$, *express*
(a) $\log_a xy^2$, *(b)* $\log_a \frac{x}{y^3}$, *(c)* $\log_a \sqrt{\frac{ax^2}{y}}$ *in terms of p and q.*

(a) $\log_a xy^2 = \log_a x + \log_a y^2 = \log_a x + 2 \log_a y = p + 2q$

(b) $\log_a \frac{x}{y^3} = \log_a x - \log_a y^3 = \log_a x - 3 \log_a y = p - 3q$

(c) $\log_a \sqrt{\frac{ax^2}{y}} = \frac{1}{2}\log_a \frac{ax^2}{y} = \frac{1}{2}(\log_a ax^2 - \log_a y)$

$$= \frac{1}{2}(\log_a a + \log_a x^2 - \log_a y)$$
$$= \frac{1}{2}(1 + 2\log_a x - q)$$
$$= \frac{1}{2}(1 + 2p - q)$$

Example 15

If $2\log_a 2 + \log_a 10 - 3\log_a 3 = 3 + \log_a 5$, find the value of a.

Collecting all the logarithms on the left hand side,

$2\log_a 2 + \log_a 10 - 3\log_a 3 - \log_a 5 = 3$

$\log_a 2^2 + \log_a 10 - \log_a 3^3 - \log_a 5 = 3$

$\log_a \frac{4 \times 10}{27 \times 5} = 3$

$\log_a \frac{8}{27} = 3$ and so $a^3 = \frac{8}{27}$ and $a = \frac{2}{3}$.

Example 16

Find the value(s) of x if
(a) $2\log_3 x = \log_3 (x+6)$,
(b) $\log_5 (x^2 - 3x + 2) = 2 + \log_5 (x - 1)$.

(a) $2\log_3 x = \log_3 (x + 6)$
$\log_3 x^2 = \log_3 (x + 6)$
$x^2 = x + 6$
$x^2 - x - 6 = 0$
or $(x - 3)(x + 2) = 0$ giving $x = 3$ or -2.

It is essential to check if any of these solutions is invalid. We see that $x = -2$ is not possible as $\log(-2)$ is undefined. Hence the only solution is $x = 3$.

(b) $\log_5 (x^2 - 3x + 2) - \log_5 (x - 1) = 2$
Then $\log_5 \frac{x^2 - 3x + 2}{x - 1} = 2$
Changing to exponential form, $\frac{x^2 - 3x + 2}{x - 1} = 5^2$
Then $x^2 - 3x + 2 = 25x - 25$ or $x^2 - 28x + 27 = 0$
i.e. $(x - 27)(x - 1) = 0$ giving $x = 27$ or 1.
Now check these results.
If $x = 1$, $\log_5 (x - 1) = \log_5 0$ which is undefined.
So the only solution is $x = 27$.

Example 17

(a) If $\log_2 p = x$ and $\log_4 q = y$, express p^2q and $\frac{p}{q^2}$ as powers of 2.

(b) If also $p^2q = 16$ and $\frac{p}{q^2} = \frac{1}{32}$, find the values of x and y.

(a) First, we express p and q as powers of 2.

If $\log_2 p = x$, then $p = 2^x$.

If $\log_4 q = y$, then $q = 4^y = (2^2)^y = 2^{2y}$.

$p^2q = (2^x)^2 \times (2^{2y}) = 2^{2x+2y}$

$\frac{p}{q^2} = \frac{2^x}{(2^{2y})^2} = 2^{x-4y}$

(b) If $p^2q = 2^{2x+2y} = 16 = 2^4$,

then $2x + 2y = 4$

or $x + y = 2$ (i)

If $\frac{p}{q^2} = 2^{x-4y} = \frac{1}{32} = 2^{-5}$,

then $x - 4y = -5$ (ii)

Solving equations (i) and (ii), $x = \frac{3}{5}$ and $y = \frac{7}{5}$.

Example 18

Find the value of x if $\log_a x$, $\log_a (x+3)$ and $\log_a (x+12)$ are three consecutive terms of an AP.

As the terms are consecutive, $\log_a x + \log_a (x + 12) = 2 \log_a (x + 3)$.

Then $\log_a x(x + 12) = \log_a (x + 3)^2$ i.e. $x(x + 12) = (x + 3)^2$

which gives $x^2 + 12x = x^2 + 6x + 9$ or $6x = 9$.

Hence $x = 1\frac{1}{2}$.

Exercise 15.5 *(Answers on page 639.)*

1 Taking scales of 2 cm for 1 unit on each axis, draw the graph of $y = 2^x$ for $-2 \le x \le 2$. Add the line $y = x$. By reflecting $y = 2^x$ in this line, draw the graph of $y = \log_2 x$ for $\frac{1}{4} \le x \le 4$.

From your graph, find approximate values for (a) $\log_2 1.5$, (b) $\log_2 3$.

2 Simplify:

(a) $\log_7 \sqrt{2}$	(b) $\log_6 36$
(c) $\log_5 27$	(d) $\log_2 25$
(e) $\log_3 125$	(f) $\log_{10} 10\,000$
(g) $\log_6 121$	(h) $\log_x x^4$

(i) $\log_5 12.5 + \log_5 10$
(j) $2 \log_7 9 - \log_7 81$
(k) $\log_3 24 + \log_3 15 - \log_3 10$
(l) $\log_7 98 - \log_7 30 + \log_7 15$
(m) $\log_5 \sqrt{6} + \log_5 9$
(n) $\dfrac{\log_a x^3}{\log_a x^2}$
(o) $\log_4 8$
(p) $\log_9 81$
(q) $\log_3 5^3$
(r) $\log_4 4^x$
(s) $\log_3 \sqrt{10}$
(t) $\log_5 10$

3 Given that $\log_3 4 = 1.262$ and that $\log_3 5 = 1.465$, find the values of:
(a) $\log_3 20$
(b) $\log_3 0.8$
(c) $\log_3 1.25$
(d) $\log_3 100$
(e) $\log_3 64$
(f) $\log_3 80$
(g) $\log_3 6.25$
(h) $\log_3 15$
(i) $\log_3 0.25$

4 Given that $\log_7 2 = 0.356$ and $\log_7 3 = 0.565$, find the values of
(a) $\log_7 6$
(b) $\log_7 9$
(c) $\log_7 18$
(d) $\log_7 24$
(e) $\log_7 4.5$
(f) $\log_7 \frac{2}{3}$
(g) $\log_7 \sqrt{3}$
(h) $\log_7 14$
(i) $\log_7 42$
(j) $\log_7 \frac{3}{7}$
(k) $\log_7 4\frac{2}{3}$

5 If $\log_2 x^3 + \log_2 x = 8$, find the value of x.

6 If $\log_5 x = a$ and $\log_5 y = b$ express (a) xy^3, (b) $\frac{5x}{y}$ in terms of a and b.

7 If $\log_a x = p$ and $\log_a y = q$, express (a) $\log_a x^2y$, (b) $\log_a \sqrt{xy}$, (c) $\log_a \frac{x^3}{y}$, (d) $\log_a \frac{a^3x}{y^2}$ in terms of p and q.

8 Find y if $\log_3 y = 2 \log_3 7$.

9 If $\log_5 p - \log_5 4 = 2$, find the value of p.

10 Given that $\log_x 8 + \log_x 4 = 5$, find the value of x.

11 Given that $\log_3 x = p$ and $\log_9 y = q$, express (a) x^3y, (b) $\frac{x}{\sqrt{y}}$ as powers of 3.

12 Given that $\log_3 4 = p$ and $\log_3 5 = q$, find the value of x if (a) $\log_3 x = p + 2q$, (b) $\log_3 x = 2p - q + 2$.

13 $\log_2 x = a$ and $\log_4 y = b$. Express x^2y and $\frac{x^3}{y}$ as powers of 2.
Given also that $x^2y = 32$ and that $\frac{x^3}{y} = \frac{1}{8}$, find the values of a and b.

14 Given that $\log_{10} 2 = h$ and $\log_{10} 7 = k$, find the value of x if (a) $\log_{10} x = 2h + k$, (b) $\log_{10} x = 3h - k + 1$.

15 If $\log_{10} x = a$ and $\log_{10} y = b$, express $\log_{10} \sqrt{\frac{10x}{y^3}}$ in terms of a and b.

16 Given that $\log_a x^2y = p$ and that $\log_a \left(\frac{x}{y^2}\right) = q$, find $\log_a x$ and $\log_a y$ in terms of p and q and hence express $\log_a xy$ in terms of p and q.

17 If $\log_4 x = p$, show that $\log_2 x = 2p$. Hence find (a) the value of k if $\log_4 k = 2 + \log_2 k$ and (b) the value of n if $\log_2 n + \log_4 n = 9$.

18 Solve the equations
(a) $2 \log_5 x = \log_5 (2x + 3)$
(b) $3 \log_2 x = \log_2 (3x - 2)$
(c) $\log_3 (x^2 + 2) = 1 + \log_3 (x + 2)$
(d) $\log_4 (x^2 + 8x - 1) = 2 + \log_4 (x - 1)$
(e) $\log_2 (2x^2 + 3x + 5) = 3 + \log_2 (x + 1)$
(f) $\log_4 (x + 17) = 2 \log_4 (x - 3)$
(g) $\log_2 (x^2 - x + 2) = 1 + 2 \log_2 x$
(h) $\log_5 x = 1 - \log_5 (x - 4)$

19 (a) If $\log_a b = x$, deduce that $x \log_b a = 1$ and hence show that $\log_a b = \frac{1}{\log_b a}$.
(b) Find $\log_2 8$ and hence state the value of $\log_8 2$.

20 If $\log_a x$, $\log_a y$ and $\log_a z$ are three consecutive terms of an AP, show that x, y and z are consecutive terms of a GP.

Common Logarithms

For practical calculation, base 10 is used and logarithms on this base are called **common logarithms.** These are written as **lg** x, which is an abbreviation for $\log_{10} x$. 10 is chosen as it is the base of the decimal system of numbers.

To see the advantage of base 10, suppose we know that lg 5.6 = 0.748. Then lg 560 = lg 5.6×10^2 = lg 5.6 + 2 lg 10 = 0.748 + 2 = 2.748 (as lg 10 = 1). The decimal part .748 is unchanged. Similarly lg 5600 would be 3.748. On any other base the logarithms of these numbers would be quite different.

Tables of common logarithms are available but they can be found directly using the LOG (or LG) key on a calculator.

There is another system of logarithms, called **natural logarithms,** written as **ln** x, which is used in Calculus. The base of natural logarithms is a certain number e (≈ 2.718). We shall see the reason for this in Chapter 18.

Logarithmic Equations

Example 19

Find $\log_2 7$.

If $\log_2 7 = x$, then $2^x = 7$.

We cannot express 7 as a power of 2 directly so we convert this equation to a **logarithmic equation** using logarithms of base 10.

Take the lg of each side.

Then lg 2^x = lg 7.

Hence x lg 2 = lg 7 and so $x = \frac{\lg 7}{\lg 2}$

$= 2.81$ (by calculator correct to 3 sig. figs.)

Hence $\log_2 7 = 2.81$.

(Verify the result by using the x^y function on the calculator).

Example 20

Find x if $3^{x-1} = 2^{x+1}$.

Convert to a logarithmic equation. Take the lg of each side.

Then $\lg 3^{x-1} = \lg 2^{x+1}$ i.e. $(x-1)\lg 3 = (x+1)\lg 2$.

Now solve for x. We do not find lg 2, lg 3 yet.

$x \lg 3 - \lg 3 = x \lg 2 + \lg 2$ which gives $x(\lg 3 - \lg 2) = \lg 3 + \lg 2$

i.e. $x \lg \frac{3}{2} = \lg 6$ or $x = \frac{\lg 6}{\lg 1.5} = 4.42$ by calculator.

$\left(\textit{Note:}\text{ The right hand side is NOT } \lg \frac{6}{1.5}\right)$.

Example 21

Find x if $\log_x 6 = 1.5$.

If $\log_x 6 = 1.5$, then $x^{1.5} = 6$.

Taking the lg of each side,

$1.5 \lg x = \lg 6$ and $\lg x = \frac{\lg 6}{1.5} = 0.5188$ (by calculator).

Hence $x = 10^{0.5188} = 3.30$ by calculator, using the x^y function.

Example 22

In Example 19, Chapter 13 we found the least value of n where $0.9^{n-1} < 0.4$. *This can also be done using logarithms.*

$\lg 0.9^{n-1} < \lg 0.4$ i.e. $(n-1)\lg 0.9 < \lg 0.4$.

However we **cannot** now write $n - 1 < \frac{\lg 0.4}{\lg 0.9}$ as $\log 0.9 < 0$ and so the inequality sign must be reversed. (Division by a negative quantity).

So $n - 1 > \frac{\lg 0.4}{\lg 0.9} = 8.696$ by calculator

and hence $n > 9.696$ and we take the integral value $n = 10$.

Note: To solve an inequality, if an integral result is required, the calculator method (using x^y) as in Chapter 13, is very suitable and quick. However to solve an *equation* such as $0.9^x = 0.4$, the logarithmic method must be used. Here $x = 8.70$ (to 3 significant figures).

Example 23

A sum of money \$P is invested at a compound interest of r% per year.

(a) Show that it will amount to $\$P(1+\frac{r}{100})^n$ *after n years.*

(b) If the rate of interest is 8%, after how many years will the sum of money be doubled?

(a) After 1 year, the amount will be $P_1 = P + \frac{Pr}{100} = P\left(1+\frac{r}{100}\right)$.

After 2 years, the amount will be $P_2 = P_1\left(1+\frac{r}{100}\right) = P\left(1+\frac{r}{100}\right)^2$ and so on.

Hence after n years the amount will be $P_n = P\left(1+\frac{r}{100}\right)^n$.

(b) $P_n = 2P$. Hence $2P = P\left(1+\frac{8}{100}\right)^n = P(1.08)^n$.

Then $1.08^n = 2$.

Taking the lg of each side, $n \lg 1.08 = \lg 2$

and $n = \frac{\lg 2}{\lg 1.08} \approx 9.01$ i.e. after 9 years.

Example 24

Given that $y = ax^b + 3$ *where* $a > 0$ *and that* $y = 8$ *when* $x = 2$ *and* $y = 48$ *when* $x = 8$*, find the values of a and b.*

Substituting $x = 2$, $8 = a \times 2^b + 3$

or $a \times 2^b = 5$ (i)

When $x = 8$, $48 = a \times 8^b + 3$ so $a \times (2^3)^b = 45$ i.e. $a \times 2^{3b} = 45$

or $a \times (2^b)^3 = 45$ (ii)

From (i), $2^b = \frac{5}{a}$ so $(2^b)^3 = \frac{125}{a^3}$.

Hence (ii) becomes $a \times \frac{125}{a^3} = 45$ so $\frac{1}{a^2} = \frac{45}{125}$

and $a^2 = \frac{125}{45} = \frac{25}{9}$ giving $a = \frac{5}{3}$ (as $a > 0$).

Then from (i), $\frac{5}{3} \times 2^b = 5$ and $2^b = 3$.

Taking the lg of each side, $b \lg 2 = \lg 3$ and $b = \frac{\lg 3}{\lg 2}$ which gives $b = 1.58$.

Example 25

Show that $\log_a b = \frac{\lg b}{\lg a}$. *Hence find the value of* $\log_2 5 \times \log_{11} 4 \times \log_5 11$.

Let $\log_a b = x$. Then $a^x = b$. Now take the lg of each side and

$$x \lg a = \lg b \text{ so } x = \frac{\lg b}{\lg a}.$$

Using this result,

$$\log_2 5 \times \log_{11} 4 \times \log_5 11 = \frac{\lg 5}{\lg 2} \times \frac{\lg 4}{\lg 11} \times \frac{\lg 11}{\lg 5}$$

$$= \frac{\lg 4}{\lg 2} = \frac{\lg 2^2}{\lg 2} = \frac{2 \lg 2}{\lg 2} = 2$$

Exercise 15.6 *(Answers on page 639.)*

Give answers correct to 3 significant figures if not exact.

1 Find the value of x if

(a) $3^x = 5$ (b) $2^{x-1} = 7$ (c) $6^{x+1} = 8$

(d) $5^{2x+1} = 3^{2-x}$ (e) $2^{2x+1} = 3^{1-x}$ (f) $2^x = 1.5$

(g) $4^{x-1} = 7$ (h) $5^{3x+2} = 7^{6x-1}$ (i) $2^{2x-1} = 3^{2-x}$

(j) $1.3^x = 5$ (k) $0.6^x = 0.4$ (l) $0.8^{x-1} = 0.2^x$

2 Calculate $\log_3 5$ and $\log_5 7$.

3 If $\log_x 3 = 17$, find the value of x.

4 What is the least number of terms of the GP 3, 4, $\frac{16}{3}$, ... that can be added for their sum to be greater than 90?

5 If the sum of n terms of the GP 8, 12, 18,... is not to exceed 500, what is the largest value of n?

6 Find the least integral value of x if (a) $1.8^{x-1} > 47$, (b) $0.75^x < 0.15$.

7 In how many years will \$3000 invested at 5% per year compound interest amount to \$5000?

8 After how many years will \$9000 amount to \$20 000 if it is invested at 4.5% per year compound interest?

9 Given that $P = 50(0.75)^n$, find (a) the value of P when $n = 4$, (b) the value of n when $P = 10$.

10 The population of a city in 1980 was 3 200 000 and this was an increase of 1.7% over the population in 1979. If this rate of increase is continued, in what year will the population first exceed 5 000 000?

11 The decay of a radioactive substance is given by the formula $M = M_0 e^{-0.2t}$ where M_0 is the initial mass, M the mass after t years and $e = 2.718$. Calculate the half-life of the substance, i.e. the number of years taken for the mass to be halved.

12 Find the value of $\log_3 49 \times \log_5 9 \times \log_7 5$.

13 Find the value of $\log_5 9 \times \log_3 7 \times \log_7 2 \times \log_2 25$.

14 If $\log_5 8 \times \log_2 x = 3$, find the value of x.

15 Given that $\lg y = 1 - 3 \lg x$, show that y can be expressed in the form $y = px^q$ and find the values of p and q.

16 Show that $\lg \frac{1}{10} = -1$. Hence find the values of x which satisfy the equation $\lg(\sin x) + 1 = 0$ for $0° < x < 180°$.

17 (a) Solve the equation $\log_x 2.5 = 8$.

(b) Find the value of x if $x^{3.2} = 10$.

18 Find x if $\log_x 12 = 5$.

19 $y = ax^b - 2$. Given that $y = 6$ when $x = 2$ and $y = 22$ when $x = 4$, find the values of a and b.

SUMMARY

- Rules for indices: $x^m \times x^n = x^{m+n}$
  $x^m \div x^n = x^{m-n}$
  $(x^m)^n = x^{mn}$
- Negative index: $x^{-n} = \dfrac{1}{x^n}$
  Zero index: $x^0 = 1$
  Fractional index: $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$
- Exponential form $y = a^x$ $\longleftrightarrow$ $x = \log_a y$ logarithmic form
  $(a > 0)$ a is the base of the logarithm

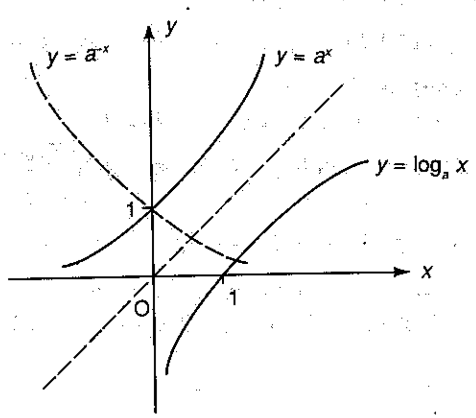


Fig.15.4

- Rules for logarithms: $\log_a PQ = \log_a P + \log_a Q$

$$\log_a \frac{P}{Q} = \log_a P - \log_a Q$$

$$\log_a P^n = n \log_a P$$

$\log_a a = 1$; $\log_a 1 = 0$

- Common logarithms have base 10 and are written as $\lg x$.

REVISION EXERCISE 15 *(Answers on page 639.)*

If not exact, give answers correct to 3 significant figures.

A

1 Find the value of (a) $8^{\frac{5}{3}}$, (b) $27^{-\frac{2}{3}}$, (c) $\left(\frac{1}{16}\right)^{-\frac{3}{2}}$

2 Solve the equations

(a) $9^x = \frac{1}{3}$ (b) $3^{x+1} = 27^{1-x}$

(c) $4^{x^2-23} = 16$ (d) $4^x - 12 \times 2^x + 32 = 0$

(e) $2^{2x-5} - 3(2^{x-3}) + 1 = 0$ (f) $4^x = 10 - 4^{x+1}$

(g) $1.6^x = 3.7$ (h) $0.35^x = 0.15$.

3 If $\log_a x^3y = u$ and $\log_a xy^2 = v$, find x in terms of a, u and v.

4 Find x if (a) $\log_3 81 = x$, (b) $\log_8 x = 2$, (c) $\log_x 8 = 2$.

5 Solve the simultaneous equations $x + y = 3$, $\log_2 x = 2 + 2 \log_2 y$.

6 (a) Given that $\log_p 7 + \log_p k = 0$, find k.

(b) Given that $4 \log_q 3 + 2 \log_q 2 - \log_q 144 = 2$, find q.

(c) Given that $\log_3 2 = 0.631$ and that $\log_3 5 = 1.465$, evaluate $\log_3 1.2$ without using tables or a calculator. (C)

7 Find x if (a) $\log_2 (x^2 + 5x - 2) = \log_2 (x^2 + 3x - 6) + \log_4 9$,

(b) $\log_3 (x - 3) + \log_3 (x + 3) = 3$.

8 Given that $xy = 2$ and $2 \lg (x - 1) = \lg y$, find the values of x and y.

9 (a) Solve the equation $5^{x+1} = 6$.

(b) Solve the equation $\log_2 x + \log_2 (x + 1) = 1$.

(c) Given that $\lg x = a$ and $\lg y = b$, express $\lg \sqrt{\frac{1000x^3}{y}}$ in terms of a and b. (C)

10 Find the value of $\log_5 9 \times \log_3 7 \times \log_7 25$.

11 (a) Given that $e = 2.718$, find the value of x such that $e^{2x} = e^x + 6$.

(b) If $y = px^q + 2$ and $y = 14$ when $x = 2$ and $y = 194$ when $x = 8$, find the values of p and of q.

(c) Given that $\log_a (xy^3) = u$ and $\log_a \left(\frac{y^2}{x}\right) = v$, express $\log_a (xy)$ in terms of u and v.

12 Solve the equations (a) $1.5^x = 10$, (b) $0.55^x = 0.1$, (c) $3.5^x = 7$.

$p \cdot x^q = 12$ $\quad \log p \cdot x^q = \log 12$

$T = T_0 \cdot e$

$(-\ln T_0 + \ln T)$

13 If $T = T_0 e^{-0.5t}$, show that $t = 2\left(\dfrac{\lg \frac{T_0}{T}}{\lg e}\right)$.

Hence find the value of t given that $e = 2.72$, $T_0 = 30$ and $T = 10$.

14 If $\log_3 (x - 6) = 2y$ and $\log_2 (x - 7) = 3y$, show that $x^2 - 13x + 42 = 72^y$. Given that $y = 1$, find the possible value(s) of x.

$(x-6)(x-7) = 72$

15 Find the relation between a and b not involving logarithms if $\log_9 a = 2 + \log_3 b$.

16 \$2000 is invested at 5% per year compound interest. After how many years will it have amounted to \$3500?

17 Inflation in a certain country is 15% per year. If this rate continues unchanged, after how many years will the cost of living have doubled?

18 Draw the graph of $y = 2^x$ for $0 \le x \le 3$ taking values of x at intervals of 0.5. By adding a suitable straight line to your graph, find an approximate solution of the equation $2^{x+1} + x = 4$.

19 Sketch the graphs of $y = \lg x$ and $y = \lg 10x$. State the coordinates of the points where each curve meets the x-axis.

20 (a) Draw the graph of $y = 2^x$ for $0 \le x \le 2$ taking scales of 2 cm for 1 unit on each axis. Add the line $y = x$ and hence draw the graph of $y = \log_2 x$ for $1 \le x \le 4$.
(b) Calculate the value of $\log_2 6$.
(c) Express $x2^x = 6$ in the form $\log_2 x = px + q$ stating the values of p and q.
(d) What is the equation of the straight line that must be added to the graph to find the solution of the equation $x2^x = 6$?
(e) Draw this line and hence solve the equation approximately.

21 (a) Solve the equations (i) $2 \times 4^{x+1} = 16^{2x}$, (ii) $\log_2 y^2 = 4 + \log_2 (y + 5)$.
(b) Given that $y = ax^n + 3$, that $y = 4.4$ when $x = 10$ and $y = 12.8$ when $x = 100$, find the values of n and of a. (C)

22 Solve the equation $\lg (\cos^2 x) + 2 = 0$ for $0° \le x \le 360°$.

23 Show that the sequence $\lg k$, $\lg 10k$, $\lg 100k$,... forms an AP and find the sum of the first 10 terms of this AP.

B

24 Solve the simultaneous equations $\log_2 x - \log_4 y = 4$, $\log_2 (x - 2y) = 5$.

25 Solve the simultaneous equations $9^x = 27^y$, $64^{xy} = 512^{x+1}$.

26 If $\log_3 2 = a$ and $\log_3 13 = b$, express $\log_{78} 52$ in terms of a and b.

27 Solve the inequality $\log_2 (\log_3 x) > 0$.

28 Given that $\log_8 (p + 2) + \log_8 q = r - \frac{1}{3}$ and that $\log_2 (p - 2) - \log_2 q = 2r + 1$, show that $p^2 = 4 + 32^r$. If $r = 1$, find the possible values of p and q.

-4 8^2 4

Conversion to Linear Form 16

In science, when two variables x and y are thought to be connected, a set of measurements is made. The results can be used to find the mathematical law connecting x and y — if there is one. When the law is found, it can be used to predict further values and these can be tested by other experiments to see if the law is still valid.

Usually the results are plotted as a graph. If this is a straight line, the relationship is easily deduced as it will be of the form $y = mx + c$, and m and c can be found from the graph.

However if the graph is not a straight line, the relationship will not be so simple. A trial formula is therefore guessed. We convert this formula to a linear form and see if the transformed values lie on a straight line graph. If they do, then we can confirm that the formula is true for these values, allowing for experimental errors.

Two very common relationships are $\boldsymbol{y = ab^x}$ and $\boldsymbol{y = ax^b}$ where a and b are constants.

Example 1

The following set of measurements of two variables x and y were obtained in an experiment. It is thought that they are related by the formula $y = ab^{-x}$. By converting this to a linear form, find whether the relationship is true for these values.

x	1.5	2.8	3.0	4.2	5.0	6.5
y	80	35	33	18	10	6

If the formula is $y = ab^{-x}$, then taking the lg of each side, $\lg y = \lg a - x \lg b$.

Now write $Y = \lg y$.

Then $Y = -x \lg b + \lg a$, which is a linear equation of the form $Y = mx + c$, where $m = -\lg b$ and $c = \lg a$.

So we plot values of Y ($= \lg y$) against x.

First we find the values of Y.

x	1.5	2.8	3.0	4.2	5.0	6.5
Y ($= \lg y$)	1.90	1.54	1.52	1.26	1	0.78

These values are plotted as shown in Fig.16.1. We see that the points lie very nearly in a straight line. Any inaccuracies can reasonably be assumed to be due to experimental errors. We draw the line which fits the points as well as we can judge. There may be some difference of opinion over the position of the line, so our results will be approximate.

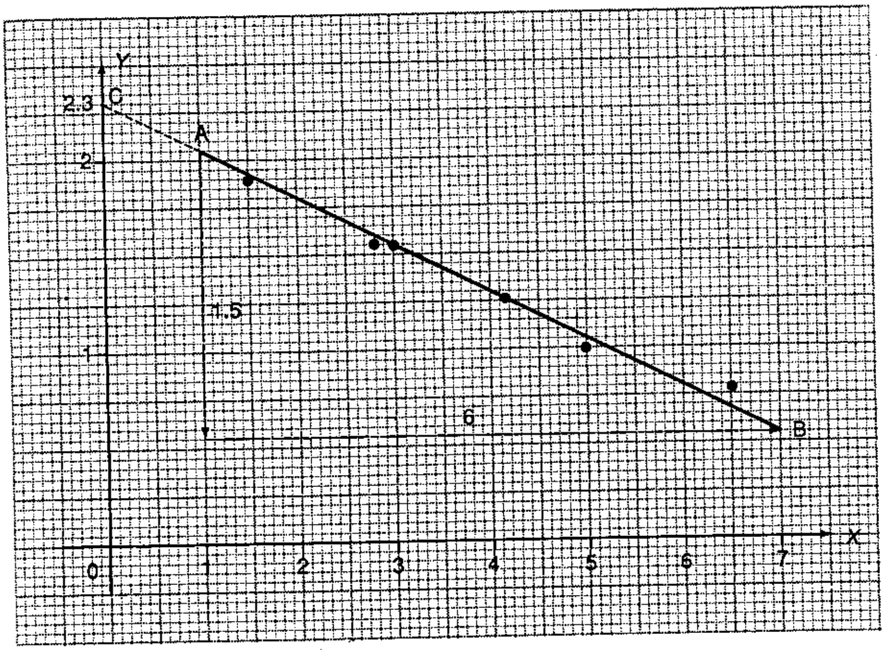


Fig. 16.1

Now find the gradient by taking two well spaced points such as A and B on the line. It helps to make the x-step between these points a convenient number. The gradient $= \frac{-1.5}{6} = -0.25$.

Then $-\lg b = -0.25$ and $b \approx 1.8$.

To find $c = \lg a$, extend the line to cut the Y-axis (point C).
Then $\lg a \approx 2.3$ giving $a \approx 200$.

Hence we find that the law relating these values is $y = 200 \times 1.8^{-x}$.

Example 2

The following set of values for two variables x and y was obtained in an experiment. It is believed that they are related by the formula $y = ax^b$. By converting to a linear form, estimate the values of a and b. From your graph, estimate the value of x for which y = 2000 and compare with the value found using the formula.

x	20	30	40	50
y	890	1640	2500	3700

If the formula is $y = ax^b$, then taking the lg of each side, $\lg y = \lg a + b \lg x$.

We write $Y = \lg y$ and $X = \lg x$.

Then $Y = bX + \lg a$ which is a linear equation of the form $y = mx + c$ where $m = b$ and $c = \lg a$.

If we plot values of Y ($= \lg y$) and X ($= \lg x$) and the graph is a straight line, then the relationship is correct.

Now find the values of X ($= \lg x$) and Y ($= \lg y$).

X	1.30	1.48	1.60	1.70
Y	2.95	3.21	3.40	3.57

These are plotted as shown in Fig.16.2. To allow space for large scales, we take X from 1.3 and Y from 2.9.

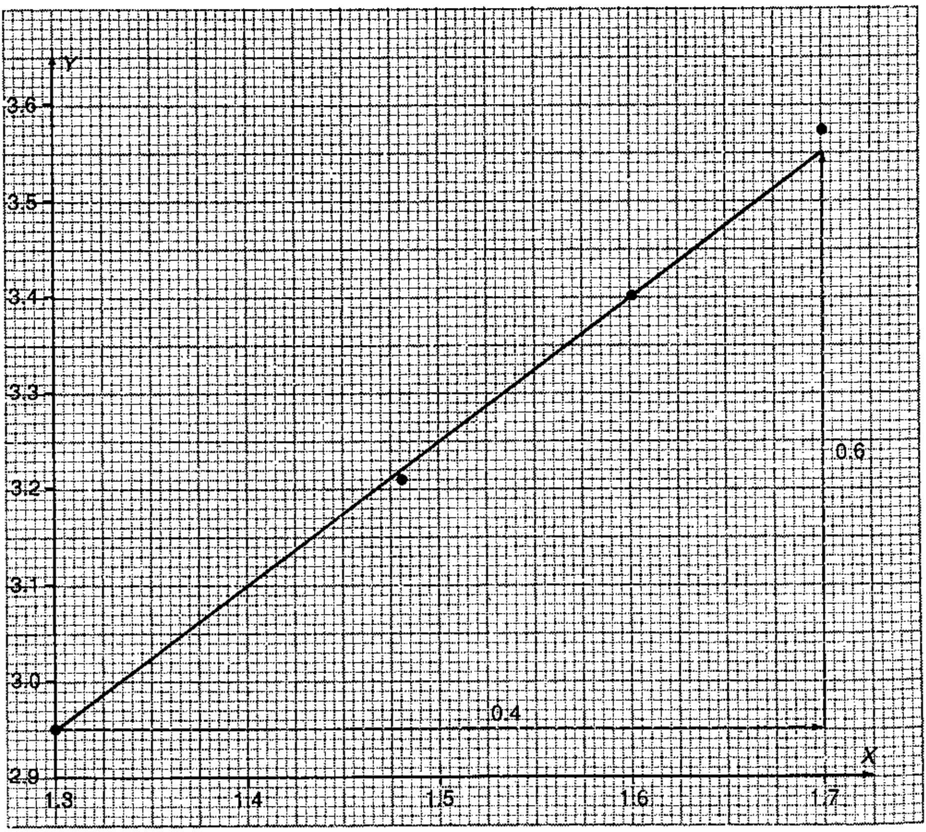


Fig. 16.2

The points lie well on a straight line and the gradient = $\frac{0.6}{0.4} = 1.5$ so $b = 1.5$.

We cannot find $c = \lg a$ from the graph as the Y-axis does not pass through $X = 0$, but we can deduce its value by using the gradient (Fig.16.3).

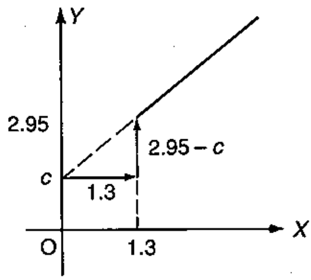


Fig. 16.3

$\frac{2.95 - c}{1.3} = 1.5$ which gives $c = 1$

Then $\lg a = 1$ and $a = 10$.

Hence the law is $y = 10x^{1.5}$.

When $y = 2000$, $Y = \lg y = 3.30$. The corresponding value of $X = 1.533$ so $\lg x = 1.533$. Hence $x = 34.1$.

Using the formula, $2000 = 10x^{1.5}$ i.e. $200 = x^{1.5}$ so $\lg 200 = 1.5 \lg x$ from which $x = 34.2$, giving good agreement.

Example 3

Two variables are known to be connected by the formula $\frac{a}{y} = \frac{b}{x^2} + 1$ where a and b are constants. The following table shows some values obtained by experiment:

x	3.5	5.0	6.5	8	10
y	0.66	1.22	1.61	2.22	2.78

By drawing a suitable straight line graph, estimate the values of a and b.

To convert $\frac{a}{y} = \frac{b}{x^2} + 1$ to a linear form, write $Y = \frac{1}{y}$ and $X = \frac{1}{x^2}$.

Then $aY = bX + 1$ i.e. $Y = \frac{b}{a}X + \frac{1}{a}$.

This is a linear equation where the gradient $m = \frac{b}{a}$ and the intercept $c = \frac{1}{a}$. We now compile a table for $X\left(=\frac{1}{x^2}\right)$ and $Y\left(=\frac{1}{y}\right)$.

X	0.082	0.04	0.024	0.016	0.01
Y	1.52	0.82	0.62	0.45	0.36

These points are plotted and the line drawn (Fig.16.4).

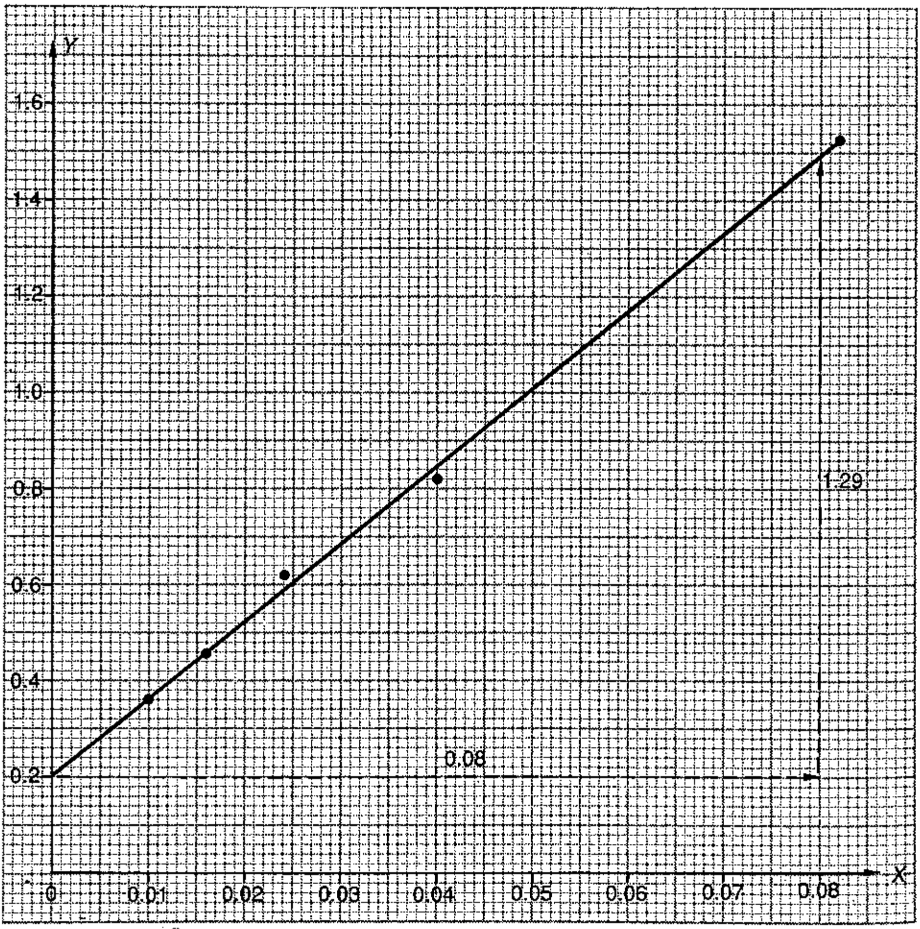


Fig. 16.4

The gradient $= \frac{1.29}{0.08} \approx 16.1 = \frac{b}{a}$.

The intercept $c = 0.2 = \frac{1}{a}$ so $a = 5$.

Hence $b = 16.1 \times 5 = 80.5$.

Example 4

The straight line in Fig.16.5 was obtained by plotting $\frac{y}{\sqrt{x}}$ against x^2. Find y in terms of x.

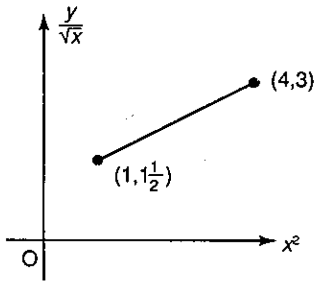


Fig. 16.5

The gradient of the line is $\frac{1\frac{1}{2}}{3} = \frac{1}{2}$.

By extending the line (Fig.16.6) to meet the $\frac{y}{\sqrt{x}}$ axis at $(0,c)$, the gradient is $\frac{1\frac{1}{2} - c}{1} = \frac{1}{2}$ so $c = 1$.

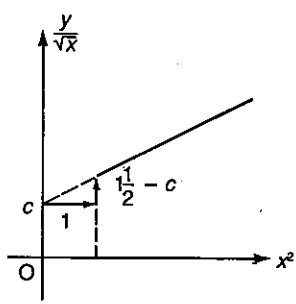


Fig. 16.6

Hence $\frac{y}{\sqrt{x}} = \frac{1}{2}x^2 + 1$ i.e. $y = \frac{1}{2}x^2\sqrt{x} + \sqrt{x}$.

Example 5

Fig.16.7 shows the straight line obtained by plotting lg y against lg x. Find

(a) lg y in terms of lg x,
(b) y in terms of x,
(c) the value of x when y = 700.

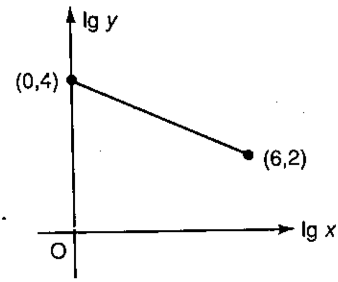


Fig. 16.7

(a) The gradient of the line $= \frac{-2}{6} = -\frac{1}{3}$ and the intercept on the lg y axis is 4. Hence the equation of the line is $\lg y = -\frac{1}{3}\lg x + 4$ which is the expression required.

(b) From (a), $\lg y = -\frac{1}{3}\lg x + 4$ i.e. $3\lg y + \lg x = 12$.

Then $\lg y^3x = \lg 10^{12}$ giving $y^3x = 10^{12}$ or $y^3 = \frac{10^{12}}{x}$.

Hence $y = \frac{10^4}{\sqrt[3]{x}} = 10^4x^{-\frac{1}{3}}$.

(c) If $y = 700$, then $700 = 10\,000x^{-\frac{1}{3}}$ and $x^{\frac{1}{3}} = \frac{100}{7}$.

Hence $x \approx 2915$.

Example 6

Convert each of the following relations to a linear form and state what functions of x and y should be plotted to obtain a straight line graph. State also the gradient and intercept of the straight line in terms of a and b.

(a) $\frac{a}{x} + \frac{b}{y} = 2$ *(b)* $y = ax + \frac{b}{x}$ *(c)* $y^2 = a + bx$

(d) $y = \frac{a}{x-b}$ *(e)* $y = a(1.5)^{-bx}$ *(f)* $y = a(x+3)^b$

(a) Take $X = \frac{1}{x}$, $Y = \frac{1}{y}$. Then $aX + bY = 2$ i.e. $Y = -\frac{a}{b}X + \frac{2}{b}$.
Plot Y against X. Gradient $= -\frac{a}{b}$, intercept $= \frac{2}{b}$.

(b) If $y = ax + \frac{b}{x}$, then $xy = ax^2 + b$. Take $X = x^2$, $Y = xy$.
This gives the linear equation $Y = aX + b$. Plot Y against X.
Gradient $= a$, intercept $= b$.

(c) Take $Y = y^2$. Then $Y = bx + a$. Plot Y against x. Gradient = b, intercept = a.

(d) $xy - by = a$ so $by = xy - a$ or $y = \frac{1}{b}xy - \frac{a}{b}$. Plot y against $X = xy$. Gradient = $\frac{1}{b}$, intercept = $-\frac{a}{b}$.

(e) $y = a(1.5)^{-bx}$. Then $\lg y = \lg a - bx \log 1.5$. Take $Y = \lg y$ then $Y = -(b \lg 1.5)x + \lg a$. Plot Y against x. Gradient = $-b \lg 1.5$ and intercept = $\lg a$.

(f) $y = a(x + 3)^b$. Then $\lg y = \lg a + b \lg(x + 3)$. Take $Y = \lg y$ and $X = \lg(x + 3)$. Plot Y against X. Gradient = b and intercept = $\lg a$.

Exercise 16.1 *(Answers on page 640.)*

1 A set of values of x and y are believed to be connected by the equation $y = ab^x$ where a and b are constants. Values of x and $\lg y$ are plotted and the graph is a straight line with gradient 0.47 and intercept -0.65. Find the value of a and of b correct to 2 significant figures.

2 A graph of $\lg y$ against $\lg x$ gives a straight line with gradient 3 and intercept 1.3. Find y in terms of x.

3 Fig. 16.8 shows the graph of $\lg y$ against $\lg x$, where $y = ax^b$. Find the value of a and of b.

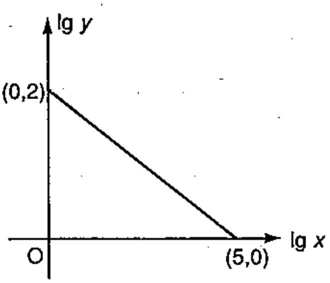


Fig. 16.8

4 Covert the equation $by = ax^2 + x$ into the linear form $Y = mX + c$, stating X and Y in terms of x and y. Y is plotted against X and the graph has a gradient of 2.3 with intercept 0.5. Find the value of a and of b.

5 The following results were obtained experimentally for two variables x and y:

x	1	2	3	4	5
y	42	120	430	920	2600

It is believed that x and y are related by the equation $y = ab^x$. By drawing a straight line graph, verify this is confirmed by the given data, except for one point. Using your graph estimate the value of a and of b and calculate a more accurate value of y for the point which did not fit.

6 The variables x and y are related in such a way that when $\frac{1}{x+1}$ is plotted against y, a straight line is obtained, passing through the points (1, 5) and (3, 11) (Fig. 16.9). Find y in terms of x.

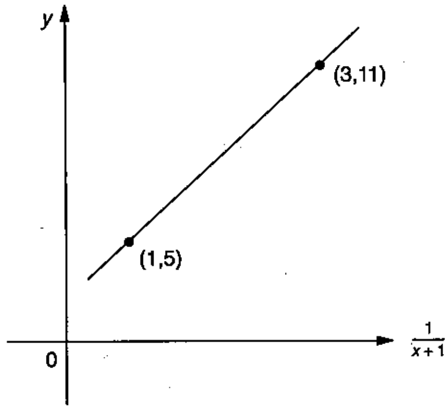


Fig. 16.9

7 It is believed that two variables u and v are related by the equation $uv^2 = av + b$, where a and b are constants.
A set of values of u and v was obtained, as in the following table:

v	1	2	5	8
u	12	3.5	0.8	0.41

By plotting uv^2 against v, verify that these values satisfy the equation and find approximate values for a and b.

8 Two variables x & y are connected by the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$. Given the following values of x and y, show how a straight line graph may be drawn

x	1	2	3	4	5
y	6.5	6.01	6.06	6.25	6.48

Draw this graph and from it, estimate the value of a and of b.

9 The following set of values of x and y obtained in an experiment are thought to be connected by the equation $\frac{p}{y} - \frac{q}{x^2} = 1$.

x	1.5	2	3.4	5
y	0.55	1	2.5	4.7

Explain how a straight line graph may be obtained and draw this graph for these values. From your graph, estimate the value of p and of q.

10 The following table gives a set of related values of x and y:

x	1.2	1.5	2	2.5	3
y	19	11	5	3	1

x and y are known to be related by the equation $x^2y = p + qx^2$. Convert this equation to linear form and draw a graph for the given values of x and y. Using the graph, find approximate values for p and q.

SUMMARY

- If $y = ax^b$, $\lg y = b \lg x + \lg a$
  Linear form $Y = bX + \lg a$
  where $Y = \lg y$, $X = \lg x$.

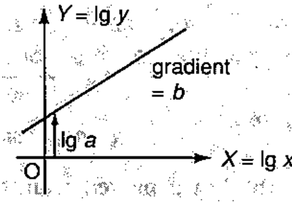


Fig. 16.10

- If $y = ab^x$, $\lg y = x(\lg b) + \lg a$
  Linear form $Y = (\lg b)x + \lg a$
  where $Y = \lg y$.

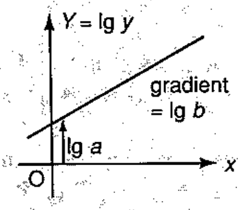


Fig. 16.11

- Other relationships can also be converted to linear form, e.g.
  $\frac{a}{y} + \frac{b}{x} = 1$ gives $Y = -\frac{b}{a}X + \frac{1}{a}$ where $Y = \frac{1}{y}$, $X = \frac{1}{x}$.
  $ay^2 = bx^2 + x$ gives $\frac{y^2}{x} = \frac{b}{a}x + \frac{1}{a}$ i.e. $Y = \frac{b}{a}x + \frac{1}{a}$ where $Y = \frac{y^2}{x}$.

REVISION EXERCISE 16 *(Answers on page 640.)*

1 Corresponding values of x and y are showing in the following table:

x	2	3	5	6	9
y	1.7	2.2	3.0	3.3	4.1

It is known that x and y are related by the equation $y^2 = a + bx$. Show that a linear equation can be derived from this and draw its graph for the above values. Hence estimate the value of a and of b and estimate the smallest possible value of x.

2 (a) Fig. 16.12 shows part of the straight line obtained by plotting y against $\frac{1}{x^2}$. Two of the points on the line are given. Find y in terms of x.

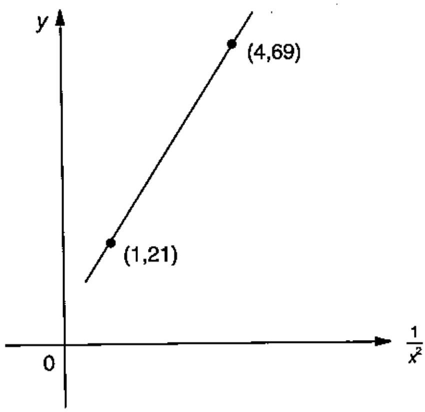


Fig. 16.12

(b) $\lg y$ is plotted against $\lg x$ and a straight line obtained, part of which is shown in Fig. 16.13. Two of the points on the line are given. Express y in terms of x.

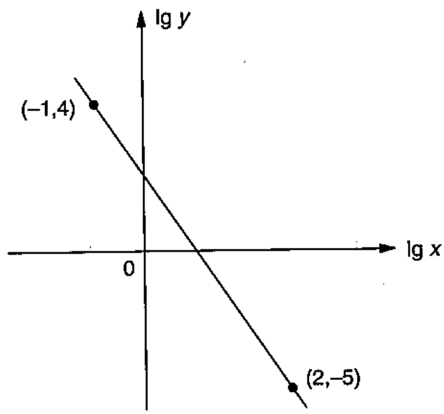


Fig. 16.13

(c) If variables x and y are connected by the equation $ax^2 - y^2 = bx$ (a and b constants) explain how the value of a and of b can be obtained from a straight line graph.

3 Measured values of x and y are given in the following table.

x	1	2	2.5	5	8
y	0.20	1.16	2.03	9.02	23.96

It is known that x and y are related by the equation $ax^2 + by = x$.
Explain how a straight line graph may be drawn to represent the given equation and draw it for the given data.
Use your graph to estimate the value of a and of b. (C)

4 (a) It is known that the variables x and y are related by the equation $y = \frac{p}{x-q}$, where p and q are unknown constants.

Express this equation in a form suitable for drawing a straight line graph, and state which variable should be used for each axis. Explain how the value of p and of q could be determined from this graph.

(b) The table shows experimental values of two variables x and y.

x	0.5	1.0	1.5	2.0	2.5	3.0
y	14.6	6.8	4.0	2.4	1.2	0.4

It is known that x and y are related by an equation of the form $y = ax + \frac{b}{x}$ where a and b are unknown constants. Plot xy against x^2 and use the graph to estimate (i) the value of a and of b, (ii) the value of y when $x = 1.2$. (C)

5 (a) The table shows experimental values of two variables x and y:

x	1.5	2.0	2.5	3.0	3.5	4.0
y	1.8	2.1	2.4	2.6	2.9	3.1

It is known that x and y are related by the equation $y = kx^n$, where k and n are constants. Draw a suitable straight line graph to represent the above data and use it to estimate k and n.

(b) The variables x and y are related in such a way that when $x + y$ is plotted against x^2 a straight line is obtained passing through $(1, -1)$ and $(5, 2)$ (Fig. 16.14). Find (i) the values of x when $x + y = 5$, (ii) y as a function of x, (iii) the values of x when $y = 0$. (C)

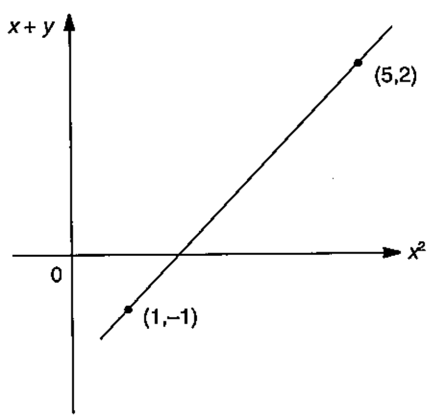


Fig. 16.14

6 (a) The variables x and y are connected by the equation $y = ax^b$ where a and b are constants. Fig. 16.15 shows the straight line graph obtained by plotting lg y against lg x.
Calculate the value of a and of b and hence find the value of y when $x = 5$.

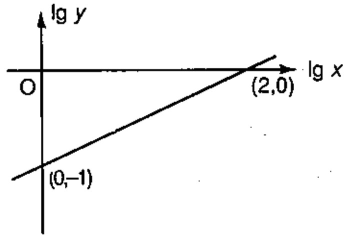


Fig. 16.15

(b) The variables x and y are connected by the equation $x + py = qxy$, where p and q are unknown constants. Explain how the value of p and of q may be obtained from a suitable straight line graph. (C)

7 It is predicted from theory that two variables P and T are related by the equation $P = a + \dfrac{b}{\sqrt{(T-3)}}$.
The following values of P and T were found by experiment:

T	10	20	30	40	50
P	13.3	10.6	9.4	8.9	8.7

By plotting P against $\dfrac{1}{\sqrt{(T-3)}}$, confirm that the equation is approximately true for these values. Use your graph to estimate the value of a and of b.

8 (a) Variables x and y are known to be related by an equation of the form $a(x + y - b) = bx^2$, where a and b are constants. Observed values of the two variables are shown in the following table.

x	1	2	3	4	5
y	0.5	0.5	1.5	3.5	6

Plot $x + y$ against x^2, draw the straight line graph and use it to estimate the value of a and of b.

(b) Variables x and y are related by the equation $\dfrac{x^2}{p^2} + \dfrac{2y^2}{q^2} = 1$ where p and q are positive constants.

When the graph of y^2 against x^2 is drawn, a straight line is obtained. Given that the intercept on the y^2-axis is 4.5 and that the gradient of the line is –0.18, calculate the value of p and of q. (C)

9 Variables x and y are related by the equation $ay = (\sqrt{x} - 1)^b$ where a and b are constants. Some values of x and y were found and are shown in the following table:

x	4	6	9	12	16
y	0.25	0.32	0.41	0.47	0.54

By drawing a suitable straight line graph, estimate the values of a and b.

10 Quantities p and v are related by the equation $pv^n = a$ where n and a are constants. The following values of p and v were found:

p	0.9	2.7	5.4	8.7
v	5	2.4	1.5	1.1

Convert the equation to linear form and draw the straight line graph. Using the graph, estimate the value of n and of a.

Calculus (4): Further Techniques: Trigonometric Functions

17

So far we can differentiate single terms such as x^5, polynomials such as $2x^3 - 3x + \frac{1}{x}$ and composite functions such as $(2x^3 - 1)^4$.
We now extend the range of functions we can deal with.

Fractional Indices

If $y = ax^n$, then you will recall that $\frac{dy}{dx} = nax^{n-1}$ and $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$.

The rules for differentiation and integration still hold when the index n, is a rational number, i.e. a fraction.

Example 1

Differentiate (a) $x^{\frac{2}{3}}$, (b) $\frac{1}{\sqrt{x}}$, (c) $\sqrt{x^2 - 2x - 3}$ wrt x.

(a) $y = x^{\frac{2}{3}}$

Then $\frac{dy}{dx} = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$

(b) $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

(c) $y = (x^2 - 2x - 3)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x^2 - 2x - 3)^{-\frac{1}{2}} \times (2x - 2)$

$= (x - 1)(x^2 - 2x - 3)^{-\frac{1}{2}}$ or $\frac{x-1}{\sqrt{x^2 - 2x - 3}}$

Example 2

Find (a) $\int x^{-\frac{1}{3}}\,dx$ *, (b)* $\int_4^9 2x^{\frac{3}{2}}\,dx$.

(a) $\int x^{-\frac{1}{3}}\,dx = \dfrac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \dfrac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}x^{\frac{2}{3}} + c$

Check by differentiating.

(b) $\int_4^9 2x^{\frac{3}{2}}\,dx = \left[\dfrac{2x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_4^9$

$= \left[\dfrac{2x^{\frac{5}{2}}}{\frac{5}{2}}\right]_4^9 = \left[\frac{4}{5}x^{\frac{5}{2}}\right]_4^9$

$= \left(\frac{4}{5} \times 9^{\frac{5}{2}}\right) - \left(\frac{4}{5} \times 4^{\frac{5}{2}}\right)$

$= \left(\frac{4}{5} \times 243\right) - \left(\frac{4}{5} \times 32\right) = \frac{4}{5}(243 - 32) = \frac{4}{5} \times 211 = 168\frac{4}{5}$

Integration of Powers of the Linear Function $ax + b$

If $y = (ax + b)^{n+1}$, then $\frac{dy}{dx} = (n + 1)a(ax + b)^n$.

Hence $\int (n + 1)a(ax + b)^n\,dx = (ax + b)^{n+1}$

and so $\quad \displaystyle\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c \quad$ where $n \neq -1$

This result only applies to a **linear function** $\boldsymbol{ax + b}$. The integration of powers of non-linear functions such as $ax^2 + b$ cannot be done in this way and is outside our work.

The case where $n = -1$ will be studied in Chapter 18.

Example 3

Find (a) $\int (2x - 1)^3\,dx$, *(b)* $\int \frac{dx}{(3x + 2)^2}$ *(c)* $\int_1^5 \sqrt{2x - 1}\,dx$.

(a) Here $a = 2$, $b = -1$ and $n = 3$.

So $\int (2x - 1)^3\,dx = \dfrac{(2x - 1)^4}{4 \times 2} + c = \frac{1}{8}(2x - 1)^4 + c$

(b) $\int \dfrac{dx}{(3x + 2)^2}$ is short for $\int \dfrac{1}{(3x + 2)^2}\,dx = \dfrac{(3x + 2)^{-1}}{(-1)(3)} + c$

$= -\frac{1}{3}(3x + 2)^{-1} + c$

$= -\dfrac{1}{3(3x + 2)} + c$

(c) $\int_1^5 \sqrt{2x-1}\, dx = \int_1^5 (2x-1)^{\frac{1}{2}}\, dx$

$$= \left[\frac{(2x-1)^{\frac{3}{2}}}{2 \times \frac{3}{2}}\right]_1^5$$

$$= \left(\frac{1}{3} \times 9^{\frac{3}{2}}\right) - \left(\frac{1}{3} \times 1^{\frac{3}{2}}\right) = 9 - \frac{1}{3} = 8\frac{2}{3}$$

Example 4

Find the area bounded by the curve $y = \frac{1}{\sqrt{2x-3}}$, the x-axis and the lines $x = 2, x = 6$.

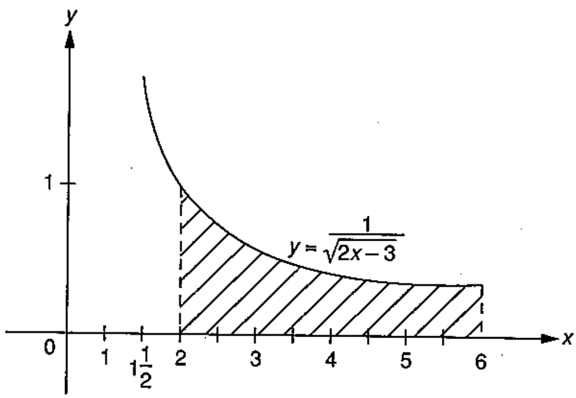


Fig.17.1

As $\sqrt{2x-3}$ is positive for $x > 1\frac{1}{2}$, y is real and positive in the area required.

$$\text{Area} = \int_2^6 (2x-3)^{-\frac{1}{2}}\, dx = \left[\frac{(2x-3)^{\frac{1}{2}}}{2 \times \frac{1}{2}}\right]_2^6$$

$$= \left[(2x-3)^{\frac{1}{2}}\right]_2^6 = \left(9^{\frac{1}{2}}\right) - \left(1^{\frac{1}{2}}\right) = 2$$

Example 5

The section of the curve $y = \frac{1}{\sqrt[3]{x-1}}$ between the lines $x = 2$ and $x = 9$ is rotated about the x-axis through 360°. Find the volume of the solid created.

You will recall that the volume of a solid of revolution $= \int \pi y^2\, dx$.

Here $y = (x-1)^{-\frac{1}{3}}$.

So the volume $= \int_2^9 \pi (x-1)^{-\frac{2}{3}}\, dx$

$$= \left[\frac{\pi(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_2^9 = \left[3\pi(x-1)^{\frac{1}{3}}\right]_2^9 = (3\pi \times 2) - (3\pi \times 1) = 3\pi$$

Example 6

If $y = 2\sqrt{9 - x^2}$, what is the approximate change in y when x is increased from 2 to 2.01?

When $x = k$, $\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \delta x$, where $\left(\frac{dy}{dx}\right)_{x=k}$ is the value of $\frac{dy}{dx}$ when $x = k$.

Here $k = 2$ and $\delta x = 0.01$.

$y = 2(9 - x^2)^{\frac{1}{2}}$ so $\frac{dy}{dx} = 2 \times \frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = -2x(9 - x^2)^{-\frac{1}{2}}$.

$\left(\frac{dy}{dx}\right)_{x=2} = -4(5)^{-\frac{1}{2}} = \frac{-4}{\sqrt{5}}$

Hence $\delta y \approx -\frac{4}{\sqrt{5}} \times 0.01 = -0.018$ (y has decreased).

Exercise 17.1 *(Answers on page 640.)*

1 Differentiate wrt x:

(a) $x^{-\frac{3}{4}}$ (b) $4\sqrt{x}$ (c) $\sqrt{4x - 3}$

(d) $2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$ (e) $\sqrt[3]{x^3 - 6x^2}$ (f) $2x^{\frac{3}{2}}$

(g) $\frac{4}{\sqrt{x}}$ (h) $\sqrt{4x^3 - 3}$ (i) $\sqrt{1 - 2x + 4x^2}$

2 Integrate wrt x:

(a) $x^{\frac{3}{4}}$ (b) $x^{-\frac{1}{2}}$ (c) $x^{\frac{2}{3}} - x^{-\frac{1}{3}}$

(d) $\frac{1}{2}x^{-\frac{2}{3}}$ (e) $\frac{x^{\frac{2}{3}} - x^{\frac{1}{2}}}{x^{\frac{1}{3}}}$ (f) $5x^{\frac{2}{3}}$

(g) $\frac{3}{\sqrt{x}}$ (h) $\frac{1}{3}x^{-\frac{1}{3}}$ (i) $\frac{x - x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

3 Evaluate

(a) $\int_1^4 \frac{\sqrt{x}}{2}\,dx$ (b) $\int_{-8}^{-1} x^{\frac{1}{3}}\,dx$ (c) $\int_0^1 x^{\frac{1}{5}}\,dx$ (d) $\int_1^4 x^{-\frac{2}{3}}\,dx$ (e) $\int_1^{16} x^{-\frac{1}{2}}\,dx$

4 Integrate wrt x:

(a) $(2x - 3)^2$ (b) $(2x + 5)^4$ (c) $(x - 2)^{-3}$

(d) $\sqrt{x - 3}$ (e) $\frac{1}{(3x - 2)^2}$ (f) $(2x + 3)^{-\frac{1}{2}}$

(g) $\frac{1}{\sqrt{2x + 3}}$ (h) $(3 - 4x)^3$ (i) $\frac{1}{\sqrt{3 - 2x}}$

(j) $(3x + 2)^4$ (k) $(4x - 1)^{\frac{1}{3}}$ (l) $(2x - 5)^{-\frac{1}{2}}$

(m) $\sqrt[3]{4x - 1}$ (n) $(1 - 2x)^{-2}$

5 Find the values of

(a) $\int_0^1 (2x + 1)^2\,dx$ (b) $\int_1^5 \sqrt{3x + 1}\,dx$

(c) $\int_0^1 (3x - 1)^2\,dx$ (d) $\int_{-5}^0 \sqrt{1 - 3x}\,dx$

(e) $\int_{\frac{2}{3}}^{\frac{4}{3}} (3x - 4)^3\,dx$ (f) $\int_{-2}^2 \sqrt{2x + 5}\,dx$

(g) $\int_1^6 \frac{dx}{\sqrt{x+3}}$ (h) $\int_1^2 (3x-2)^3\,dx$

6 Calculate the area bounded by the curve $y = (3x-1)^{-2}$, the x-axis and the lines $x = 1$, $x = 3$.

7 The part of the curve $y = \frac{1}{2x-3}$ between $x = 2$ and $x = 3$ is rotated about the x-axis through 360°. Find the volume of the solid of revolution.

8 If $y = 3\sqrt{x}$, find the approximate change in y when x is increased from 4 to 4.01.

9 Given that $y = 3\sqrt{9+x^2}$, find the change in y approximately when x is decreased from 4 to 3.99.

10 Given that $T = 9r^{\frac{4}{3}}$ and that r is increased from 8 to 8.01, find the approximate change in T.

11 If $P = kv^{\frac{2}{3}}$, where k is a constant, find the approximate percentage change in P if v is increased by 3% when it is 5.

12 If $V = 10x^{\frac{3}{2}}$, find the approximate change in V when x is decreased from 4 to 3.998.

Differentiation of the Product of Two Functions

$y = (3x-1)^3(x^2+5)^2$ is a product of two functions of x, $(3x-1)^3$ and $(x^2+5)^2$. Each of these can be differentiated but how can we find $\frac{dy}{dx}$? As we shall see, the result is NOT the product of their derivatives.

Let $y = uv$ where u and v are each functions of x.

Suppose x has an increment δx. This will produce increments δu in u and δv in v and finally produce an increment δy in y.

So $y + \delta y = (u+\delta u)(v+\delta v) = uv + u\delta v + v\delta u + (\delta u)(\delta v)$

Then $\delta y = u\delta v + v\delta u + (\delta u)(\delta v)$

and $\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v$

Now let $\delta x \to 0$. Consequently $\delta u \to 0$, $\delta v \to 0$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$.

So, as $\delta x \to 0$, $\frac{dy}{dx} \to u\frac{dv}{dx} + v\frac{du}{dx}$.

Hence we have the **product rule** for $y = uv$:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

where u and v are functions of x.

As the result is symmetrical in u and v, it does not matter which function is chosen as u or v.

Example 7

Differentiate $(3x-2)(x^3+4)$ wrt x.

Take $u = 3x - 2$, $v = x^3 + 4$.

$$\frac{du}{dx} = 3,\ \frac{dv}{dx} = 3x^2$$

Then
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= (3x - 2) \times 3x^2 + (x^3 + 4) \times 3$$
$$= 9x^3 - 6x^2 + 3x^3 + 12 = 12x^3 - 6x^2 + 12$$

Example 8

Differentiate $x^3(2x - 1)^4$ wrt x.

Take $u = x^3$, $v = (2x - 1)^4$.

$$\frac{dy}{dx} = \underset{\substack{\uparrow \\ u}}{x^3} \times \underbrace{4(2x-1)^3 \times 2}_{\substack{\uparrow \\ \frac{dv}{dx}}} + \underset{\substack{\uparrow \\ v}}{(2x-1)^4} \times \underset{\substack{\uparrow \\ \frac{du}{dx}}}{3x^2}$$

In this example, we simplify as far as possible and leave the result in factor form.

$$\frac{dy}{dx} = x^2(2x - 1)^3[x \times 8 + (2x - 1)3]$$
$$= x^2(2x - 1)^3(14x - 3)$$

Example 9

Differentiate $(3x - 1)^3(x^2 + 5)^2$ wrt x.

$$\frac{dy}{dx} = (3x - 1)^3 \times 2(x^2 + 5) \times 2x + (x^2 + 5)^2 \times 3(3x - 1)^2 \times 3$$
$$= (3x - 1)^2(x^2 + 5)[(3x - 1) \times 4x + (x^2 + 5) \times 9]$$
$$= (3x - 1)^2(x^2 + 5)(12x^2 - 4x + 9x^2 + 45)$$
$$= (3x - 1)^2(x^2 + 5)(21x^2 - 4x + 45)$$

Exercise 17.2 *(Answers on page 641.)*

1 Differentiate each of the following products wrt x. Leave the answers in simplified factor form.

(a) $x(x - 2)^2$
(b) $x^2(x^2 - 1)$
(c) $(x^2 + 1)(x^3 - 1)$
(d) $(x + 1)^2(x - 2)^3$
(e) $x^5(1 - 2x)^2$
(f) $(1 - x)^2(3 - x)^3$
(g) $x^2(x^2 - x - 1)^3$
(h) $x^2 (x^2 - 3)^3$
(i) $(3x - 2)^2(2x^2 - 1)$
(j) $(x^2 + 1)^2(2x - 1)^3$
(k) $\sqrt{x}(x^3 - 1)^2$
(l) $x(\sqrt{x} - 1)^2$
(m) $2x(1 - 2x)^3$
(n) $\sqrt{x - 1}(x + 1)^4$
(o) $(x^2 - x - 2)(x + 1)^3$
(p) $(3x - 1)^2(2x + 3)^3$

2 Find the equation of the tangent to the curve $y = (x + 1)(x - 2)^3$ at the point where $x = 1$.

3 Given that $y = ax^2(x-1)^3$ and that $\frac{dy}{dx} = 32$ when $x = 2$, find the value of a.

4 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y = x(x+1)^3$.

5 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y = x(2x-1)^4$.

6 For what values of x does the curve $y = x^2(x-1)^3$ have stationary points?

7 f, g and h are three functions of x. By taking f as u and gh as v, show that $\frac{d(fgh)}{dx} = gh\frac{df}{dx} + f\frac{d(gh)}{dx}$ and deduce that $\frac{d(fgh)}{dx} = gh\frac{df}{dx} + fh\frac{dg}{dx} + fg\frac{dh}{dx}$.
Hence differentiate $x(x+1)^2(x+2)^3$ wrt x.

Differentiation of the Quotient of Two Functions

We can also find a formula for differentiating a quotient of two functions, such as $\frac{x+1}{x-1}$.
Suppose $y = \frac{u}{v}$ where u and v are functions of x.
Let x have an increment δx. Then u and v will have increments δu, δv respectively and y will have an increment δy.

Now $y + \delta y = \frac{u + \delta u}{v + \delta v}$ and therefore

$$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{uv + v\delta u - uv - u\delta v}{v(v+\delta v)} = \frac{v\delta u - u\delta v}{v(v+\delta v)}$$

Then $$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta u}{\delta x}}{v(v+\delta v)}$$

Now let $\delta x \to 0$. Then $\delta u \to 0$, $\delta v \to 0$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$.

So we obtain the **quotient formula** for the differentiation of $y = \frac{u}{v}$:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This formula is not symmetrical in u and v so it is important to get the terms in the right order. An informal way of remembering this formula is '*bottom dee top minus top dee bottom all over bottom squared*' where 'top' = u, 'bottom' = v.
Note: It is unnecessary to use this formula for functions with a constant numerator such as $\frac{k}{\sqrt{x-2}}$ (take as $k(x-2)^{-\frac{1}{2}}$) or functions with a single term denominator such as $\frac{x-2x^3}{x^2}$ (divide first).

Example 10

Differentiate (a) $\frac{x-1}{x+1}$, (b) $\frac{x^2+1}{x^2-x-1}$, (c) $\frac{x}{\sqrt{x+1}}$ *wrt x.*

(a) Here $u = x-1$, $v = x+1$, $\frac{du}{dx} = 1$, $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(b) $u = x^2 + 1,\ v = x^2 - x - 1,\ \frac{du}{dx} = 2x,\ \frac{dv}{dx} = 2x - 1$

$$\frac{dy}{dx} = \frac{(x^2 - x - 1)(2x) - (x^2 + 1)(2x - 1)}{(x^2 - x - 1)^2}$$

$$= \frac{2x^3 - 2x^2 - 2x - 2x^3 - 2x + x^2 + 1}{(x^2 - x - 1)^2} = \frac{-x^2 - 4x + 1}{(x^2 - x - 1)^2}$$

(c) $u = x,\ v = (x + 1)^{\frac{1}{2}},\ \frac{du}{dx} = 1,\ \frac{dv}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{(x + 1)^{\frac{1}{2}}(1) - x\frac{1}{2}(x + 1)^{-\frac{1}{2}}}{[(x + 1)^{\frac{1}{2}}]^2} = \frac{(x + 1)^{\frac{1}{2}} - \frac{x}{2}(x + 1)^{-\frac{1}{2}}}{x + 1}$$

To simplify this, multiply the numerator and denominator by $2(x + 1)^{\frac{1}{2}}$.

$$\frac{dy}{dx} = \frac{2(x + 1) - x}{2(x + 1)(x + 1)^{\frac{1}{2}}} = \frac{x + 2}{2(x + 1)^{\frac{3}{2}}}$$

Example 11

If $y = \frac{x}{3x + 2}$ *, show that* $\frac{dy}{dx} = \frac{2}{(3x + 2)^2}$.

Hence or otherwise find $\int_1^3 \frac{dx}{(3x + 2)^2}$.

$$\frac{dy}{dx} = \frac{(3x + 2)(1) - x(3)}{(3x + 2)^2} = \frac{2}{(3x + 2)^2}$$

Hence means that we should use the above result and notice that

$$\int_1^3 \frac{dx}{(3x + 2)^2} = \frac{1}{2}\int_1^3 \frac{2}{(3x + 2)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{x}{3x + 2}\right]_1^3 \quad \text{(i)}$$

$$= \frac{1}{2}\left(\frac{3}{11}\right) - \frac{1}{2}\left(\frac{1}{5}\right) = \frac{2}{55}$$

Otherwise means that another method can be used. We must notice that it is the integral of a linear function.

$$\int_1^3 (3x + 2)^{-2}\,dx = \left[\frac{(3x + 2)^{-1}}{-3}\right]_1^3$$

$$= \left[\frac{-1}{3(3x + 2)}\right]_1^3 \quad \text{(ii)}$$

$$= \left(-\frac{1}{33}\right) - \left(-\frac{1}{15}\right) = \frac{2}{55}$$

Note: The two integrals (i) and (ii) look different but they only differ by a constant.

$$\frac{1}{2}\left(\frac{x}{3x + 2}\right) = \frac{1}{6}\left(\frac{3x + 2 - 2}{3x + 2}\right) = \frac{1}{6}\left(1 - \frac{2}{3x + 2}\right) = \frac{1}{6} - \frac{-1}{3(3x + 2)}.$$

The constant $\frac{1}{6}$ disappears when the limits are substituted.

Exercise 17.3 *(Answers on page 641.)*

1 Differentiate wrt x, simplifying where possible:

(a) $\frac{x}{x+2}$ (b) $\frac{x+1}{x+2}$ (c) $\frac{x-2}{2x+1}$

(d) $\frac{3x-2}{x^2+1}$ (e) $\frac{x^2+2}{x-1}$ (f) $\frac{x^2+x-1}{1-x}$

(g) $\frac{x}{\sqrt{x-2}}$ (h) $\frac{2x}{\sqrt{2x+1}}$ (i) $\frac{x^3}{x^2+1}$

(j) $\frac{x}{\sqrt{x^2-2}}$ (k) $\frac{x^2}{3x+1}$ (l) $\frac{x-1}{2-x}$

(m) $\frac{3x-4}{x^2+1}$ (n) $\frac{x+1}{x^2+1}$

2 If $y = \frac{x}{x+1}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that $(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

3 If $y = \frac{x}{2x-1}$, show that $\frac{dy}{dx} = \frac{-1}{(2x-1)^2}$. Hence or otherwise find $\int_1^5 \frac{dx}{(2x-1)^2}$.

4 Given that $y = \frac{x}{2x+3}$, find $\frac{dy}{dx}$.

Hence or otherwise evaluate $\int_1^2 \frac{dx}{(2x+3)^2}$.

5 If $y = \sqrt{\frac{x}{x+1}}$, find $\frac{dy}{dx}$. (Take $y = \frac{\sqrt{x}}{\sqrt{x+1}}$.)

Hence evaluate $\int_{-\frac{4}{3}}^{-\frac{9}{8}} \frac{dx}{x^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$.

6 Differentiate $\frac{x}{\sqrt{x^2+3}}$ wrt x. Hence find $\int_{-1}^{1} \frac{dx}{(x^2+3)^{\frac{3}{2}}}$.

7 Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{x+1}{x-1}}$.

8 Find the values of x which give stationary points on the curve $y = \frac{x^2}{x-2}$.

9 (a) Given that $y = \frac{x+a}{x+2}$ and that $\frac{dy}{dx} = -\frac{1}{25}$ when $x = 3$, find the value of a.

(b) If $\int_0^1 \frac{1}{(2x+k)^2}\,dx = \frac{1}{3}$, where k is a constant, find the value of k.

10 If $y = \frac{x}{\sqrt{x^2+x+1}}$, find $\frac{dy}{dx}$.

Hence find the x-coordinate of the stationary point on the curve.

Differentiation of Implicit Functions

All the functions we have met so far have been in the form $y = f(x)$ i.e. they have been **explicit** functions. y has been given directly in terms of x. A function may however be stated **implicitly**, as for example $x^3 + y^3 = 3xy$, where it would be difficult to make y the subject. Using the product rule we can differentiate such functions and *then* find $\frac{dy}{dx}$.

Example 12

Find $\frac{dy}{dx}$ *if* $x^2y - 3x = 5$.

(It would be easy to make y the subject here and then differentiate but we use this as a simple example of differentiating an implicit function.)

We differentiate each term wrt x.
x^2y is treated as a product.

$\frac{d(x^2y)}{dx} = x^2\frac{dy}{dx} + y\frac{dx^2}{dx} = x^2\frac{dy}{dx} + y \times 2x$

Hence differentiating the complete equation, $x^2\frac{dy}{dx} + 2xy - 3 = 0$ and so $\frac{dy}{dx} = \frac{3 - 2xy}{x^2}$.

Note that $\frac{dy}{dx}$ is now expressed in terms of x and y.

Example 13

Find $\frac{dy}{dx}$ *if* $x^3 + y^3 = 3xy$.

Differentiate each term wrt x.
y^3 is treated as a composite function and xy as a product.

$3x^2 + \frac{dy^3}{dx} = 3x\frac{dy}{dx} + 3y\frac{dx}{dx}$

i.e. $3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$ (as $\frac{dx}{dx} = 1$)

Dividing through by 3 and rearranging the terms,

$(y^2 - x)\frac{dy}{dx} = y - x^2$

and $\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$.

Example 14

Find the equations of the tangents at the points where $x = 2$ *on the curve* $x^2 + y^2 + 3x - 4y = 7$.

First we find $\frac{dy}{dx}$.

Differentiating each term of the equation, $2x + 2y\frac{dy}{dx} + 3 - 4\frac{dy}{dx} = 0$

and hence $(4 - 2y)\frac{dy}{dx} = 2x + 3$ giving $\frac{dy}{dx} = \frac{2x + 3}{4 - 2y}$.

We now find the coordinates.

Substituting $x = 2$ in the equation, $4 + y^2 + 6 - 4y = 7$ i.e.
$y^2 - 4y + 3 = 0$ or $(y - 3)(y - 1) = 0$ and $y = 1$ or 3.

There are two points where $x = 2$, whose coordinates are (2,3) and (2,1).

At (2,3), $\frac{dy}{dx} = \frac{4+3}{4-6} = -\frac{7}{2}$ and the equation of the tangent there is

$y - 3 = -\frac{7}{2}(x-2)$ i.e. $2y + 7x = 20$.

At (2,1), $\frac{dy}{dx} = \frac{4+3}{4-2} = \frac{7}{2}$ and the equation of the tangent there is

$y - 1 = \frac{7}{2}(x-2)$ i.e. $2y = 7x - 12$.

Exercise 17.4 *(Answers on page 641.)*

1 Find $\frac{dy}{dx}$ in terms of x and y for the following curves:

(a) $2x^2 + xy = 5$
(b) $xy = 7$
(c) $x^2y + 2x = 5$
(d) $xy^2 + 2y = 1$
(e) $xy - y^2 = 3x$
(f) $xy + x + y = 5$
(g) $x^2 + y^2 = 10$
(h) $x^2 - xy = 4y^2$
(i) $\frac{x^2}{2} - \frac{y^2}{3} = 1$
(j) $\sqrt{x} + \sqrt{y} = 2$
(k) $y^2 = 2x - 3$
(l) $x^2 + y^2 - 2x + 5y = 12$
(m) $x^2y + xy^2 = x - y$
(n) $2x^2 - 3y^2 = 2x + 3y - 1$
(o) $xy + y^2 = x$
(p) $x^2 - y^2 = 2xy$
(q) $y^2 = 3x^3 - 2xy$
(r) $x^2 = 8y^3$
(s) $\sqrt{x} - \sqrt{y} = 1$

2 Find the gradient of the following curves at the point given:

(a) $x^2 + y^2 = 13$ at (3,2)
(b) $y^2 = 8x$ at (2,4)
(c) $\sqrt{x} + \sqrt{y} = 5$ at (4,1)
(d) $x^3 + y^3 = 9$ at (2,1)
(e) $x^2 + 4y^2 - 2x - 3y = 1$ at the point (2,–1).
(f) $xy^2 = y(2x-1) + 27$ at the point (2,–3).

3 Find the equations of the tangent and the normal to the curve $y^2 = 4x$ at the point (1,2).

4 Find the equations of the tangents to the curve $x^2 + y^2 = 3x - y$ at the points where $x = 1$.

5 Find the equation of the normal to the curve $x^2 + 3y = 2xy$ at the point where $x = 2$.

6 If $y^2(x+3) = 20$, find the values of $\frac{dy}{dx}$ when $x = 2$.

7 The point (2,1) lies on the curve $Ax^2 + By^2 = 11$ where A and B are constants. If the gradient of the curve at that point is 6, find the value of A and of B.

8 Find the equation of the tangent to the curve $x^3 + y^3 = 9xy$ at the point (2, 4).

Revision of Calculus Methods

At this point, before going on to the calculus of the trigonometric functions, it would be valuable to practise the techniques already studied. Here is an exercise for this.

Exercise 17.5 *(Answers on page 641.)*

1 Use any appropriate method to differentiate the following functions. Simplify where possible.

(a) $3x^5 - 2x^{-\frac{2}{3}} - \frac{1}{x}$ (b) $x\sqrt{x} - 5$ (c) $(2x^3 - 1)^3$

(d) $\frac{x}{1 + x^2}$ (e) $\sqrt{2x^2 - 4x + 3}$ (f) $\frac{\sqrt{x}}{1 + x}$

(g) $\frac{1}{\sqrt{4x - 3}}$ (h) $x - \frac{1}{\sqrt{x}}$ (i) $\left(x - \frac{2}{x}\right)^2$

(j) $\frac{x - 2}{x + 1}$ (k) $\sqrt{x} + \frac{1}{\sqrt{x}}$ (l) $\sqrt{x^4 + 2x + 1}$

(m) $\frac{1}{x\sqrt{x}}$ (n) $(x - 1)^3(3 - 2x)^2$ (o) $\frac{x^4 - 5x + 3}{3x^3}$

(p) $(x^2 - 2x - 1)^3$ (q) $\frac{1}{\sqrt{2x^2 - 4x + 5}}$ (r) $(3x^2 - 4)^{\frac{3}{2}}$

2 Integrate wrt x: (a) $x^{-\frac{3}{4}}$ (b) $\sqrt{4x + 3}$ (c) $\frac{1}{\sqrt{1 - 4x}}$ (d) $\frac{1}{(3x + 4)^3}$ (e) $\frac{2}{\sqrt{4x - 3}}$

3 Evaluate: (a) $\int_1^2 \frac{dx}{\sqrt{3x - 2}}$ (b) $\int_1^3 (2x - 1)^2\, dx$ (c) $\int_{-1}^0 \frac{dx}{(3 - 4x)^2}$

4 Find the equation of the tangent to the curve $y = \frac{x + 3}{x - 1}$ where $x = 5$.

5 If $x^2 - 2y^2 = 14$, find the values of $\frac{dy}{dx}$ when $x = 4$.

6 Given that $x^2 + 3xy - y^2 = 12$, find the equations of the tangents at the points where $x = 2$.

7 Given that $y = \frac{x}{1 - 2x}$, find $\frac{dy}{dx}$.

Hence or otherwise evaluate $\int_{-2}^{-1} \left(\frac{2}{1 - 2x}\right)^2 dx$.

8 Differentiate $\frac{x^2}{x + 2}$ wrt x and hence evaluate $\int_1^3 \frac{x(x + 4)}{(x + 2)^2}\, dx$.

9 If $y = (x + 2)\sqrt{1 - x}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence find the position and nature of the stationary point on the curve.

10 Find the equation of the normal to the curve $y = \frac{\sqrt{x}}{x - 1}$ at the point $\left(4, \frac{2}{3}\right)$.

11 Given that $y = \sqrt{3x + 2}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.

12 If $y = \frac{x}{2x + 1}$, find the value of $\frac{\frac{d^2y}{dx^2}}{[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}}}$ when $x = 0$.

Differentiation of sin *x*: An Important Limit

In order to differentiate $\sin x$, we must first find the limit of $\frac{\sin x}{x}$ as $x \to 0$, where x is in radians, as the result will be needed.

Using a calculator, the following values of $\sin x$ and x were obtained:

x (radians)	$\sin x$
0.2	0.198669
0.1	0.09983
0.05	0.049979
0.01	0.0099998
0.001	0.0009999

This shows that when x is small, $\sin x \approx x$. It would suggest that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Here is a simple proof of this.

In Fig. 17.2, OAB is a sector of a circle centre O, radius r and angle x radians. AC is perpendicular to OA. Then AC = $r \tan x$.

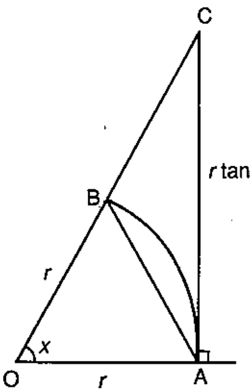


Fig. 17.2

Area of $\triangle$AOB < area of sector AOB < area of $\triangle$AOC,

i.e. $\frac{1}{2}r^2 \sin x \quad < \quad \frac{1}{2}r^2 x \quad < \quad \frac{1}{2}r^2 \tan x$

Hence $\sin x < x < \tan x$.

Dividing by $\sin x$, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Now as $x \to 0$, $\cos x \to 1$ and $\frac{1}{\cos x} \to 1$.

The left hand term is fixed at 1 and the right hand term $\to 1$. Hence the middle term must $\to 1$. Therefore $\lim_{x \to 0} \frac{x}{\sin x} = 1$.

In a more convenient form,

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Note: For this result to be valid, x must be in radians.

We can now find the derivative of $\sin x$ (x in radians).

Let $y = \sin x$. If x takes an increment δx, the corresponding increment in y will be δy.

Then $y + \delta y = \sin(x + \delta x)$ so $\delta y = \sin(x + \delta x) - \sin x$.

To simplify this, recall that

$$\begin{aligned}
&\sin(A + B) - \sin(A - B) \\
&\quad = \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B \\
&\quad = 2 \cos A \sin B \qquad \text{(i)}
\end{aligned}$$

Now if we take $A + B = x + \delta x$ and $A - B = x$, $A = x + \frac{\delta x}{2}$, $B = \frac{\delta x}{2}$.

Hence, using the result (i), $\delta y = 2 \cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)$.

Then $\frac{\delta y}{\delta x} = \frac{2 \cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x} = \cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$

If $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\cos\left(x + \frac{\delta x}{2}\right) \to \cos x$ and the limit of $\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$ is 1 $\left(\text{as } \frac{\delta x}{2} \text{ also} \to 0\right)$. Hence $\frac{dy}{dx} = \cos x$.

$$\boxed{\frac{d}{dx} \sin x = \cos x} \qquad x \text{ in radians.}$$

Note: $\frac{d}{dx} \sin x$ is another way of writing $\frac{d \sin x}{dx}$.

The gradient at any point on the sine curve is the value of $\cos x$ at that point (Fig. 17.3).

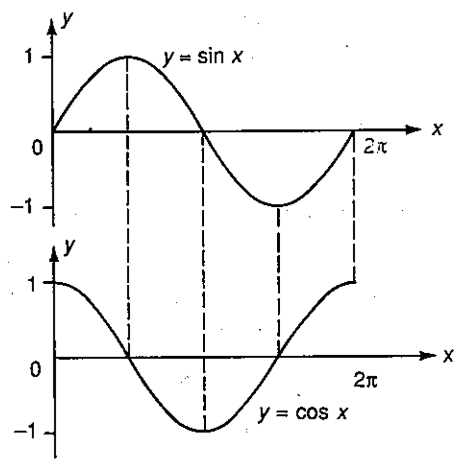


Fig. 17.3

Example 15

Differentiate (a) $\sin 3x$, *(b)* $\sin(ax + b)$, *(c)* $\sin^2 x$, *(d)* $\sin^3(3x - 2)$.

(a) $y = \sin 3x$

We treat this as a composite function, i.e. $y = \sin u$ where $u = 3x$.

$\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 3$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times 3 = 3 \cos 3x$.

Note that the function **sin** is differentiated first to give cos, then the angle $\mathbf{3x}$ is differentiated to give 3.

(b) $y = \sin(ax + b)$

Taking $y = \sin u$ where $u = ax + b$,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times a = a\cos(ax + b)$

Note this result for future use:

$$\frac{d}{dx} \sin(ax + b) = a \cos(ax + b)$$

First differentiate the function, then the angle.

(c) $y = \sin^2 x$

Treat this as a power of the function $\sin x$.

Take $y = u^2$ where $u = \sin u$.

First differentiate as a power, i.e. $\frac{dy}{du}$, then differentiate the function sin, i.e. $\frac{du}{dx}$.

$$\frac{dy}{dx} = \underbrace{2\sin x}_{\substack{\text{differentiate}\\ \sin^2 x \text{ to get}\\ 2\sin x}} \times \underbrace{\cos x}_{\substack{\text{differentiate}\\ \sin x \text{ to get}\\ \cos x}} = 2\sin x \cos x$$

(The result could also be written as $\sin 2x$).

(d) $y = \sin^3 (3x - 2)$

First differentiate as a **power,** then differentiate **sin**, then the angle.

$$\frac{dy}{dx} = \underbrace{3\sin^2(3x-2)}_{\substack{\text{differentiate } \sin^3\\ \text{to get } 3\sin^2}} \times \underbrace{\cos(3x-2)}_{\substack{\text{differentiate sin}\\ \text{to get cos}}} \times \underbrace{3}_{\substack{\text{differentiate}\\ 3x-2 \text{ to get } 3}}$$

The sequence is

power first

$\sin^3(3x - 2)$

function second — variable last

Hence $\frac{dy}{dx} = 9 \sin^2(3x - 2) \cos(3x - 2)$

Example 16

Differentiate sin $x°$ wrt x.

We must first convert the angle to radians.

$x° = \frac{\pi x}{180}$ radians

If $y = \sin \frac{\pi x}{180}$, then

$\frac{dy}{dx} = \cos \frac{\pi x}{180} \times \frac{\pi}{180}$ or $\frac{\pi}{180} \cos x°$.

Note that the result is NOT $\cos x°$. All formulae in calculus for trigonometrical functions are only true for radian measure. Angles in degrees must be converted to radians.

Example 17

Differentiate (a) x sin x, (b) $\sqrt{1 - \sin x}$ wrt x.

(a) This is a product of x and $\sin x$.

If $y = x \sin x$, then $\frac{dy}{dx} = x \cos x + \sin x$.

(b) If $y = (1 - \sin x)^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{1}{2}(1 - \sin x)^{-\frac{1}{2}} \times (-\cos x)$

$$= \frac{-\cos x}{2\sqrt{1 - \sin x}}$$

Example 18

Find the values of x for $0 < x < \pi$ which satisfy the equation $\frac{d}{dx}(x - \sin 2x) = \sin^2 x$.

$\frac{d}{dx}(x - \sin 2x) = 1 - 2 \cos 2x = 1 - 2(1 - 2 \sin^2 x) = 4 \sin^2 x - 1$

Hence $4 \sin^2 x - 1 = \sin^2 x$ or $\sin x = \pm \frac{1}{\sqrt{3}}$.

Solving this equation, $x = 0.62$ or 2.52 radians ($x < \pi = 3.14$).

Differentiation of cos *x*

Using the formula for sin(A – B),

$$\sin\left(\tfrac{\pi}{2} - x\right) = \sin\tfrac{\pi}{2}\cos x - \cos\tfrac{\pi}{2}\sin x = \cos x.$$

Similarly, verify that $\cos\left(\tfrac{\pi}{2} - x\right) = \sin x$.

$$\text{So } \frac{d(\cos x)}{dx} = \frac{d\sin\left(\frac{\pi}{2} - x\right)}{dx}$$

$$= \cos\left(\tfrac{\pi}{2} - x\right) \times (-1) = -\cos\left(\tfrac{\pi}{2} - x\right)$$

$$= -\sin x$$

$$\boxed{\frac{d}{dx}\cos x = -\sin x} \qquad x \text{ in radians}$$

It follows that $\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$

Example 19

Differentiate (a) cos 5x, (b) $\cos^2\left(\frac{x}{2} + 3\right)$, *(c)* $\frac{\sin 2x}{\cos 3x}$.

(a) $y = \cos 5x$
Using the same procedure as before, $\frac{dy}{dx} = -\sin 5x \times 5$
$= -5\sin 5x$.

(b) $y = \cos^2\left(\frac{x}{2} + 3\right)$

$$\frac{dy}{dx} = 2\cos\left(\tfrac{x}{2} + 3\right) \times \left(-\sin\left(\tfrac{x}{2} + 3\right)\right) \times \tfrac{1}{2}$$

$$= -\sin\left(\tfrac{x}{2} + 3\right)\cos\left(\tfrac{x}{2} + 3\right)$$

(c) $y = \frac{\sin 2x}{\cos 3x}$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{(\cos 3x)(2\cos 2x) - (\sin 2x)(-3\sin 3x)}{\cos^2 3x}$$

$$= \frac{2\cos 3x\cos 2x + 3\sin 3x\sin 2x}{\cos^2 3x}$$

Example 20

Find the values of θ ($0 < \theta < 2\pi$) for which $y = \frac{\sin\theta}{3 - \cos\theta}$ is stationary.

Using the quotient rule,

$$\frac{dy}{d\theta} = \frac{(3 - \cos\theta)(\cos\theta) - \sin\theta(\sin\theta)}{(3 - \cos\theta)^2} = \frac{3\cos\theta - (\cos^2\theta + \sin^2\theta)}{(3 - \cos\theta)^2}$$

$$= \frac{3\cos\theta - 1}{(3 - \cos\theta)^2}$$

Then $\frac{dy}{d\theta} = 0$ when $3\cos\theta - 1 = 0$ i.e. when $\cos\theta = \frac{1}{3}$
and therefore $\theta = 1.23$ or $2\pi - 1.23 = 5.05$ radians.

Example 21

Differentiate $\sec x$.

Let $y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$.

$$\frac{dy}{dx} = -1(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

Example 22

Show that $\frac{d}{dx}(\sin 2x \cos x) = 2\cos x(1 - 3\sin^2 x)$.

If $y = \sin 2x \cos x$, using the product rule,

$$\frac{dy}{dx} = (\sin 2x)(-\sin x) + (\cos x)(2\cos 2x)$$

$$= -2\sin x \cos x \sin x + (2\cos x)(1 - 2\sin^2 x)$$

$$= -2\sin^2 x \cos x + 2\cos x - 4\sin^2 x \cos x$$

$$= 2\cos x - 6\sin^2 x \cos x$$

$$= (2\cos x)(1 - 3\sin^2 x)$$

Differentiation of tan *x*

If $y = \tan x = \frac{\sin x}{\cos x}$, then using the quotient rule,

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad x \text{ in radians}$$

Then $\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$

Example 23

Differentiate wrt x (a) x tan 2x, (b) sin x tan x.

(a) If $y = x \tan 2x$, $\dfrac{dy}{dx} = x \sec^2 2x \times 2 + \tan 2x$
$= 2x \sec^2 2x + \tan 2x.$

(b) If $y = \sin x \tan x$, $\dfrac{dy}{dx} = \sin x \sec^2 x + \tan x \cos x$
$= \sin x \sec^2 x + \sin x$
$= (\sin x)(\sec^2 x + 1)$

Exercise 17.6 *(Answers on page 642.)*

1 Differentiate wrt x:

(a) $\sin 3x$ (b) $\sin \frac{x}{2}$ (c) $\cos \frac{x}{4}$
(d) $\tan 3x$ (e) $\operatorname{cosec} x$ (f) $x \sin x$
(g) $x^2 \sin 2x$ (h) $\cos(2x^2 - 1)$ (i) $\sin\left(\frac{\pi}{3} - x\right)$
(j) $\tan \frac{x}{2}$ (k) $x \sin x + \cos x$ (l) $\dfrac{\cos x}{2 - \sin x}$
(m) $\cos^3 2x$ (n) $x \cos x - \sin 2x$ (o) $\sin 3x \cos 2x$
(p) $\sqrt{4 + \sin^2 2x}$ (q) $\cos^3(1 - 3x^2)$ (r) $\sqrt{\tan 2x}$

2 Differentiate wrt x:

(a) $\cos 3x$ (b) $\sin \frac{x}{3}$ (c) $\cos(2x^2 - 1)$
(d) $\sin^3 2x$ (e) $\tan\left(\frac{x}{3} - 2\right)$ (f) $\sin \frac{x}{2} \cos 2x$
(g) $\dfrac{1 - \sin x}{1 + \sin x}$ (h) $x^2 \tan \frac{x}{2}$ (i) $\cos x^2$
(j) $x(\cos 2x - \sin x)$

3 If $y = \sin 2x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, and show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

4 If $y = x \sin 2x$, find the value of $\dfrac{dy}{dx}$ when $x = \frac{\pi}{2}$.

5 Given $y = A \cos 2x + B \sin 2x$, where A and B are constants, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.
If also $y = 3$ when $x = \frac{\pi}{2}$ and $\dfrac{dy}{dx} = 4$ when $x = 0$, find the value of A and of B.

6 If $y = \cos\theta + 2\sin\theta$, find the values of θ $(0 < \theta < 2\pi)$ for which $\dfrac{dy}{d\theta} = 0$.

7 Find $\dfrac{dy}{dx}$ if $y = (\sin x + \cos 2x)^2$.

8 Solve the equation $\dfrac{d}{dx}(x + \sin 2x) = 2$ for $0 < x < \pi$.

9 Differentiate $\dfrac{\sin x}{1 + \cos x}$ wrt x and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x}$.

10 (a) Show that if $y = 2 \sin x - \cos x$, then $\dfrac{dy}{dx} = 0$ when $\tan x = -2$.
Hence find the values of x $(0 < x < 2\pi)$ where y has stationary values.

(b) Find the value of x $(0 < x < 2\pi)$ for which $y = \dfrac{3 \cos x}{2 - \sin x}$ is stationary. Hence find the maximum and minimum values of y.

11 Find the equations of the tangents to the curve $y = \sin x$ where $x = 0$ and $x = \pi$.

12 Find the equation of the tangent to the curve $y = \cos x$ where $x = \frac{\pi}{2}$.

Integration of Trigonometric Functions

If $y = \sin(ax + b)$, then $\frac{dy}{dx} = a\cos(ax + b)$.

Therefore $$\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$$

If $y = \cos(ax + b)$, then $\frac{dy}{dx} = -a\sin(ax + b)$.

Therefore $$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

If $y = \tan(ax + b)$, then $\frac{dy}{dx} = a\sec^2(ax + b)$.

Therefore $$\int \sec^2(ax + b)\,dx = \frac{1}{a}\tan(ax + b) + c$$

For all these results, x must be in radians.

Example 24

Integrate (a) $\int \sin 3x\,dx$, (b) $\int \cos\frac{x}{2}\,dx$.

(a) $\int \sin 3x\,dx = \frac{-\cos 3x}{3} + c = -\frac{1}{3}\cos 3x + c$

(b) $\int \cos\frac{x}{2}\,dx = \frac{\sin\frac{x}{2}}{\frac{1}{2}} + c = 2\sin\frac{x}{2} + c$

Example 25

Find the area of the shaded region in Fig.17.4 between the part OA of the curve $y = \sin x$ and the line OA, where O is the origin and A is the point $(\frac{\pi}{2}, 1)$.

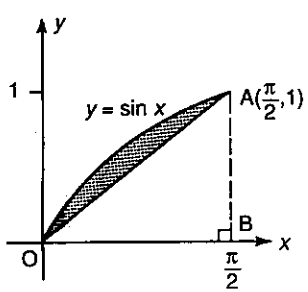


Fig. 17.4

The shaded area is the area under the curve minus the area of $\triangle$OBA where AB is perpendicular to x-axis.

Area $= \int_0^{\frac{\pi}{2}} \sin x \, dx - \frac{1}{2} \times \frac{\pi}{2} \times 1$

$= \left[-\cos x\right]_0^{\frac{\pi}{2}} - \frac{\pi}{4}$

$= \left(-\cos \frac{\pi}{2}\right) - (-\cos 0) - \frac{\pi}{4} = 0 - (-1) - \frac{\pi}{4} = 1 - \frac{\pi}{4}$ units^2

Example 26

Find $\int \sin^2 x \, dx$.

We cannot find $\int \sin^2 x \, dx$ directly as it is not in the form $\sin(ax + b)$.

We use the formula $\cos 2x = 1 - 2\sin^2 x$ to convert it to a suitable form.

Then $\int \sin^2 x \, dx = \int \frac{(1 - \cos 2x)}{2} \, dx$

$= \int \left(\frac{1}{2} - \frac{\cos 2x}{2}\right) dx = \frac{x}{2} - \frac{\sin 2x}{2 \times 2} + c = \frac{x}{2} - \frac{\sin 2x}{4} + c.$

The same method is used to find $\int \cos^2 x \, dx$.

Example 27

Sketch the curve $y = 1 + \cos x$ for $0 \le x \le \pi$. This curve is rotated about the x-axis through 2π radians. Find the volume created in terms of π.

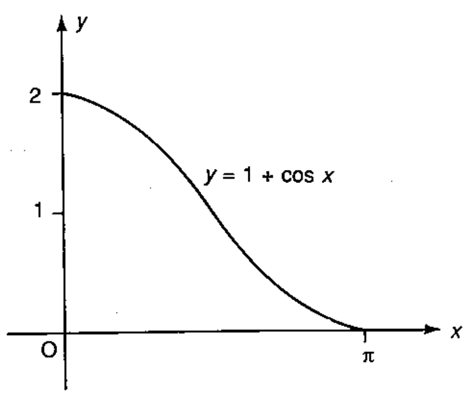


Fig. 17.5

Fig. 17.5 shows the curve, which is $y = \cos x$ moved up 1 unit.

The volume $= \int_0^{\pi} \pi(1 + \cos x)^2 \, dx$

$$= \pi \int_0^{\pi} (1 + 2\cos x + \cos^2 x) \, dx$$

$$= \pi \int_0^{\pi} \left(1 + 2\cos x + \frac{1 + \cos 2x}{2}\right) dx$$

$$= \pi \int_0^{\pi} \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x\right) dx$$

$$= \pi \left[\frac{3x}{2} + 2\sin x + \frac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= \pi \left(\frac{3\pi}{2} + 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - \pi(0) = \frac{3\pi^2}{2} \text{ units}^3$$

Exercise 17.7 *(Answers on page 642.)*

1 Integrate wrt x:

(a) $\sin 2x$ (b) $\cos 4x$ (c) $\sin \frac{x}{2}$

(d) $3\sin 3x$ (e) $\sec^2 3x$ (f) $\cos 2x - \sin x$

(g) $\sin x + \cos x$ (h) $\cos^2 \frac{x}{2}$ (i) $\cos 5x$

(j) $\sin\left(\frac{\pi}{4} - x\right)$ (k) $\sec^2 \frac{x}{2}$ (l) $\cos 2x - \sin x$

(m) $(\cos x - \sin x)^2$ (n) $2\sin x + \frac{1}{2}\sin 2x$

2 Evaluate

(a) $\int_0^{\frac{\pi}{2}} \cos x \, dx$ (b) $\int_0^{\frac{\pi}{2}} \sin x \, dx$

(c) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (d) $\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

(e) $\int_0^{\frac{\pi}{2}} (\sin 2x - \cos x) \, dx$ (f) $\int_0^{\frac{\pi}{2}} (\cos x + \sin x)^2 \, dx$

(g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \frac{3x}{2} \, dx$ (h) $\int_0^{\frac{\pi}{3}} \sin 3x \, dx$

(i) $\int_0^{\pi} \cos \frac{x}{2} \, dx$ (j) $\int_0^{\pi} \cos 2x \, dx$

(k) $\int_0^{\frac{5\pi}{12}} \cos^2 x \, dx$

3 If $\frac{dy}{d\theta} = \frac{1}{\theta^2} + \frac{1}{2}\cos 2\theta$, find y if $y = 1$ when $\theta = \frac{\pi}{2}$.

4 Find (a) the area of the region enclosed by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ and (b) the volume created if this region is rotated about the x-axis.

5 Differentiate $\frac{1}{1 + \cos x}$ wrt x. Hence find the area of the region under the curve $y = \frac{\sin x}{(1 + \cos x)^2}$ between $x = 0$ and $x = \frac{\pi}{2}$.

6 The region bounded by the x-axis and the part of the curve $y = 2 \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis through 360°. Find the volume of the solid generated.

7 Sketch the curves $y = \cos x$ and $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

Find (a) the value of x where the curves intersect, (b) the area of the region bounded by the two curves and the x-axis. (c) If this region is rotated about the x-axis through 360°, find the volume of the solid created.

8 Find the area of the region enclosed by the x-axis, the y-axis, the curve $y = \cos x$ and the line $x = \frac{\pi}{6}$. If this region is rotated about the x-axis through 360°, find the volume created.

9 Using an identity for $\cos 4x$, find $\int \cos^2 2x \, dx$.

10 Sketch the curves $y = \cos x$ and $y = \sin 2x$ for $0 \leq x \leq \frac{\pi}{2}$. Find
(a) the value of x where the curves intersect $\left(\text{apart from } x = \frac{\pi}{2}\right)$ and
(b) the area of the region enclosed by the two curves and the x-axis.

11 (a) Show that $\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \sec^2 x - 1$.
(b) Hence find the value of $\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} \, dx$.

12 By writing $3x$ as $2x + x$, show that $\cos 3x \equiv 4 \cos^3 x - 3 \cos x$.
Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$.

SUMMARY

- $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c \; (n \neq -1)$
- *Product rule:* If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
- *Quotient rule:* If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- For x in radians, a and b constants:

  $\frac{d}{dx} \sin(ax + b) = a \cos(ax + b)$, $\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + c$

  $\frac{d}{dx} \cos(ax + b) = -a \sin(ax + b)$, $\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + c$

  $\frac{d}{dx} \tan(ax + b) = a \sec^2(ax + b)$, $\int \sec^2(ax + b) \, dx = \frac{1}{a} \tan(ax + b) + c$
- To integrate $\sin^2 x$ or $\cos^2 x$, use the identity for $\cos 2x$.

REVISION EXERCISE 17 *(Answers on page 642.)*

A

1 If $y = \frac{x}{1+x^2}$, find $\frac{dy}{dx}$.
Hence find the value of $\int_0^2 \frac{1-x^2}{(1+x^2)^2}\,dx$.

2 If $y = x \sin x$, find $\frac{d^2y}{dx^2}$.

3 Sketch the curve $y = 1 + 2\sin\frac{x}{2}$ for $0 \le x \le 2\pi$. If this part of the curve is rotated about the x-axis through 2π radians, find the volume produced.

4 Evaluate $\int_0^{\alpha} \cos x\,dx$ when $\tan\alpha = \frac{3}{4}\left(0 < \alpha < \frac{\pi}{2}\right)$.

5 (a) Differentiate with respect to x (i) $(4x+1)^3$, (ii) $x\tan 3x$.

(b) Given that $xy^3 + y = x^3$, obtain $\frac{dy}{dx}$ in terms of x and y.

(c) Given that $y = \frac{\sqrt{x}}{x-2}$, prove that $\frac{dy}{dx} = -\frac{x+2}{2\sqrt{x}\,(x-2)^2}$.

Hence obtain the equation of the normal to the curve $y = \frac{\sqrt{x}}{x-2}$ at the point on the curve where $x = 4$. (C)

6 Calculate the area of the region enclosed by the curve $y = \sin x$, the tangent at the point (0,0) to this curve and the line $x = \frac{\pi}{2}$.

7 Sketch the curve $y = 1 + \sin x$ for $0 \le x \le \frac{3}{2}\pi$.
Show that $(1+\sin x)^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$.
Hence show that the volume of the solid of revolution formed when the region $\left(0 \le x \le \frac{3}{2}\pi\right)$ bounded by the curve $y = 1 + \sin x$, the y-axis and the x-axis is rotated through one revolution about the x-axis is $\frac{1}{4}\pi(9\pi + 8)$. (C)

8 Differentiate wrt x (a) $\frac{x}{\cos x}$, (b) $\frac{1+\sin x}{1-\sin x}$, (c) $\frac{\sin x}{\sin x + \cos x}$, (d) $2x\sqrt{(2x-1)^3}$.

9 Find the equations of the two tangents to the curve $x^2 + y^2 = 10$ which are parallel to the line $y + 3x = 1$.

10 (a) Express $\sin^2 x$ in terms of $\cos 2x$ and hence evaluate $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 4\sin^2 x\,dx$.

(b) Find the area of the region bounded by the curve $y = \frac{1}{\sqrt{3-2x}}$, the x-axis and the lines $x = -3$ and $x = 1$.

(c) The part of the curve $y = \frac{1}{x-1}$ between $x = 2$ and $x = 4$ is rotated through 360° about the x-axis. Find the volume of the solid created.

11 If $y = \frac{1}{\sqrt{1+x^2}}$, find the approximate change in y if x is increased by 2% when it has the value 0.75.

12 Calculate the gradient of the curve $y = \frac{2x-3}{x+2}$ at the point where it meets the x-axis.

13 Sketch the curve $y = |\sin x|$ for $0 \le x \le 2\pi$.
Evaluate $\int_0^{2\pi} |\sin x|\,dx$.

14 If $r = a\sqrt{1 - \cos\theta}$, show that $\frac{dr}{d\theta} = \frac{a}{\sqrt{2}}\cos\frac{\theta}{2}$.

15 (a) Given that $y = \sin^2 x \cos 2x$, find the values of x $(0 \le x \le \pi)$ for which $\frac{dy}{dx} = 0$.

(b) Find the values of x $(0 \le x \le 2\pi)$ for which $y = \frac{1 + \sin x}{\sin x + \cos x}$ is stationary. State the maximum and minimum values of y.

16 If $y = \sin 2\theta$, find the approximate change in y when θ is increased from $\frac{\pi}{6}$ to $\frac{\pi}{6} + 0.01$.

17 Show that the function $\frac{x}{x^2 - 1}$ is always decreasing for $x > 1$.

18 Find $\frac{dy}{dx}$ in terms of x and y if $2xy^2 + y + 2x = 8$. Hence find the gradient of the curve at the points where $x = 1$.

19 If $y = a \sin 2x$, where a is a constant, satisfies the equation $\frac{d^2y}{dx^2} + 8y = 4 \sin 2x$, find the value of a.

20 Given that $r^2(1 + \cos\theta) = k$, where k is a constant, show that $\frac{dr}{d\theta} = \frac{r}{2}\tan\frac{\theta}{2}$.

B

21 Solve the equation $\int_x^{2x} \sin\frac{t}{2}\, dt = 0$ for $0 \le x \le 2\pi$.

22 Find $\frac{dy}{dx}$ for each of the functions $xy = a$ and $y = \sqrt{k^2 + x^2}$ where a and k are constants. Hence show that the tangents at the point of intersection of the curves are perpendicular.

23 A particle moves in a straight line and its distance s from a fixed point O of the line at time t is given by $s = 4 \sin 2t$.

(a) Show that its velocity v and its acceleration a at time t are given by $v = 2\sqrt{16 - s^2}$ and $a = -4s$.

(b) Find the greatest distance from O reached by the particle.

24 At a certain port the height h metres of the tide above the low water level is given by $h = 2(1 + \cos\theta)$ where $\theta = \frac{\pi t}{450}$ and t is the time in minutes after high tide.

(a) What length of time is there between high and low tide?

(b) At what rate is the tide falling, in metres per minute, 75 minutes after high tide?

(c) A bridge is 10 metres above the low water level. A boat can only sail under this bridge when the distance between the water and the bridge is not less than 7 metres. How long after high tide will it be before the boat can sail under the bridge?

25 (a) Differentiate $\cot\theta$ wrt θ.

(b) A cone has a base radius r and a semi-vertical angle θ. Show that its volume $V = \frac{1}{3}\pi r^3 \cot\theta$.

(c) r is fixed but θ is measured as 45° with an error of 4%. Find the percentage error in the calculated value of V.

Calculus (5) : e^x and $\ln x$

18

The function $\frac{1}{x}$ cannot be integrated by the usual rule. So if this function is to have an integral it cannot be an algebraic one but some other type of function. The story of its discovery is beyond our work but we can start with the origin of an important number in the story.

This comes from asking what happens to the value of $\left(1 + \frac{1}{t}\right)^t$ as $t \to \infty$ i.e. what is the value (if any) of $\lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t$?

We shall not be able to *prove* what this limit is but the following set of values made by a calculator will suggest an answer.

t	$(1 + \frac{1}{t})^t$
100	$(1.01)^{100} \approx 2.7048$
1000	$(1.001)^{1000} \approx 2.7169$
10 000	$(1.0001)^{10\,000} \approx 2.71815$
100 000	$(1.00001)^{100\,000} \approx 2.71827$
1 000 000	$(1.000001)^{1\,000\,000} \approx 2.71828$
10 000 000	$(1.0000001)^{10\,000\,000} \approx 2.71828$

As x increases, it appears that $\left(1 + \frac{1}{t}\right)^t$ tends to a value which is approximately 2.71828. This is true and we denote this limit by the letter **e**. Like π, e is an irrational number. Its importance is that it is taken as the base of **natural logarithms**, i.e. $\log_e x$ (written as $\ln x$.)

If $y = \log_e x = \ln x$ then $e^y = x$
If $y = e^x$ then $\ln y = x$

Similar to other logarithms, $\ln 1 = 0$, $\ln e = 1$ and if $0 < x < 1$, $\ln x$ is negative. If $x \le 0$, $\ln x$ is undefined.

So for example, if $y = e^{2x+3}$ then $\ln y = 2x + 3$; if $y = 2e^{3x}$ then $\ln y = \ln 2 + \ln e^{3x} = \ln 2 + 3x$.

We shall now see why such a strange number is chosen as a base for logarithms.

$\frac{d}{dx} \ln x$

Take $y = \ln x$ and let x have an increment δx. Consequently, y has an increment δy.

Then $y + \delta y = \ln(x + \delta x)$ and $\delta y = \ln(x + \delta x) - \ln x = \ln\left(\frac{x + \delta x}{x}\right)$.

Hence $\frac{\delta y}{\delta x} = \frac{1}{\delta x} \ln\left(\frac{x + \delta x}{x}\right) = \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{1}{\delta x}}$.

To make use of the above limit, write $\frac{\delta x}{x} = \frac{1}{t}$ so $\frac{1}{\delta x} = \frac{t}{x}$.

Then $\frac{\delta y}{\delta x} = \ln\left(1 + \frac{1}{t}\right)^{\frac{t}{x}} = \frac{1}{x} \ln\left(1 + \frac{1}{t}\right)^t$.

Now let $\delta x \to 0$. Consequently, $\delta y \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $t \to \infty$ and $\left(1 + \frac{1}{t}\right)^t \to e$.

Using these, we have $\frac{dy}{dx} = \frac{1}{x} \ln e = \frac{1}{x}$ as $\ln e = \log_e e = 1$.

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

This is a very important and simple result. It is the basis of the work in this chapter and shows why e is taken as the base of natural logarithms.

Now using the rule for a composite function, we can differentiate $\ln f(x)$.

$\frac{d}{dx} \ln f(x)$

Suppose $y = \ln f(x)$.

Take $u = f(x)$ and so $y = \ln u$.

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = \frac{d}{dx} f(x) = f'(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times f'(x) = \frac{f'(x)}{f(x)}$

$$\boxed{\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}}$$

Example 1

Differentiate wrt x (a) $\ln(ax + b)$, (b) $\ln(x^2 - 3x + 1)$, (c) $\ln \sin 3x$, (d) $x^2 \ln x$, (e) $\frac{\ln x}{x + 1}$

(a) Here $f(x) = ax + b$.

$\frac{d}{dx} \ln(ax + b) = \frac{a}{ax + b}$ as $f'(x) = \frac{d}{dx}(ax + b) = a$.

(b) $\frac{d}{dx} \ln(x^2 - 3x + 1) = \frac{2x - 3}{x^2 - 3x + 1}$ (as $f'(x) = 2x - 3$)

(c) $\frac{d}{dx} \ln \sin 3x = \frac{3 \cos 3x}{\sin 3x} = 3 \cot 3x$

(d) $y = x^2 \ln x$ is a product of x^2 and $\ln x$.

So $\frac{dy}{dx} = 2x \ln x + x^2 \frac{d}{dx} \ln x = 2x \ln x + x^2 \times \frac{1}{x} = 2x \ln x + x$

(e) $y = \frac{\ln x}{x+1}$ is a quotient of two functions.

Hence $\frac{dy}{dx} = \frac{(x+1)\frac{1}{x} - \ln x}{(x+1)^2} = \frac{x+1-x\ln x}{x(x+1)^2}$

Example 2

Find $\frac{dy}{dx}$ if $y = \ln(x+1)(3x-2)^2$.

We could differentiate directly but it is easier to simplify the logarithm first. $\ln(x+1)(3x-2)^2 = \ln(x+1) + \ln(3x-2)^2 = \ln(x+1) + 2\ln(3x-2)$. Now differentiate:

$$\frac{dy}{dx} = \frac{1}{x+1} + 2\left(\frac{3}{3x-2}\right) = \frac{3x-2+6(x+1)}{(x+1)(3x-2)} = \frac{9x+4}{(x+1)(3x-2)}$$

Example 3

Fig. 18.1 shows part of a straight line obtained by plotting ln y against x with two of the points on the line marked. Express y in terms of x and find (a) the value of y when x = –2, (b) the value of x when (i) y = 1, (ii) y = 2.

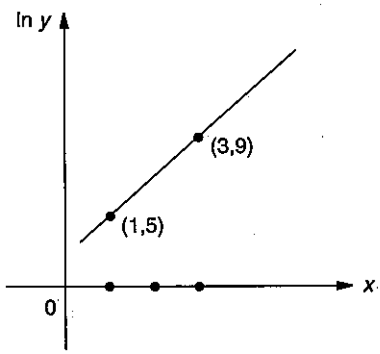


Fig. 18.1

The equation of the line will be of the form $\ln y = mx + c$.

Then $5 = m + c$ ($x = 1$, $\ln y = 5$) and $9 = 3m + c$ ($x = 3$, $\ln y = 9$)

Hence $m = 2$ and $c = 3$, giving $\ln y = 2x + 3$ so $y = e^{2x+3}$.

(a) $y = e^{-4+3} = e^{-1}$ (≈ 0.37 by calculator)

(b) (i) $1 = e^{2x+3}$ so $2x + 3 = 0$ and $x = -1.5$

(ii) $2 = e^{2x+3}$ so $\ln 2 = 2x + 3$ and $x = \frac{\ln 2 - 3}{2}$ (≈ -1.15).

Exercise 18.1 *(Answers on page 643.)*

1 Differentiate the following wrt x and simplify where possible:

(a) $\ln 5x$ (b) $\ln x^2$ (c) $\ln(3x - 1)$
(d) $\ln(\sin x)$ (e) $\ln(x + \tan x)$ (f) $\ln(\cos 2x)$
(g) $\ln(\cos^2 x)$ (h) $\ln\left(\sin \frac{x}{2}\right)$ (i) $\ln(2x^2 - 4x - 1)$
(j) $\ln \sqrt{2x - 5}$ (k) $\ln\left(\frac{1}{\sqrt{x}}\right)$ (l) $x \ln x$
(m) $\frac{\ln x}{x^2}$ (n) $\ln(x \cos x)$ (o) $\cos(\ln x)$
(p) $\ln x \ln 3x$ (q) $(x^2 + 1)\ln(x - 1)$ (r) $(\ln x)^2$
(s) $\ln(\cos 3x)$ (t) $(x - 1) \ln 2x$ (u) $\ln (x + \sin x)$
(v) $\ln(x + 3)(2x - 1)$ (w) $\ln\left(\frac{x - 4}{x + 3}\right)$

2 If $y = \ln(x + 1)(x - 2)$, show that $\frac{dy}{dx} = \frac{2x - 1}{x^2 - x - 2}$.

3 If $y = \ln(3x + 1)(2x - 1)$, find and simplify $\frac{dy}{dx}$.

4 Given that $y = \ln\left(\frac{x - 2}{x + 1}\right)$, find $\frac{dy}{dx}$ in its simplest form.

5 Fig. 18.2 shows parts of two straight lines obtained by plotting $\ln y$ against x for two different functions. Each has two points marked. Find for each function, (a) y in terms of x, (b) the value of x when $y = 1$.

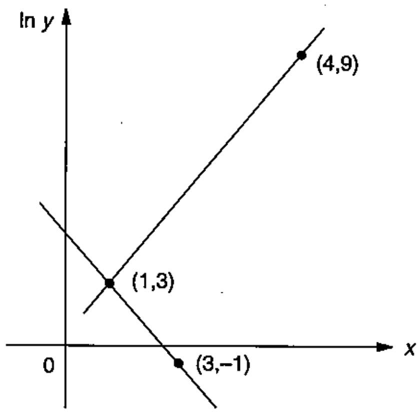


Fig. 18.2

6 State how the functions (a) $y = e^{3x-2}$ and (b) $y = 3e^{-2x}$ can each be represented by a straight line graph and give the equation of each line.

7 On graph paper, draw the graph of $y = e^x$ for $0 \le x \le 2$. By adding a suitable straight line, find an approximate solution to the equation $e^x + x = 5$.

8 Given that $e^{5x} = e^{3(y+1)}$ and that $\ln(3x + 4y) = 2 \ln 5$, form two simultaneous equations and hence find the value of x and of y.

9 Differentiate $\ln(x - \sin x)$. Hence find the gradient on the curve $y = \ln(x - \sin x)$ where $x = \pi$.

10 Differentiate $\frac{\sin x}{1 - \cos x}$. Hence show that $\frac{d}{dx} \ln\frac{\sin x}{1 - \cos x} = -\text{cosec } x$.

11 Differentiate $\frac{\cos x}{1 + \sin x}$. Hence show that $\frac{d}{dx} \ln\frac{\cos x}{1 + \sin x} = -\sec x$.

$\frac{d}{dx} e^x$

If $y = e^x$, then $\ln y = x$. Differentiating both sides wrt x, $\frac{1}{y}\frac{dy}{dx} = 1$ so $\frac{dy}{dx} = y = e^x$.

Hence
$$\frac{d}{dx} e^x = e^x$$

This result makes e^x a unique function. It is the only function whose derivative is itself. The gradient at a point on the curve $y = e^x$ equals the value of y at that point. (This was suggested in Question 3 of Exercise 15.3).

$\frac{d}{dx} e^{f(x)}$

We can also differentiate composite functions of the type $e^{f(x)}$.

If $y = e^{f(x)}$ and $u = f(x)$, then $y = e^u$.

$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$.

Hence $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u f'(x) = f'(x)e^{f(x)}$.

$$\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$$

Example 4

Differentiate (a) e^{3x-2}, (b) $e^{\sin 2x}$, (c) xe^{-2x}, wrt x.

(a) $\frac{de^{3x-2}}{dx} = 3e^{3x-2}$ as $\frac{d(3x-2)}{dx} = 3$

(b) $\frac{d}{dx} e^{\sin 2x} = (2\cos 2x)\, e^{\sin 2x}$

(c) $y = xe^{-2x}$ is a product.

$\frac{dy}{dx} = e^{-2x} + x(-2e^{-2x}) = e^{-2x}(1 - 2x)$

Example 5

Find the coordinates of the point of intersection of the curves $y = e^{2x-1}$ and $y = e^{2-x}$ and the gradient of each curve at that point.

At the point of intersection, $e^{2x-1} = e^{2-x}$ so $2x - 1 = 2 - x$ and $x = 1$. The coordinates of the point are (1,e).

For $y = e^{2x-1}$, $\frac{dy}{dx} = 2e^{2x-1} = 2e$ when $x = 1$.

For $y = e^{2-x}$, $\frac{dy}{dx} = -e^{2-x} = -e$ when $x = 1$.

Exercise 18.2 *(Answers on page 643.)*

1 Differentiate wrt x:

(a) e^{4x}
(b) e^{5x-1}
(c) e^{5-3x}
(d) e^{x^2}
(e) $e^{\cos x}$
(f) xe^x
(g) $(2x-4)e^{-\frac{x}{2}}$
(h) e^{ax+b}
(i) e^{x^2+2x-1}
(j) $e^x \sin x$
(k) $\frac{e^x+1}{e^x}$
(l) $\frac{e^x}{x+1}$
(m) x^2e^{-x}
(n) $e^x - e^{-x}$
(o) $e^{-x}(\cos x - \sin x)$
(p) $(3x+2)e^{-2x}$
(q) $e^{2x} \ln x$
(r) $\frac{e^x}{e^x-1}$
(s) $\frac{xe^x}{x-1}$
(t) $e^{2x} \cos 2x$
(u) $\frac{e^x}{x}$
(v) x^3e^{2x}
(w) $(e^x - e^{-x})^2$

2 Find the coordinates of the point where the curves $y = e^{5x-2}$ and $y = e^{4-x}$ meet and the gradient of each curve at that point.

3 Find the range of values of x for which $(x-3)e^{-2x}$ is increasing.

4 If $y = xe^{3x}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence find the value of x for which y has a stationary point and state the nature of that point.

5 Given that $y = x^2e^{2x}$, find the values of x for which y is stationary.

6 If $y = (x^2-3)e^{-x}$, find the values of x where y is stationary and the nature of these points.

7 If $y = e^x \cos x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use these to find the values of x $(0 < x < 2\pi)$ where y has stationary points and state the nature of these points.

8 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y = e^x(\cos x + \sin x)$.
Hence find the values of x $(0 < x < 2\pi)$ where y is stationary and the nature of the stationary points.

9 Given that $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

10 Find the gradient on the curve $y = e^{2x} \cos x$ where $x = 0$.

11 If $y = a^x$, show that $\ln y = x \ln a$ and hence find $\frac{dy}{dx}$.

12 If $\frac{d}{dx}\left(\frac{\sin x}{e^x}\right) = \frac{f(x)}{e^x}$, find f($x$).

Integration of $\dfrac{1}{ax + b}$ and $e^{ax + b}$

We can now find an answer for $\int \frac{1}{x}\, dx$.

We know that $\frac{d}{dx} \ln x = \frac{1}{x}$ so $\int \frac{1}{x}\, dx = \ln x + c$.

However we must be careful. If $x < 0$, $\ln x$ is undefined. We can guard against this by writing

$$\int \frac{1}{x}\, dx = \ln |x| + c$$

This is justified as shown in Fig. 18.3 which shows the two branches of the curve $y = \frac{1}{x}$.

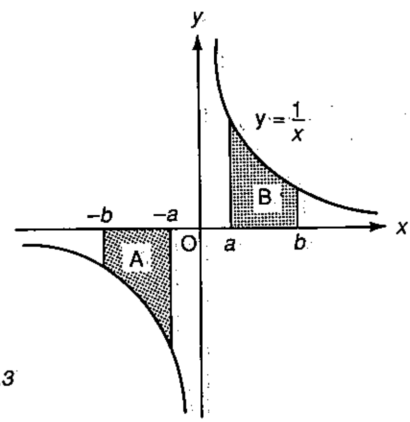


Fig. 18.3

The area $A = \int_{-b}^{-a} \frac{1}{x}\, dx = \left[\ln x\right]_{-b}^{-a}$ which is undefined.

By symmetry however, area A = area B = $\left[\ln x\right]_{a}^{b} = \left[\ln |x|\right]_{-a}^{-b}$.

Further, since $\frac{d}{dx} \ln(ax + b) = \frac{a}{ax + b}$,

then $$\int \frac{dx}{ax + b} = \frac{1}{a} \ln |ax + b| + c$$

As $\frac{d}{dx}\, e^{ax+b} = ae^{ax+b}$,

then $$\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + c$$

Note: These results only apply to the linear function $ax + b$.

Example 6

Find (a) $\int (3x-2)^{-1}\,dx$, *(b)* $\int_{-2}^{-1} \frac{dx}{2-3x}$, *(c)* $\int_{2}^{3} e^{2-x}\,dx$.

(a) $\int \frac{1}{3x-2}\,dx = \frac{1}{3}\ln|3x-2| + c$

(b) $\int_{-2}^{-1} \frac{dx}{2-3x} = \left[-\frac{1}{3}\ln|2-3x|\right]_{-2}^{-1}$

$= \left(-\frac{1}{3}\ln|5|\right) - \left(-\frac{1}{3}\ln|8|\right)$

$= -\frac{1}{3}\ln 5 + \frac{1}{3}\ln 8 = \frac{1}{3}\ln\frac{8}{5} \approx 0.16$

Such results however are usually left in terms of ln.

(c) $\int_{2}^{3} e^{2-x}\,dx = \left[-e^{2-x}\right]_{2}^{3}$

$= (-e^{-1}) - (-e^{0}) = -e^{-1} + 1 = 1 - \frac{1}{e}$.

Example 7

Find the area of the region enclosed by the curve $y = \frac{1}{x}$, *the x-axis and the lines* $x = 1, x = 3$.

Area $= \int_{1}^{3} \frac{1}{x}\,dx = \left[\ln|x|\right]_{1}^{3} = \ln 3 - \ln 1 = \ln 3$ units^2

Example 8

The part of the curve $y = \frac{1}{\sqrt{2-x}}$ *between* $x = -3$ *and* $x = -1$ *is rotated about the x-axis through 360°. Find the volume of the solid created.*

Volume $= \int_{-3}^{-1} \pi y^2\,dx = \pi\int_{-3}^{-1} \frac{1}{2-x}\,dx$

$= \pi\left[-\ln|2-x|\right]_{-3}^{-1} = \pi(-\ln|3|) - \pi(-\ln|5|) = \pi\ln\frac{5}{3}$ units^3

Example 9

The region enclosed by the curves $y = e^x$ *and* $y = e^{2x}$ *and the lines* $x = 1, x = 2$, *is rotated about the x-axis through 360°. Find, in terms of e, the volume of the solid formed.*

Volume $= \pi\int_{1}^{2} [(e^{2x})^2 - (e^x)^2]\,dx$

$$= \pi \int_1^2 (e^{4x} - e^{2x})\,dx = \pi \left[\tfrac{1}{4}e^{4x} - \tfrac{1}{2}e^{2x}\right]_1^2$$

$$= \pi\left(\frac{e^8}{4} - \frac{e^4}{2}\right) - \pi\left(\frac{e^4}{4} - \frac{e^2}{2}\right) = \frac{\pi}{4}(e^8 - 3e^4 + 2e^2)$$

$$= \frac{\pi e^2}{4}(e^6 - 3e^2 + 2) \text{ units}^3$$

Exercise 18.3 *(Answers on page 643.)*

1 Find

(a) $\int e^{3x}\,dx$ (b) $\int e^{-x}\,dx$ (c) $\int e^{2-x}\,dx$

(d) $\int \frac{dx}{2x+3}$ (e) $\int \frac{dx}{3-2x}$ (f) $\int (4-x)^{-1}\,dx$

(g) $\int \frac{x+1}{x}\,dx$ (h) $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$ (i) $\int e^{-3x}\,dx$

(j) $\int e^{1-2x}\,dx$ (k) $\int \frac{dx}{3x+1}$ (l) $\int \frac{x+2}{x^2}\,dx$

2 Evaluate the following, giving the result in terms of e:

(a) $\int_0^1 e^{4x}\,dx$ (b) $\int_{-1}^0 e^{-x}\,dx$ (c) $\int_{-3}^{-1} e^{1-x}\,dx$

(d) $\int_1^3 e^{2x-1}\,dx$ (e) $\int_{-2}^0 \frac{2}{e^{2x}}\,dx$ (f) $\int_1^2 e^{-2x}\,dx$

(g) $\int_{-2}^0 e^{-\frac{x}{2}}\,dx$ (h) $\int_{-1}^1 e^{2-x}\,dx$

3 Express the following in terms of ln:

(a) $\int_1^2 \frac{dx}{x-3}$ (b) $\int_{-2}^{-1} \frac{dx}{3-x}$ (c) $\int_0^2 \frac{dx}{2x+3}$

(d) $\int_0^1 (2-x)^{-1}\,dx$ (e) $\int_{-\frac{4}{3}}^{-\frac{1}{3}} \frac{dx}{2-3x}$ (f) $\int_{\frac{1}{2}}^2 \frac{3x+1}{2x}\,dx$

(g) $\int_{-2}^0 (5-x)^{-1}\,dx$ (h) $\int_1^2 \frac{dx}{3-4x}$ (i) $\int_{-4}^0 \frac{dx}{1-2x}$

4 Show that $\frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$. Hence evaluate $\int_2^3 \frac{dx}{x^2-1}$.

5 Show that $\frac{1}{x-4} - \frac{1}{x-2} = \frac{2}{x^2-6x+8}$ and hence evaluate $\int_5^6 \frac{dx}{x^2-6x+8}$.

6 P is a function of t such that $\frac{dP}{dt} = e^{-3t}$ and $P = 3$ when $t = 0$. Find P in terms of t.

7 Calculate the area of the region enclosed by the curves $y = e^x$ and $y = e^{2x}$ and the line $x = 1$.

8 Find $\int \frac{e^x+1}{e^x}\,dx$.

9 The part of the curve $y = \frac{1}{\sqrt{x+2}}$ between $x = 1$ and $x = 3$ is rotated about the x-axis through 360°. Find the volume formed.

10 (a) The gradient of a curve is given by $\frac{dy}{dx} = e^{2x}$ and it passes through the point $\left(0, \frac{1}{2}\right)$. Find the equation of the curve.

(b) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$.

11 (a) Sketch the curves $y = e^x$ and $y = \ln x$.

(b) Differentiate $x \ln x - x$ wrt x and hence find $\int \ln x\, dx$.

(c) Hence find the area of the region enclosed by these curves, the x-axis, the y-axis and the line $x = 2$.

12 (a) Find the equation of the tangent to the curve $y = \ln x$ at the point where $x = 1$.

(b) Find the area between this tangent, the curve and the line $x = 2$. (Use the result of Question 11(b).)

SUMMARY

- $\ln x = \log_e x$; $e \approx 2.7183$
- If $y = \ln x$, $x = e^y$; if $y = e^{ax+b}$, $\ln y = ax + b$
- $\frac{d}{dx} \ln x = \frac{1}{x}$ $\qquad$ $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ $\qquad$ $\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$
- $\int \frac{dx}{x} = \ln|x| + c$ $\qquad$ $\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + c$
- $\int e^x\, dx = e^x + c$ $\qquad$ $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + c$

Note: $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ for all values of n *except* $n = -1$.

REVISION EXERCISE 18 *(Answers on page 643.)*

A

1 Evaluate

(a) $\int_0^1 e^{1-2x}\, dx$ $\qquad$ (b) $\int_1^2 \frac{dx}{x-5}$ $\qquad$ (c) $\int_{-2}^0 e^{1-\frac{x}{2}}\, dx$

(d) $\int_a^b e^{kt}\, dt$ (k is a constant) $\qquad$ (e) $\int_{-3}^{-1} e^{2-x}\, dx$ $\qquad$ (f) $\int_{-2}^{-1} \frac{dx}{4-x}$

(g) $\int_0^1 \frac{e}{e^{2x}}\, dx$

2 Find the gradient of the curve $y = \ln(x + \sin 2x)$ where $x = \frac{\pi}{2}$.

3 Sketch the curve $y = e^{-2x}$ for $x > 0$ and find the area of the region enclosed by the curve, the y-axis, the x-axis and the line $x = 1$.

4 Fig. 18.4 shows part of the curves $y = e^{x-2}$ and $y = e^{-x}$ which intersect at P. Find (a) the coordinates of P and (b) the area of the shaded region in terms of e.

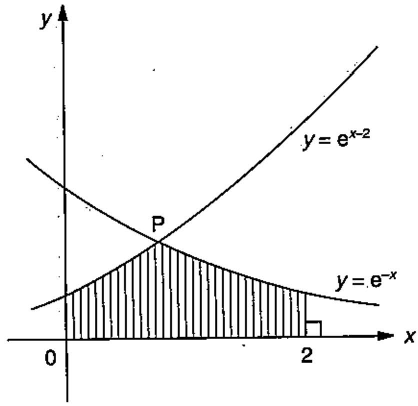


Fig. 18.4

5 Find the values of x at the turning points on the curve $y = (x^2 + x - 1)e^{-x}$ and determine the nature of these points.

6 If $\frac{d}{dx}\left(\frac{x^2+1}{e^x}\right) = \frac{f(x)}{e^x}$, find f($x$).

7 Show that $\int_0^2 \frac{4}{3x+2}\, dx \approx 1.85$.

8 (a) Differentiate $\ln(x\sqrt{x+1})$ wrt x simplifying your result.

(b) Hence show that $\int \frac{3x+2}{x^2+x}\, dx = \ln x^2(x+1) + c$.

9 Find the value of x at the turning point on the curve $y = \frac{\ln x}{x}$ and find whether it is a maximum or a minimum point.

10 Find the approximate change in $\ln x$ if x is increased from 2 to 2.01.

11 If $\frac{dy}{dx} = \frac{1}{3x-2}$ and $y = 0$ when $x = 2$, find the value of y when $x = 1$.

12 If $y = e^{3x} - 2e^{-3x}$, show that $\frac{d^2y}{dx^2} = 9y$.

13 Find the value of k if $y = e^{kx} \sin 2x$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$.

14 If $y = \frac{e^{-x}}{\cos x}$, find the value of $\frac{dy}{dx}$ when $x = 0$.

15 Given that $y = e^{ax} \sin bx$, where a and b are constants, show that $\frac{d^2y}{dx^2} = e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$.

16 Differentiate $e^{2x}(2x - 1)$ wrt x. Hence find the value of $\int_0^1 xe^{2x}\, dx$.

17 The part of the curve $y = \ln x$ between $x = 1$ and $x = e$ is rotated about the y-axis through 360°. Find the volume of the solid formed.

18 Show that $\frac{x-2}{x-1} = 1 - \frac{1}{x-1}$.
Hence show that the area of the region enclosed by the curve $y = \frac{x-2}{x-1}$, the x-axis and the lines $x = 2$, $x = 4$ is $2 - \ln 3$.

19 Differentiate $\ln\left(\cos \frac{x}{2}\right)$ wrt x. Hence evaluate $\int_0^{\frac{\pi}{2}} \tan \frac{x}{2}\, dx$.

20 If $y = e^{3x} \cos 2x$, find $\frac{dy}{dx}$ and express it in the form $Re^{3x} \cos(2x + \alpha)$, stating the values of R and α. Hence express $\frac{d^2y}{dx^2}$ in a similar form.

21 Show that $y = x^2 \ln\left(\frac{1}{x}\right)$ has only one turning point and determine if it is a maximum or minimum point. Also find the value of y at the point.

B

22 Differentiate (a) $\ln x^x$, (b) x^x wrt x.

23 If $xe^y = x + 1$, find $\frac{dy}{dx}$.

24 Given that $y = \cos(e^x)$, show that $\frac{d^2y}{dx^2} - \frac{dy}{dx} + ye^{2x} = 0$.

25 Sketch the curves $y = e^x$ and $y = 2 + 3e^{-x}$ and show that the coordinates of their point of intersection are $(\ln 3, 3)$. Hence find the area of the region enclosed by the two curves and the y-axis.

26 Differentiate $\ln [x + \sqrt{x^2 - 1}]$ wrt x. Hence find the value of $\int_2^3 \frac{dx}{\sqrt{x^2 - 1}}$.

27 (a) Sketch on the same diagram the curves $y = e^x$ and $y = e^{-x}$.
(b) Add a sketch of the curve $y = \frac{1}{2}(e^x + e^{-x})$.
(c) If $y = \frac{1}{2}(e^x + e^{-x})$, find $\frac{dy}{dx}$ and $\left(\frac{dy}{dx}\right)^2$
(d) Show that $1 + \left(\frac{dy}{dx}\right)^2 = y^2$.
(e) The length s of the arc of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Find the length of the arc of $y = \frac{1}{2}(e^x + e^{-x})$ from $x = -1$ to $x = 1$..

Parametric Equations

19

In previous work, the equations of curves have been expressed in **Cartesian** form, i.e. as a relation between the coordinates x and y. Another method of stating the equation of a curve is to use a third variable, called a **parameter**. x and y are then each expressed in terms of this parameter. The equation of the curve is now given by *two* **parametric equations.** For example, the parametric equations of a curve could be $x = 2t$, $y = t^2$ where t is the parameter. This seems to be a more complicated way of describing a curve (two equations instead of one) but for many curves it can be more convenient .

The parameter can be any suitable variable such as a number, an angle, a length etc. It must however satisfy two conditions:

(1) each point on the curve must be related to a unique value of the parameter;
(2) each value of the parameter must give the coordinates of only one point of the curve.

The Cartesian equation of the curve is found by eliminating the parameter between the parametric equations.

Example 1

The parametric equations of a curve are $x = 2t$, $y = t^2$. Sketch the curve and find its Cartesian equation.

As $y = t^2$, y is never negative. Also the curve is symmetrical about the y-axis.

$t = 0$ gives the point (0,0), $t = 1$ gives the point (2,1), $t = -2$ gives the point (–4,4) and so on.

If $t > 0$, $x > 0$ and x and y both increase as t increases.

If $t < 0$, $x < 0$. As t decreases, x decreases but y increases.

As t takes values from $-\infty$ to $+\infty$, the point (x,y) moves along the curve as shown in Fig.19.1.

If $x = 2t$, $t = \frac{x}{2}$.

Then $y = t^2 = \left(\frac{x}{2}\right)^2$ and $4y = x^2$.

The curve is a parabola.

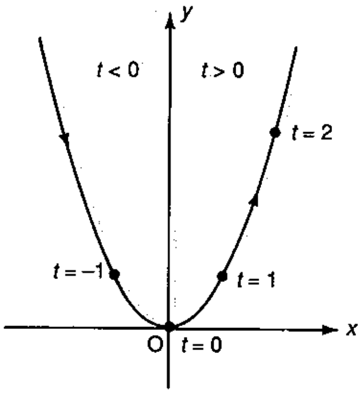


Fig. 19.1

Example 2

The parametric equations of a curve are $x = \frac{t}{1+t}$, $y = \frac{t^2}{1+t}$. *Find its Cartesian equation.*

We find t in terms of x and y first.

If $x = \frac{t}{1+t}$, then $(1+t)x = t$ (i)

If $y = \frac{t^2}{1+t}$, then $(1+t)y = t^2$ (ii)

Divide (ii) by (i). Then $\frac{y}{x} = t$. Now substitute in (1).

$\left(1 + \frac{y}{x}\right)x = \frac{y}{x}$ which gives $x + y = \frac{y}{x}$ or $x^2 + xy = y$.

This could also be written as $y = \frac{x^2}{1-x}$.

Example 3

Find the Cartesian equations of the curves given by (a) $x = \sec\theta + 2$, $y = 3\tan\theta$, *(b)* $x = 3\sin\theta$, $y = 2\cos 2\theta$.

(a) To eliminate the parameter θ, we use the identity $\sec^2\theta = 1 + \tan^2\theta$.
So rearranging and squaring the two parametric equations, we have

$(x-2)^2 = \sec^2\theta$

$\left(\frac{y}{3}\right)^2 \quad = \tan^2\theta$

Subtracting, we get

$(x-2)^2 - \left(\frac{y}{3}\right)^2 = 1$

which reduces to $9x^2 - y^2 - 36x + 27 = 0$.

(b) We use the identity $\cos 2\theta = 1 - 2\sin^2\theta$

Then $\frac{y}{2} = 1 - 2\left(\frac{x}{3}\right)^2$ or $9y = 18 - 4x^2$.

To find the gradient from the parametric equations

Given the parametric equations for a curve $y = F(x)$ we can find $\frac{dy}{dx}$ by using the rule from Chapter 10: $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Hence the gradient $\frac{dy}{dx}$ on the curve $y = F(x)$ given by the parametric equations $x = f(t)$, $y = g(t)$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 4

A curve is given by the parametric equations $x = \frac{1}{1+t}$, $y = \frac{t^2}{1+t}$. Find

(a) $\frac{dy}{dx}$ in terms of t,

(b) the coordinates of the point(s) where the gradient is -3,

(c) the equations of the tangent and the normal at the point where $t = 2$.

(a) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\frac{dy}{dt} = \frac{(1+t)2t - t^2}{(1+t)^2} = \frac{t(t+2)}{(1+t)^2}$$

and $\frac{dx}{dt} = -\frac{1}{(1+t)^2}$

Hence $\frac{dy}{dx} = \frac{t(t+2)}{(1+t)^2} \times \frac{(1+t)^2}{-1} = -t(t+2)$

(b) If the gradient $= -3$, then $-t(t+2) = -3$.

So $t^2 + 2t - 3 = 0$ i.e. $(t+3)(t-1) = 0$ giving $t = -3$ or $t = 1$.

There are two points where the gradient $= -3$.

When $t = -3$, $x = -\frac{1}{2}$, $y = -\frac{9}{2}$ i.e. the point $\left(-\frac{1}{2}, -\frac{9}{2}\right)$.

When $t = 1$, $x = \frac{1}{2}$, $y = \frac{1}{2}$, i.e. the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(c) When $t = 2$, $\frac{dy}{dx} = -8$ and $x = \frac{1}{3}$, $y = \frac{4}{3}$.

The equation of the tangent at $\left(\frac{1}{3}, \frac{4}{3}\right)$ is $y - \frac{4}{3} = -8\left(x - \frac{1}{3}\right)$

i.e. $y + 8x = 4$.

The equation of the normal is $y - \frac{4}{3} = \frac{1}{8}\left(x - \frac{1}{3}\right)$

i.e. $24y - 3x = 31$.

Example 5

The parametric equations of a curve are $x = 2 + \cos\theta$, $y = 3 + \sin\theta$.

(a) Find the Cartesian equation.

(b) Show that the curve is a circle with centre (2,3) and radius 1 and find the meaning of the parameter θ.

Find (c) the equation of the chord joining the points with parameters 0 and $\frac{\pi}{2}$ and (d) the equation of the tangent where $\theta = \frac{3\pi}{4}$.

(a) From the parametric equations we obtain $\cos\theta = x - 2$ and $\sin\theta = y - 3$. We use the identity $\sin^2\theta + \cos^2\theta = 1$ to eliminate θ.

Then $(y-3)^2 + (x-2)^2 = 1$ i.e. $x^2 + y^2 - 4x - 6y + 12 = 0$.

(b) The equation $(x-2)^2 + (y-3)^2 = 1$ states that the distance of the point (x,y) from the fixed point (2,3) is always 1. Hence the point $P(x,y)$ must move on a circle with centre C(2,3) and radius 1 (Fig.19.2).

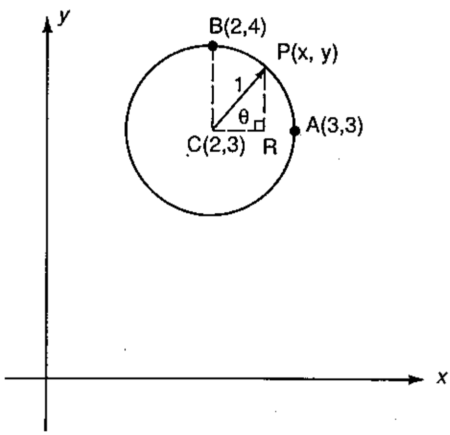


Fig. 19.2

If we draw the lines CR and PR parallel to the x– and y-axes respectively and take $\angle PCR = \theta$, then $CR = \cos\theta$ and $RP = \sin\theta$. The coordinates of P will then be $(2 + \cos\theta, 3 + \sin\theta)$. θ is the angle between CP and the x-axis. As θ varies from 0 to 2π, P describes the circumference of the circle.

(c) If $\theta = 0$, $x = 3$, $y = 3$ i.e. the point A(3,3).
If $\theta = \frac{\pi}{2}$, $x = 2$, $y = 4$ i.e. the point B(2,4).

The equation of AB will be $\frac{y-3}{4-3} = \frac{x-3}{2-3}$ i.e. $x + y = 6$.

(d) $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$

When $\theta = \frac{3\pi}{4}$, $\frac{dy}{dx} = -\cot\frac{3\pi}{4} = 1$, $x = 2 + \cos\frac{3\pi}{4} = 1.29$,

$y = 3 + \sin\frac{3\pi}{4} = 3.71$.

So the equation of the tangent is $y - 3.71 = x - 1.29$ i.e. $y = x + 2.42$.

Example 6

The parametric equations of a curve are $x = 2t - 1$, $y = t^2 + 1$.

(a) Find the Cartesian equation.

(b) If the tangent at a point P (parameter p) passes through the point (2,3), find the value(s) of p.

(c) Find the equation of the tangent to the curve which is parallel to the line $y + 2x = 3$.

(a) If $x = 2t - 1$, then $t = \frac{x+1}{2}$.

Then $y = \left(\frac{x+1}{2}\right)^2 + 1$ i.e. $4y = x^2 + 2x + 5$.

(b) $\frac{dy}{dt} = 2t$ and $\frac{dx}{dt} = 2$

$\frac{dy}{dx} = \frac{2t}{2} = t$

So the gradient of the tangent at P = p.

The coordinates of P are $(2p - 1, p^2 + 1)$ and the equation of the tangent is $y - (p^2 + 1) = p(x - 2p + 1)$ and this passes through (2,3).
Hence $3 - p^2 - 1 = p(2 - 2p + 1)$ which reduces to $p^2 - 3p + 2 = 0$
i.e. $(p - 2)(p - 1) = 0$ giving $p = 1$ or 2.

(c) The gradient of $y + 2x = 3$ is -2 and the gradient of the tangent to the curve is t.
Hence $t = -2$.

When $t = -2$, $x = -5$ and $y = 5$.

So the equation of the tangent is $y - 5 = -2(x + 5)$ i.e. $y + 2x = -5$.

Example 7

A curve is given by $x = \frac{t+3}{t^2-1}$, $y = \frac{3t}{t^2-1}$.
Find the values of t and the coordinates of the points where the curve meets the line given by $x = 2p - 1$, $y = 2 - p$.

We find the Cartesian equation of the line first. $x = 2(2 - y) - 1$ giving $x + 2y = 3$.
Now substitute $x = \frac{t+3}{t^2-1}$ and $y = \frac{3t}{t^2-1}$ in this equation.

$$\frac{t+3}{t^2-1} + \frac{6t}{t^2-1} = 3$$

i.e. $t + 3 + 6t = 3t^2 - 3$ which reduces to $3t^2 - 7t - 6 = 0$.

Hence $(3t + 2)(t - 3) = 0$ giving $t = -\frac{2}{3}$ or 3.

The points are therefore $\left(\frac{-\frac{2}{3}+3}{\frac{4}{9}-1}, \frac{-2}{\frac{4}{9}-1}\right) = \left(-\frac{21}{5}, \frac{18}{5}\right)$ and $\left(\frac{6}{8}, \frac{9}{8}\right) = \left(\frac{3}{4}, \frac{9}{8}\right)$.

Example 8

The equation of a curve is $x^2 - y^2 = 4$. *If x is expressed as* $t + \frac{1}{t}$ *in terms of a parameter t, find the parametric equation for y.*

We have $\left(t + \frac{1}{t}\right)^2 - y^2 = 4$ i.e. $y^2 = t^2 + 2 + \frac{1}{t^2} - 4 = \left(t - \frac{1}{t}\right)^2$

Example 9

In Fig.19.3, A is the point (3,1) and B (t, 0) is a variable point on the x-axis. BCD is perpendicular to AB where C lies on the y-axis and BC = CD.

(a) Find the coordinates of C in terms of t.

(b) Hence state the parametric equations of the locus of D as t varies and find its Cartesian equation.

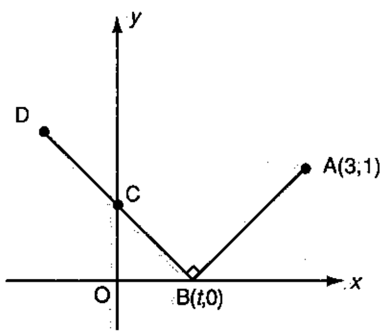


Fig. 19.3

(a) The coordinates of B are $(t,0)$ where t is a parameter.

The gradient of AB $= \frac{1}{3-t}$ so the gradient of BCD $= t-3$.

The equation of BCD is $y = (t-3)(x-t)$ and this meets the y-axis where $x = 0$. The coordinates of C are therefore $(0, -t(t-3))$.

(b) A **locus** is the set of all the possible positions a point can take. In this case, the locus of D will be the curve on which D lies. If the coordinates of D are (p,q) and C is the midpoint of BD, then

$0 = \frac{1}{2}(p+t)$ i.e. $p = -t$

and $-t(t-3) = \frac{1}{2}(q+0)$ i.e. $q = -2t(t-3)$.

Hence the parametric equations of the locus of D are $x = -t$ and $y = -2t(t-3)$.

Substituting $t = -x$ in the equation for y, the Cartesian equation of the locus of D will be $y = 2x(-x-3)$ i.e. $y = -2x(x+3)$.

Exercise 19.1 *(Answers on page 644.)*

1 Find, in as simple a form as possible, the Cartesian equations of the following curves:

(a) $x = 3\cos\theta,\ y = 2\sin\theta$
(b) $x = 1 + 2\cos t,\ y = 1 - 3\sin t$
(c) $x = t^3,\ y = t^2$
(d) $x = t + 1,\ y = t^2 - 1$
(e) $x = t + \frac{1}{t},\ y = t - 1$
(f) $x = \sin 2\theta,\ y = \cos\theta$
(g) $x = t(t-1),\ y = t^2$
(h) $x = \frac{2t}{1+t},\ y = \frac{1-t}{1+t}$
(i) $x = 2t^2 - 1,\ y = 1 - t$
(j) $x = 2\cos 2\theta,\ y = 1 + \cos\theta$
(k) $x = t - \frac{1}{t},\ y = t + \frac{1}{t}$
(l) $x = t,\ y = \sqrt{t-1}$
(m) $x = \frac{1}{t+1},\ y = 2t - 3$
(n) $x = e^{-t},\ y = 3e^{2t}$

2 Find the gradient on the curve $x = \theta - \cos\theta,\ y = 1 - 2\sin\theta$ at the point where $\theta = \pi$.

3 Find $\frac{dy}{dx}$ in terms of t for the curve $x = \frac{t^2}{1+t},\ y = \frac{1-t}{1+t}$.
Hence find the equation of the tangent and the normal where $t = 2$.

4 The parametric equations of a curve are $x = \frac{1}{1+t},\ y = (2t-1)^2$.
Find the equation of the normal where $t = 1$.

5 By finding $x^2 + y^2$, deduce the Cartesian equation of the curve given by

$x = \frac{t}{\sqrt{t^2+1}},\ y = \frac{1}{\sqrt{t^2+1}}$.

Describe the curve.

6 Find the values of t and the coordinates of the points where the curve given by $x = t^2 - 1,\ y = 3t$ meets the line given by $x = m - 1,\ y = 2m + 1$. State the values of m at these points.

7 A curve is given by $x = t - 1,\ y = t^2 + t$. Find the values of t where

(a) the normal to the curve is parallel to the line $3x + y = 5$,
(b) the turning point is.

8 A curve is given as $x = 3 - 2t$, $y = t(1 - t)$. Find (a) $\frac{dy}{dx}$ in terms of t, (b) the equation of the normal which is parallel to the line $x + y = 2$.

9 A is the point $(2t,0)$ and B the point $(0,t)$, where t is a parameter. P is the midpoint of AB and the line through P perpendicular to AB meets the y-axis at Q. (a) Find the equation of PQ in terms of t and (b) the coordinates of Q. (c) Hence show that the lengths of AB and PQ are equal for all values of t.

10 The parametric equations of a curve are $x = t^2 + t$, $y = 2t - 1$.
Find the values of t and the coordinates of the points where the curve meets the line $y = 2x - 3$.

11 A curve is given in terms of a parameter θ as $x = 2\cos\theta - \sin\theta$, $y = \cos\theta + \sin\theta$. Express $\sin\theta$ and $\cos\theta$ in terms of x and y and hence find the Cartesian equation of the curve.

12 The equation of a curve is $(x - 1)y = 2x^2$. By taking $y = tx$, find parametric equations for the curve in terms of t.

13 P is the point $(2t,t)$, where t is a parameter. The line through P with gradient t meets the axes at A and B.

(a) Write down the equation of the line APB.
(b) Find the coordinates of A and B.
(c) Hence find parametric equations (in terms of t) for the locus of M, the midpoint of AB.
(d) Obtain the Cartesian equation of this locus.

14 Show that the equations $x = 2 + 5\cos\theta$, $y = -1 + 5\sin\theta$ represent a circle and state its radius and the coordinates of its centre.

15 The Cartesian equation of a curve is $y^2 - 3y - 2x + 2 = 0$. If the parametric equation for y is $y = 1 - t$, find the corresponding equation for x.

16 (a) The parametric equations of a curve are $x = 1 + t\cos 45°$, $y = 2 + t\sin 45°$. Find the Cartesian equation and state what type of curve this is. Interpret the meaning of the parameter t.
(b) Find parametric equations for the straight line through the point $(-2,3)$ whose gradient is $\frac{2}{3}$.

17 A straight line with gradient $-t$ passes through the variable point $(2t,0)$. Another line with gradient t passes through the point $(0,2t)$. If the lines intersect at a point P, find parametric equations for the curve on which P will lie as t varies.

18 A curve is given by $x = t - 3$, $y = t^2 + 8t$. Show that the line $y = 4x + 8$ is a tangent to the curve and find the coordinates of the point of contact.

19 The line $4y + x = 16$ is a tangent to the curve given by $x = 4t$, $y = \frac{4}{t}$. Find the coordinates of its point of contact.

20 The parametric equations of a curve are $x = \frac{t^2 + 2}{t + 2}$, $y = \frac{3t}{t + 2}$.
(a) Find the values of t where the curve meets the line $x + y = 4$.
(b) Find $\frac{dy}{dx}$ in terms of t and hence find the values of t where the tangent is parallel to the line $y = 2x + 3$.

21 A curve is given by $x = t$, $y = t^2 + 2$.
(a) Find the equation of the tangent where $x = t$.
(b) If this tangent passes through the point (3,2), find the values of t.

22 If the parametric equations of a curve are $x = t^2 - \frac{1}{t}$, $y = t^2 + \frac{1}{t}$, find (a) the Cartesian equation, (b) the equation of the tangent at the point where $x = 0$.

23 A line with variable gradient m passes through the point (4,2) for all values of m and meets the y-axis at P and the x-axis at Q.
(a) Find the coordinates of P and Q in terms of m.
(b) If M is the midpoint of PQ, find parametric equations in terms of m for the curve on which M will lie.
(c) Obtain the Cartesian equation of this curve.

24 P is a fixed point with coordinates (4,3). A line through P with gradient t meets the y-axis at Q and a second line through P with gradient $\frac{1}{t}$ meets the x-axis at R. Find
(a) the equations of PQ and PR, and
(b) the coordinates of Q and R in terms of t.
(c) M is the midpoint of QR. Obtain parametric equations for the locus of M and also the Cartesian equation of this locus.

SUMMARY

- The Cartesian equation of a curve is a relation between the coordinates x and y.
- Parametric equations of a curve are in the form $x = f(t)$, $y = g(t)$ where t is a parameter. Any value of t gives one point on the curve and each point of the curve has a unique value of t.
- For parametric equations, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

REVISION EXERCISE 19 *(Answers on page 645.)*

A

1 The parametric equations of a curve are $x = 3t^2$, $y = 2t^3$ where t is a parameter.
(a) Find the equation of the tangent and of the normal at the point with parameter -2.
(b) Derive the Cartesian equation of the curve.

2 Find the Cartesian equations of the following curves:

(a) $x = 2\cos\theta,\ y = 3\sin^2\theta$

(b) $x = 2t(t+1),\ y = 1 - t$

(c) $x = 1 - \frac{1}{t},\ y = 2t + \frac{1}{t}$

(d) $x = e^{2t+1},\ y = 2e^{-t-1}$

(e) $x = 2e^{3t},\ y = e^{-2t}$

3 The equation of a curve is given in terms of a parameter θ as $x = \sec\theta + 2\tan\theta$, $y = 3\sec\theta - \tan\theta$. Express $\sec\theta$ and $\tan\theta$ in terms of x and y and hence find the Cartesian equation of the curve.

4 (a) A curve is defined parametrically by

$$x = 3\left(t + \frac{1}{t}\right), \qquad y = 2\left(t - \frac{1}{t}\right).$$

(i) Write down the gradient of the curve at the point whose parameter is 2.

(ii) Write down and simplify expressions for $2x + 3y$ and $2x - 3y$ in terms of t and hence obtain the Cartesian equation of the curve.

(b) The straight line $3x - 4y - 15 = 0$ intersects the curve whose parametric equations are $x = 5t^2$, $y = 10t$ at two points.

(i) Calculate the value of t at each of these points of intersection.

(ii) Prove that the tangents to the curve at the points of intersection are perpendicular to each other. (C)

5 A curve is defined parametrically by the equations $x = \frac{t}{1+t}$, $y = \frac{t^2}{1+t}$.

Show that $\frac{dy}{dx} = t(t+2)$.

Find (i) the value of t at the point where the tangent is parallel to the tangent at the point where $t = -3$, (ii) the equation of the normal at the point where $t = -3$. (C)

6 A curve is given by the parametric equations $x = 2t^2 - 3t$, $y = t^2 - 8t + 1$.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find the value of t at the turning point and the equation of the tangent at that point.

(c) Find the value of t at the point where the tangent is parallel to the line $y + 2x = 3$.

7 (a) The parametric equations of a curve are $x = 3t + 1$, $y = 2t^2$.
A point P on the curve has parameter p. Given that the tangent at P passes through the point $(1, -8)$ calculate the possible values of p.

(b) The Cartesian equation of a curve is $y(y - 2) = x$. Given that x is defined parametrically by $x = t^2 - 1$ and that $y = 4$ when $t = 3$, express y in terms of t.

(c) The parametric equations of a curve are $x = t^3 - t$, $y = t^2 + t$.
Express $\frac{x}{y}$ in terms of t in the simplest possible form. Hence, or otherwise, find the Cartesian equation of the curve. (C)

8 The line $y = tx$ meets a line through the point $(t, 0)$ at right angles at the point P. Find (a) the coordinates of P in terms of t and (b) the Cartesian equation of the locus of P.

9 A curve is given parametrically by the equations $x = \frac{6}{(1+t)^2}$, $y = \frac{3t}{1+t}$.

Given that P and Q are points on the curve with parameters 0 and 2 respectively, (i) find the equation of the chord PQ, (ii) show that $\frac{dy}{dx} = -\frac{(1+t)}{4}$ and hence find the equation of the normal at P.

Find the points of intersection of the curve with the line $y = x$. (C)

10 The parametric equations of a curve are $x = t + \frac{1}{t}$, $y = t(t-2)$.

Find the values of t where the tangent is parallel to the line $y = x + 2$.

11 (a) Obtain the Cartesian equation of the curve whose parametric equations are $x = 1 + \frac{1}{t}$, $y = 2t(t+1)$.

(b) For the curve whose equation is $x^2 + 4y^2 - 8y = 0$, a parametric form for y is given by $y = 1 + \sin\theta$. Obtain, in its simplest form, a corresponding parametric form for x.

(c) The parametric equations of a curve are $x = t^2 + t$, $y = 2t + 1$.

Obtain the equation of the tangent at each of the two points where this curve meets the y-axis. Calculate the coordinates of the point of intersection of these tangents. (C)

12 A curve has parametric equations $x = 4 - 3t$, $y = 9t(1-t)$. Find the equations of the tangents which pass through the point (2,6).

13 (a) The coordinates of a point are given parametrically by the equations $x = \frac{3t}{t-2}$, $y = \frac{2t+4}{t-2}$.

Find $\frac{dy}{dx}$ and hence, or otherwise, obtain the Cartesian equation which corresponds to the above parametric equations.

(b) Write down the equation of the straight line having a gradient of $-t$ and passing through the point $(0,t)$. This line meets the line $x + t(y+1) = 0$ in the point (X,Y). Obtain expressions for X and Y in terms of t and hence evaluate X + Y. (C)

14 The parametric equations of a curve are given as $x = ae^{-t}\cos t$, $y = ae^{-t}\sin t$, where a is a constant and t is a parameter. Show that $\frac{dy}{dx} = \tan\left(t - \frac{\pi}{4}\right)$.

15 The position vector $\mathbf{r}$ of a point P is given by $\mathbf{r} = (t-1)\mathbf{i} + (t^2+2)\mathbf{j}$ where t is a parameter.

(a) State the parametric equations for the locus of P and obtain the Cartesian equation of this locus.

(b) Find the position vectors of the points where the curve meets the line $2x + y = 3$.

B

16 Fig. 19.4 shows a vertical circular disc centre C and radius 2 which is rolled along the horizontal line AB. P is the point of contact with AB at the start and as the disc is rolled, P describes a curve known as a **cycloid**. When the disc has turned through an angle θ, P is at the position P′.

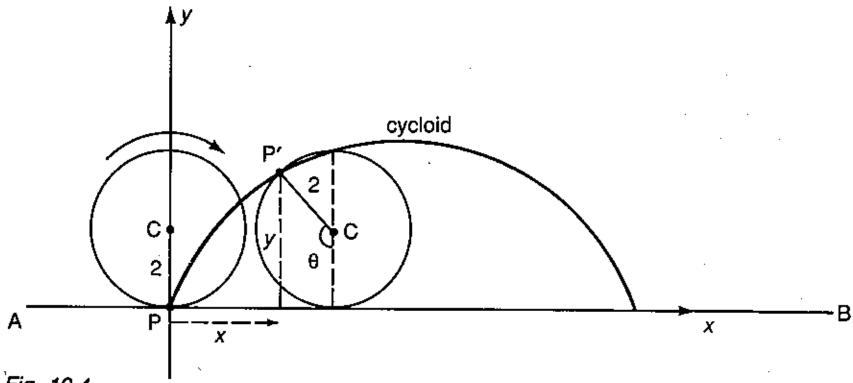


Fig. 19.4

(a) Taking P as the origin of coordinates, show that the parametric equations for P′ are $x = 2(\theta - \sin\theta)$, $y = 2(1 - \cos\theta)$.

(b) What is the value of θ when P reaches the line AB again for the first time after the start?

(c) Show that $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 16\sin^2\frac{\theta}{2}$.

(d) The length s of an arc of a curve, given in terms of a parameter θ is $\int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ where a and b are the values of the parameter at the ends of the arc. Using this, find the length of one arch of the cycloid.

17 By taking $y = tx$, find parametric equations for the curve $x^3 + y^3 = 3xy$. Hence find the equation of the tangent at the point where $t = 1$.

18 A curve is given by $x = 2\cos^3 t$, $y = 2\sin^3 t$ where t is a parameter.

(a) Find $\frac{dy}{dx}$ in terms of t and show that the equation of the tangent at the point with parameter p is $x\sin p + y\cos p = \sin 2p$.

(b) This tangent meets the axes at P and Q. Show that the length of PQ is constant whatever value p has.

(c) Obtain the Cartesian equation of the curve.

Revision Papers 6 – 10

PAPER 6 *(Answers on page 645.)*

1 (a) Find x if $(1.2)^x = (2.1)^{2x-1}$.
(b) Solve the equation $\log_5 x + \log_5(6x - 5) = 2$.

2 Evaluate (a) $\int_{-5}^{0} \frac{dx}{(5-x)^2}$, (b) $\int_{-5}^{0} \frac{dx}{5-x}$, (c) $\int_{-5}^{0} \frac{dx}{\sqrt{5-x}}$.

3 Given that $2x^3 + ax^2 + bx + 6$ has factors $(2x - 1)$ and $(x + 2)$ find the value of a and of b. With these values, find the remaining factor.

4 (a) Sketch the curve $y = 1 + e^x$.
Find (b) the equation of the tangent to this curve at the point $(1, 1 + e)$ and (c) the coordinates of the point where this tangent cuts the y-axis. (d) Hence find the area enclosed by the curve, this tangent and the y-axis. [Leave your answer in terms of e.]

5 Express $\cos x + 2 \sin x$ in the form $R \sin(x + \alpha)$, where α is acute. Hence solve the equation $2 \cos x + 4 \sin x = 1$ for $0° < x < 360°$.

6 (a) Find $\frac{dy}{dx}$ if $y = \ln\left(\frac{x-2}{x+2}\right)$, simplifying your answer.

(b) Hence evaluate $\int_0^1 \frac{1}{x^2 - 4}\, dx$.

7 The values of x and y in the table below are believed to fit the equation $y = ax^n$.

x	1.5	1.7	1.9	2.1	2.3
y	8.6	10.4	12.3	14.3	16.4

By drawing a suitable straight line graph, estimate the values of a and n to 2 significant figures.

8 Fig. R6 shows part of the graphs of $y = \sin x$ and $y = \cos 2x$ for $0 < x < \frac{\pi}{2}$. Find (a) the x-coordinate of A and (b) the shaded area.

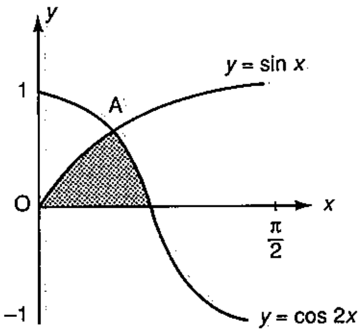


Fig. R6

9 (a) Find an expression for the sum of the first n terms of the AP 8, 11, 14, ... and find the value of n if the sum is 148.
(b) The sum of the first three terms of a GP is 35 and the 3rd term is greater than the first term by 15. Find the first term and the common ratio.

10 A curve is given by the parametric equations $x = t^2 + 1 - \frac{1}{t}$, $y = t + \frac{1}{t}$.
Find the coordinates of the turning points on the curve.

PAPER 7 *(Answers on page 645.)*

1 (a) Differentiate $x \sin x + \cos x$ wrt x.
Hence evaluate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$.
(b) Given that $\frac{dy}{dx} = \frac{1}{\sqrt{2 - x}}$, find y if $y = -2$ when $x = 1$.

2 A straight line with variable gradient m is drawn through the point (1,1) and meets the x-axis at A and the y-axis at B. The rectangle AOBT is drawn, where O is the origin.
(a) Find the coordinates of T in terms of m.
(b) Hence find the Cartesian equation of the locus of T as m varies.

3 (a) Find the values of x at the turning points on the curve $y = (x^2 - 2)e^{-2x}$ and the nature of these points.
(b) Given that $e^{3(x-3)} = e^{2y}$ and that $\ln(8x + 3y) = 2 \ln 7$ find the value of x and of y.
(c) Find and simplify $\frac{dy}{dx}$ if $y = (2x - 3)(x + 1)^3$.

4 (a) Prove that $2(3 \cos A - 2 \sin A)(3 \cos A + 2 \sin A) = 13 \cos 2A + 5$.
(b) Solve the equations
(i) $\cos \frac{\theta}{2} = 0.4$,
(ii) $2 \sin^2 \theta = 3(1 + \cos \theta)$ for $0° \le \theta \le 360°$.

5 (a) Solve the equation $2x^3 - 9x^2 + 12x - 4 = 0$.

(b) If the remainder when $f(x)$ is divided by $x - 2$ is 3, what is the remainder when $f(x + 1)$ is divided by $x - 1$?

6 Values of x and y were found by experiment and are given in the following table:

x	2	4	6	8	10	12
y	4.6	12.4	23.4	37.6	55.0	75.6

It is known that these values satisfy the equation $y = ax^2 + bx$ where a and b are constants. By drawing a suitable straight line graph, find approximate values for a and b.

7 (a) Show that the tangent to the curve $y = e^{x+1}$ where $x = 1$ passes through the origin and find the area enclosed by this tangent, the curve and the y-axis in terms of e.

(b) Differentiate (i) $\frac{\ln x}{x^2}$, (ii) $\frac{e^{-x}}{x+1}$, wrt x.

(c) If $\int_a^{-1} \frac{dx}{5+x} = \ln 2$, find the value of a.

8 (a) If the nth term of an arithmetic progression is $4n - 7$, find the sum of the first 40 terms.

(b) A geometric progression has first term a and common ratio r.
Given that the sum of n terms is 422, show that

$$ar^{n-1} = \frac{422(r-1)+a}{r}.$$

If, in addition, the first and nth terms are 32 and 162 respectively, find r and n.
(C)

9 (a) Find $\frac{dy}{dx}$ if $xy^2 + 3x = 8 - 2y$.

(b) The parametric equations of a curve are $x = 2t - 3$, $y = t^2$. Find (i) the Cartesian equation of the curve, (ii) the values of k if the tangent at the point with parameter k passes through the point (2,4).

10 (a) Prove the identity $\cot\theta - \cot 2\theta \equiv \operatorname{cosec} 2\theta$.
Hence solve the equation $\cot\theta = \cot 2\theta + 3$ for $0° \leq \theta \leq 360°$.

(b) If $\cos 2A = \frac{3}{5}$, where A is an acute angle, find the value of tan A without using tables or a calculator.

PAPER 8 *(Answers on page 645.)*

1 (a) Evaluate (i) $\int_0^2 e^{1-\frac{x}{2}}\,dx$, (ii) $\int_{-1}^0 \frac{dx}{x-2}$.

(b) Differentiate $\ln x\sqrt{x^2+1}$ wrt x, simplifying your answer.

(c) If $\frac{a}{3-x} + \frac{b}{x+2} = \frac{3x+1}{(3-x)(x+2)}$ for all values of x except 3 and -2, find the values of a and b.
Hence find the value of $\int_1^2 \frac{3x+1}{6+x-x^2}\,dx$.

2 Express $\frac{3}{2}$ $\sin x + 2\cos x$ in the form $R\cos(x - \alpha)$ where α is an acute angle. Hence find the maximum and minimum values of $3\sin x + 4\cos x$ and the values of x ($0° \le x \le 360°$) where they occur.

3 (a) If $T = \ln(2x - 1)$, find the approximate change in T when x is increased from 2 to 2.01.

(b) Sketch the curve $y = 1 + 2\sin\frac{x}{2}$ for $-\pi \le x \le \pi$.

Given that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, find the area of the region bounded by the curve, the x-axis and the lines $x = -\pi$, $x = \pi$. (Leave the answer in terms of π).

4 (a) Given that A is an acute angle and that $\tan A = \frac{4}{3}$, find (without using tables or a calculator) the values of (i) $\sin A$, (ii) $\cos 2A$, (iii) $\sin\frac{A}{2}$.

(b) By expanding both sides of the equation $2\sin(x + 60°) = \cos(x - 30°)$, show that $\tan x = -\cot 30°$. Hence solve the equation for $0° \le x \le 360°$.

5 (a) When $\ln y$ is plotted against x for a certain function, a straight line is obtained passing through the points (1,3) and (3,–1). Express y in terms of x.

(b) A production line assembling computers is to be run down. Production started at 500 per week but this is reduced by 15% each week. When production first reaches 100 or less computers in a week, the line will be shut down at the end of that week. For how many weeks will it be operated?

(c) Solve the equation $x^3 + 12 = x^2 + 8x$.

6 (a) A GP with $r > 0$ is such that the sum of the first two terms is 7 and the third term is $2\frac{1}{4}$. Find r and the sum to infinity.

(b) The fourth term of an AP is 14 and the eleventh (and last) term is 35. Find the sum of the last 6 terms.

7 Measured values of x and y are given in the following table:

x	1	2	3	4	5	6	7
y	5.1	4.6	4.2	3.8	3.2	2.4	1.4

It is known that x and y are related by the equation $y^2 = a + bx$.
Explain how a straight line graph may be drawn to represent the given equation and draw it for the values given.

Use the graph to estimate the value of a and of b. Estimate the greatest possible value of x.

(C)

8 (a) Solve the equation $3^{2x+2} - 10(3^x) + 1 = 0$.

(b) On the same axes, draw the graphs of $y = \ln(1 + x)$ and $y = \frac{1}{x}$ for $\frac{1}{2} \le x \le 4$. From your graph, find approximately

(i) the solution of the equation $x\ln(1 + x) = 1$ and

(ii) the value of $e^{\frac{1}{2}}$.

9 (a) If $y = \ln(\tan x)$, show that $\frac{dy}{dx} = \frac{2}{\sin 2x}$. Hence find $\int \operatorname{cosec} 2x \, dx$.

(b) Calculate the area of the region enclosed by the curve $y = \frac{1}{2x-3}$, the x-axis and the lines $x = 2$, $x = 4$.

(c) Given that $\sin y = e^x$, show that $\frac{dy}{dx} = \tan y$.

10 The parametric equations of a curve are $x = t^2$, $y = t - \frac{1}{3}t^3$. Find

(a) $\frac{dy}{dx}$ in terms of t,

(b) the coordinates of the turning points of the curve,

(c) the values of t where the tangent to the curve is parallel to the line $3y = 4x + 2$,

(d) the Cartesian equation of the curve.

PAPER 9 *(Answers on page 646.)*

1 (a) Differentiate wrt x (i) $\frac{1-2x}{x+2}$, (ii) $(x-2)^2(2x-3)^3$ simplifying your answers.

(b) Evaluate (i) $\int_{-1}^{-2} \frac{dx}{3x+2}$, (ii) $\int_0^{\frac{\pi}{2}} 2 \cos 3x \, dx$.

2 (a) The first 3 terms of a GP are $x + 1$, $x - 3$ and $x - 6$. Find (i) the value of x, (ii) the sum to infinity of the GP.

(b) Find the sum to infinity of the GP $\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^3} + \ldots$

(c) The sum of the first five terms of an AP is 55 and the sum of the four terms from the 6th to the 9th (inclusive) is 116. Find the AP.

3 (a) Fig. R7 shows part of the curve $y = \cos x$. Find the shaded area.

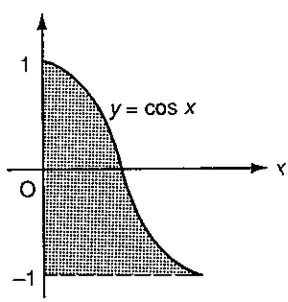


Fig. R7

(b) Sketch the curve $y = e^{-x}$. Find

(i) the equation of the tangent to the curve where $x = 0$,

(ii) the x-coordinate of the point where this tangent meets the x-axis,

(iii) the area of the region enclosed by this tangent, the curve, the x-axis and the line $x = 2$.

[Leave your answer in terms of e].

4 (a) A particle moves in a straight line from a point O in the line so that, t seconds after leaving O, its velocity, v m s^{-1}, is given by $v = 10(1 - e^{-t})$. Calculate
(i) the acceleration of the particle when $t = 2$,
(ii) the displacement of the particle from O when $t = 2$.

(b) Show that $\int_0^3 \frac{12}{2x+3}\,dx = 6.59$ approximately.

(c) Fig. R8 shows part of the curve $y^2 = x + 1$ which intersects the axes at A and B. Calculate the shaded area enclosed between the curve and the line AB. (C)

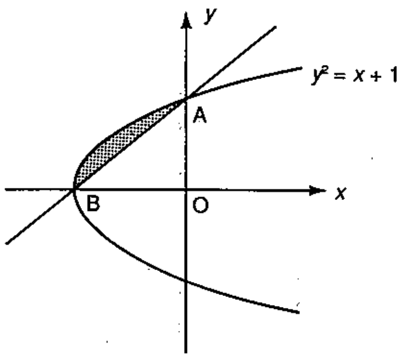


Fig. R8

5 Solve the equations
(a) $\log_7(2x - 3) + \log_7(x + 1) = 1$.
(b) $2^x + 2^{3-x} = 6$
(c) $3^{x-1} = 2^{x+1}$.

6 (a) Find the values of t and the coordinates of the points where the curve given by $x = t(2t + 1)$, $y = t + 1$ meets the line given by $x = 2 - 4T$, $y = 2T + 1$.

(b) Given that $f(x) = x^3 + ax^2 + bx + 3$ and $\frac{d}{dx}f(x)$ each has $x - 1$ as a factor, find the values of a and b and hence factorize $f(x)$.

7 Sketch the curves $y = \frac{1}{x}$ and $y^2 = x$ for $x > 0$ and $y > 0$.
(a) Find the coordinates of the point of intersection of the curves.
(b) Calculate the area of the region enclosed by the curves and the line $x = 4$.
(c) If this region is rotated about the x-axis through 360°, find the volume created.

8 The equation of a curve is $x(y^2 - 1) = 3$.
(a) If the parametric equation for y is $y = \frac{1}{t}$, find the corresponding equation for x.
(b) Find the equation of the tangent to the curve at the point where $t = 2$.

9 Variables x and y are believed to be connected by the equation $\frac{a}{x} + \frac{b}{y} = 1$. The following set of values was obtained in an experiment to test the equation:

x	2	3	4	6	8
y	8.8	5.8	5	4.4	4.1

By drawing a suitable straight line graph for these values, show that the equation is true and estimate the values of a and of b.

10 (a) Solve, for $0° \leq x \leq 360°$, the equations
(i) $3 \cos 2x = 2 + \cos x$, (ii) $\cot \frac{x}{2} = -1.15$.
(b) If $\tan A = 2 \tan B$, show that $\tan(A - B) = \frac{\sin 2B}{3 - \cos 2B}$.

PAPER 10 *(Answers on page 646.)*

1 (a) The first term of a GP is 3 and the common ratio is $\frac{4}{3}$.
Find the greatest number of terms which can be added for their sum to be less than 200.
(b) An AP has 15 terms and the last one is 44. The sum of the last ten terms is 305. Find the AP.

2 (a) The polynomial $2x^3 + x^2 + ax + b$ has $(x - 2)$ as a factor and leaves a remainder of -4 when it is divided by $(x - 1)$. Find the values of a and of b and hence factorize the polynomial.
(b) Without using tables or a calculator, find the value of
$\log_2 24 - \log_2 6 + \log_4 64 - 2 \log_2 4$.
(c) Solve the equation $2^{x-2} + 1 = 2^{3-x}$.

3 (a) Differentiate wrt x: (i) $\tan \frac{x}{2}$ (ii) $\sin^2 2x$ (iii) $\sqrt{1 + \sin x}$
(b) Given the curve $y = x^2 + \frac{4}{x}$, find (i) the equation of the tangent where $x = 2$,
(ii) the coordinates of the point where this tangent meets the curve again.
(iii) Find the area enclosed by this tangent, the curve and the line $x = 1$.

4 (a) (i) If $t = \tan \frac{x}{2}$, show that $\sin \frac{x}{2} = \frac{t}{\sqrt{1 + t^2}}$ and find a similar expression for $\cos \frac{x}{2}$ in terms of t.
(ii) Hence show that $\sin x = \frac{2t}{1 + t^2}$ and that $\cos x = \frac{1 - t^2}{1 + t^2}$.
(iii) Using these results show that the equation $\cos x = 2(1 + \sin x)$ reduces to the equation $3t^2 + 4t + 1 = 0$ and hence find the values of x which satisfy the equation for $0° \leq x \leq 360°$.
(b) (i) Differentiate $\tan^3 x$ wrt x.
(ii) Show that $\tan^4 x = \sec^2 x \tan^2 x - \sec^2 x + 1$.
(iii) Hence, using these results, find $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$.

5 (a) Differentiate $\dfrac{x}{\sqrt{2-x}}$ wrt x.

Hence evaluate $\displaystyle\int_{-2}^{0} \frac{4-x}{(2-x)^{\frac{3}{2}}}\,dx$.

(b) Given that $y = (2x-1)^3(x+1)^4$, find the values of x for which $\frac{dy}{dx} = 0$.

(c) Given that $x^2y + xy^2 = 1 - y$, find $\frac{dy}{dx}$ in terms of x and y.

6 (a) A particle moves on a straight line so that its velocity v m s^{-1} t seconds after passing through a fixed point O on the line is given by $v = 12e^{-\frac{t}{3}}$. (i) State its velocity at O. Find (ii) its acceleration 3 seconds after passing O and (iii) its displacement from O at that time.

(b) Find the values of (i) $\displaystyle\int_1^2 \frac{x^2-1}{x^3}\,dx$, (ii) $\displaystyle\int_0^1 \frac{e^x-1}{e^{2x}}\,dx$.

7 (a) Sketch the graph of $y = \ln x$ for $x > 0$. Express $xe^x = 7.39$ in the form $\ln x = ax + b$ and state the values of a and of b. Insert on your sketch the additional graph required to illustrate how a graphical solution of the equation $xe^x = 7.39$ may be obtained.

(b) Given that $\log_3 x = r$ and $\log_9 y = s$, express xy^2 and $\frac{x^2}{y}$ as powers of 3. Hence, given that $xy^2 = 81$ and $\frac{x^2}{y} = \frac{1}{3}$, determine the value of r and of s. (C)

8 A curve is given by the parametric equations $x = (t-2)^2$, $y = t - 4$. Find

(a) the equation of the normal at the point where $t = 4$,

(b) the value of t where this normal meets the curve again,

(c) the Cartesian equation of the curve.

9 Differentiate $(x+1)\ln(x+1) - x$ wrt x and hence find $\int \ln(x+1)dx$.

Sketch the curve $y = e^x - 1$. Using the result found above, find the area of the region between the part of the curve $y = e^x - 1$ from $x = 0$ to $x = 1$ and the y-axis.

10 The parametric equations of the curve on which a point P lies are $x = 10t$, $y = 30t - 5t^2$.

(a) If O is the origin, state the gradient of OP in terms of t.

(b) Find the gradient of the tangent to the curve at P.

(c) Find the values of t for which this tangent is perpendicular to OP.

PART III

Particle Mechanics

Kinematics

20

STRAIGHT LINE MOTION

Kinematics is a branch of Applied Mathematics that deals with the study of the motion of a body without taking into consideration how the motion is caused. In this chapter, we will study the motion of a particle **moving in a straight line.**

We have already discussed in Chapter 10 the concept of velocity as a rate of change of distance in a given direction, and acceleration as the rate of change of velocity. We will now develop further the relations between distance, velocity and acceleration.

Consider a particle P moving along a straight line Y'OY (Fig. 20.1).

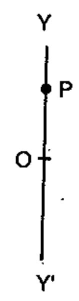


Fig. 20.1

The position of the particle is given by its *displacement* from a given point O. The term displacement, if you recall in Chapter 8, is a vector defining the distance in a given direction. In Fig. 20.1, the displacements of the particle above O are taken to be positive and those below O are taken to be negative. Similarly, the *velocity* and *acceleration* of the particle would be directed quantities, being positive if they are in the direction OY and negative if they are in the direction OY'.

s–t GRAPHS

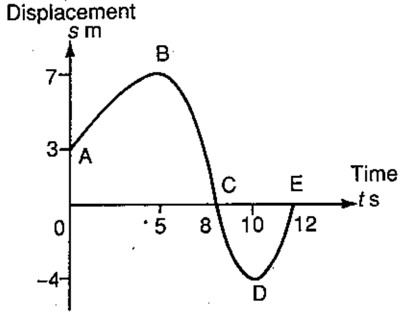


Fig. 20.2

Fig. 20.2 is a graph showing the relation between the distance or displacement s (in metres) and time t (in seconds). It is called an **s–t graph.** We have already learnt that the gradient of the graph $\frac{ds}{dt}$ represents the velocity in m s^{-1}. From the graph, we note the following facts.

1 The starting or initial position A of the body is 3 m from a reference position O. The initial velocity is given by the gradient at A and is positive. The body is moving away from O in the positive direction.

2 From A to B, the body moves further away from O but its velocity is decreasing, as the gradient is decreasing.

3 At B, 7 m from O and 5 s from the start, the gradient is zero, i.e. the velocity is zero. The body is momentarily at rest.

4 From B to C, the gradient is negative. Hence the velocity is negative, i.e. the body is moving back towards O and arrives there after 8 s from the start. Note that the *distance* travelled up to this time is $4 + 7 = 11$ m but the *displacement* from O at this time is zero. As this graph always shows the displacement of the body at any time, it is better described as a **displacement–time graph.**

5 The body reaches position D (4 m from O in the opposite direction to B) after 10 s, and is again momentarily at rest.

6 From D to E, the body is now moving in the positive direction and reaches O again after 12 s from the start.

If s is given as a function of t, the velocity can be found by obtaining the value of $\frac{ds}{dt}$. Otherwise, an approximate value can be obtained by fitting a tangent to the curve and measuring its gradient.

v–t GRAPHS

Another useful graph is the **v–t graph,** which relates velocity to time. Fig. 20.3 shows an example of such a graph.

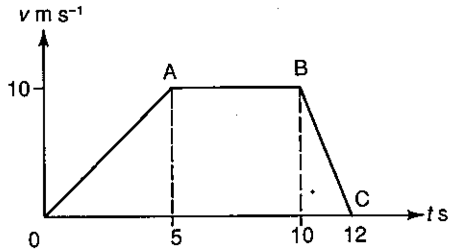


Fig. 20.3

The gradient of the graph, $\frac{dv}{dt}$, represents the acceleration. There is no indication of the displacement of the body. From the graph, we note the following facts.

1 The body starts at $t = 0$ from rest (i.e. with zero initial velocity). From O to A, the velocity increases until it reaches 10 m s^{-1} at time 5 s. Since OA is a straight line, the acceleration is constant or **uniform** and is equal to $\frac{10}{5} = 2$ m s^{-2}. At A, the acceleration ceases.

2 From A to B, the body moves with uniform velocity (10 m s^{-1}).

3 From B to C, the velocity decreases steadily. The acceleration is negative. A negative acceleration is called a **deceleration** or a **retardation.** In this casr, the retardation is uniform and is equal to $\frac{10}{2} = 5$ m s^{-2}. The body comes to rest again at $t = 12$ s.

Area under the v–t Graph

The area under a v–t graph provides us with additional information. In the centre part of the graph in Fig. 20.3, the body is moving with a uniform velocity of 10 m s^{-1} for 5 s. Hence the distance covered is 50 m. The area under this part of the curve is 10 m $s^{-1} \times 5$ s = 50 m. So the area under a v–t graph is numerically equal to the distance covered by the body (strictly speaking, the displacement of the body).

If v is measured in m s^{-1} and t in s, the area will give the distance in m; if v is in km h^{-1} and t in h, the area represents the distance measured in km.

(In calculus, the area under the graph would be $\int v\ dt = \int \frac{ds}{dt}\ dt = \int ds = s$ with the appropriate limits taken).

Example 1

A train starts from rest from station P and accelerates uniformly for 2 min reaching a speed of 60 km h^{-1}. It maintains this speed for 10 min and then retards uniformly for 3 min to come to rest at station Q. Find

(a) the distance PQ in km,

(b) the average speed of the train,

(c) the acceleration in m s^{-2}.

The v–t graph is shown in Fig. 20.4. OA is a straight line as the acceleration is uniform. AB is a straight section parallel to the t-axis as the velocity is constant at 60 km h^{-1}. BC is also straight, with negative gradient, as the retardation is uniform. v is marked in km h^{-1} and, to be consistent, t is also marked in h.

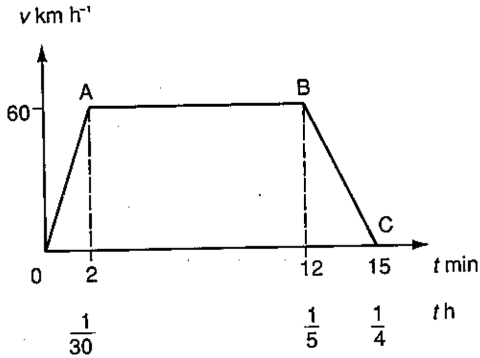


Fig.20.4

(a) The area under the graph represents the distance travelled.

Area of trapezium OABC $= \frac{1}{2}(\text{AB} + \text{OC}) \times 60$

$$= \frac{1}{2}\left(\frac{10}{60} + \frac{15}{60}\right) \times 60 = 12.5$$

Hence the distance PQ is 12.5 km.

(b) The average speed $= \dfrac{\text{total distance travelled}}{\text{time taken}}$

$$= \frac{12.5}{\frac{1}{4}} = 50 \text{ km h}^{-1}$$

(c) To find the acceleration in m s^{-2}, we work in m and s.

Now 60 km $h^{-1} = \dfrac{60 \times 1000}{60 \times 60} = \dfrac{50}{3}$ m s^{-1}

Hence the acceleration $= \dfrac{\frac{50}{3}}{120} \approx 0.14$ m s^{-2}.

Example 2

An MRT train starts from station X and accelerates uniformly to a speed of 20 m s⁻¹. It maintains this speed and then retards uniformly until it comes to rest at station Y. The distance between the stations is 2 km. The total time taken is 2 min. If the retardation is twice the acceleration in magnitude, find

(a) the time for which the train is travelling at constant speed,

(b) the acceleration.

We first draw the v–t graph (Fig. 20.5). OA is a straight line as the acceleration is uniform. AB is a straight line parallel to the t-axis as the speed is constant. BC is also straight, with a negative gradient, as the retardation is uniform. v is marked in m s^{-1}, while t is marked in s to be consistent with v.

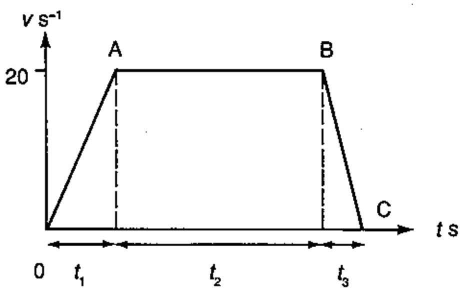


Fig. 20.5

Suppose the train takes t_1, t_2 and t_3 seconds for the 3 parts of the journey as indicated in Fig. 20.5.

(a) The train travels at constant speed for t_2 seconds.
The total time taken is 120 s, i.e. $t_1 + t_2 + t_3 = 120$ s.

The area under the graph represents the distance travelled, i.e. area of trapezium OABC represents the distance between station X and station Y or 2000 m.

Therefore $\frac{1}{2}(t_2 + 120) \times 20 = 2000$

$$t_2 + 120 = 200$$
$$t_2 = 80$$

Hence the train travels at constant speed for 80 s.

(b) Let the acceleration be a m s^{-2}.
Then the retardation is $2a$ m s^{-2}.
The gradient of OA gives the acceleration

$$\frac{20}{t_1} = a$$

or $\quad t_1 = \frac{20}{a}$

Similarly, retardation is given by

$$\frac{20}{t_3} = 2a$$

or $\quad t_3 = \frac{10}{a}$

We know that the journey takes 120 s

i.e. $t_1 + t_2 + t_3 = 120$

so $\frac{20}{a} + 80 + \frac{10}{a} = 120$

which simplifies to $\frac{2}{a} + \frac{1}{a} = 4$ or $\frac{3}{a} = 4$

giving $a = \frac{3}{4}$.

Hence the acceleration is $\frac{3}{4}$ m s^{-2}.

Example 3

A car is travelling at a constant speed of 72 km h^{-1} and passes a stationary police car. The police car immediately gives chase, accelerating uniformly to reach a speed of 90 km h^{-1} in 10 s and continues at this speed until he overtakes the other car. Find

(a) the time taken by the police to catch up with the car,

(b) the distance travelled by the police car when this happens.

The v–t graphs of the car and the police car are shown in Fig. 20.6. The constant speed of the car is represented by the straight line PQ. OAB is the graph of the speed of the police car.

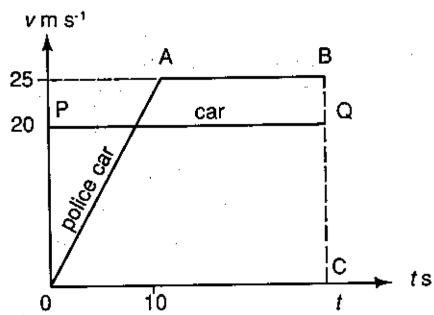


Fig. 20.6

We work in m and s, so 72 km h^{-1} = 20 m s^{-1} and 90 km h^{-1} = 25 m s^{-1}.

(a) Let the time taken by the police car to overtake the other car be t s.
When this happens, both cars would have covered the same distance, therefore area of OABC = area of OPQC

i.e. $\frac{1}{2}[t + (t - 10)] \times 25 = 20 \times t$

Verify that this gives $t = 25$ s.
Hence the police car takes 25 s to catch up with the car.

(b) The distance travelled by the police car is the same as the distance covered by the other car, and this is $20 \times 25 = 500$ m.

Exercise 20.1 *(Answers on page 647.)*

1 A car starts from rest, accelerates at 0.8 m s^{-2} for 10 s and then continues at a steady speed for a further 20 s. Draw the v–t graph and find the total distance travelled.

2 A car starts from rest, accelerating at 1 m s^{-2} for 10 s. It then continues at a steady speed for a further 20 s and decelerates to rest in 5 s. Find (a) the distance travelled in m, (b) the average speed in m s^{-1} and (c) the time taken to cover half the distance.

3 Two cars start from the same place. One accelerates at 1 m s^{-2} for 10 s, the other accelerates at 0.8 m s^{-2} for 20 s. Both cars continue with the speed then reached. How long after the start will the second car overtake the first and in what distance?

4 A car accelerates from rest to reach a certain speed in 10 min. It then continues at this speed for another 10 min and decelerates to rest in a further 5 min. The total distance covered is 17.5 km. Find the steady speed reached.

5 Two trains A and B, starting together from rest, arrive together at rest 10 min later. Train A accelerates uniformly at 0.125 m s^{-2} for 2 min, continues at the steady speed reached for another 4 min and then retards uniformly to rest. Train B accelerates uniformly for 5 min and then retards uniformly to rest.
Draw both journeys on the same v–t graph and find
(a) the distance (in m) travelled,
(b) the acceleration of train B,
(c) the distance between the two trains after 3 min.

6 A car travelling at a constant velocity of 20 m s^{-1} passes a stationary sports car. Ten seconds afterwards the sports car accelerates uniformly at 3 m s^{-2} to reach a speed of 30 m s^{-1} with which it continues. Draw the v–t graphs of both cars together and find when and where the sports car overtakes the first car.

7 In rising from rest to rest in 8 s, a lift accelerates uniformly to its maximum speed and then retards uniformly. The retardation is one-third the acceleration and the distance travelled is 20 m. Find the acceleration, the retardation and the maximum speed reached.

8 An electric train takes 3 min to travel between two stations 2970 m apart. The train accelerates uniformly to a speed of 18 m s^{-1} and then travels for a time at this speed before retarding uniformly to rest at the second station.
If the acceleration and retardation are in the ratio 2:3, calculate the times for which the train was accelerating and travelling at steady speed.

9 An electric train accelerates uniformly from rest to a speed of 20 m s^{-1} which it maintains until the brakes are applied. It is then brought to rest by a uniform retardation equal in magnitude to twice its former acceleration. The total distance covered is 7.8 km and the total time taken is 7 minutes. Sketch a velocity-time diagram. Calculate
(a) the time for which the train is travelling at constant speed,
(b) the initial acceleration in m s^{-2}. (C)

10 A cyclist starting from rest accelerates uniformly to his maximum speed of 10 m s^{-1} which he then maintains for the next 3 km. He then applies his brakes and decelerates to rest at a rate numerically equal to four times his previous acceleration. Sketch a velocity-time diagram.
Given that the total distance travelled by the cyclist is 3.1 km, calculate
(a) the time during which acceleration takes place,
(b) the time during which deceleration takes place,
(c) the total time for which the cyclist is in motion. (C)

11 A car accelerates uniformly from rest to reach a speed of V m s^{-1} in 5 seconds. It travels at this speed for 20 seconds and then decelerates uniformly to come to rest in a further t seconds.
(a) Sketch the v–t graph for the motion.
(b) If the distance travelled while decelerating is $\frac{4}{5}$ of the distance travelled while accelerating, find the value of t.
(c) Given that the total distance travelled was 637 m, find the value of V.

12 A train moves with a constant speed of 20 m s^{-1} for 10 seconds. It then accelerates uniformly during the next 5 seconds to reach a speed of V m s^{-1} when it decelerates uniformly to rest in the next 4 seconds.
(a) Sketch a v–t graph for the journey.
(b) If the total distance travelled was 385 m, find the value of V.

13 Due to track repairs, an electric train has to decelerate uniformly from a speed of 20 m s^{-1} to a speed of 10 m s^{-1}. It then travels at this speed for a certain time after which it accelerates uniformly at half the magnitude of the rate of deceleration to reach a speed of 30 m s^{-1}.
(a) Sketch a v–t graph.
(b) Given that the total distance travelled was 3100 m and that the total time taken was 220 seconds, calculate the times during which the train was decelerating and accelerating.

14 A train travels from station A to station B. It accelerates uniformly from rest at A to reach a maximum speed of 90 km h^{-1} in 30 seconds, then travels at this speed for 190 seconds, after which it slows down uniformly to come to a stop at B. The rate of deceleration is $1\frac{1}{2}$ times the magnitude of the rate of acceleration. Calculate
(a) the rate of acceleration in m s^{-2},
(b) the distance (in km) between the stations.

CONSTANT ACCELERATION

Let us consider the special case of motion in a straight line when the acceleration (or retardation) is constant. We can then derive a set of equations which will be useful in solving problems connected with constant acceleration. Suppose a body, moving with constant acceleration a, has initial velocity u. At time t, let its displacement be s and its velocity be v (all in consistent units). The v–t graph is shown in Fig. 20.7.

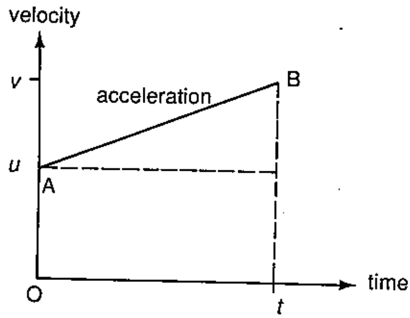


Fig. 20.7

1 Acceleration = gradient of AB

i.e. $a = \frac{v - u}{t}$

or $at = v - u$

Rearranging,

$$v = u + at \qquad \text{(i)}$$

2 The area under the graph represents the displacement.
Hence

$$s = \frac{1}{2}(u + v) \times t \qquad \text{(ii)}$$

3 Substituting for v in equation (ii),

$s = \frac{1}{2}(u + u + at) \times t$

Simplifying,

$$s = ut + \frac{1}{2}at^2 \qquad \text{(iii)}$$

4 From equation (i), $t = \left(\frac{v - u}{a}\right)$

Substituting for t in equation (ii),

$s = \frac{1}{2}(u + v)\left(\frac{v - u}{a}\right)$

i.e. $2as = v^2 - u^2$

or

$$v^2 = u^2 + 2as \qquad \text{(iv)}$$

These four equations are applicable *only* to a body travelling in a straight line with constant acceleration (or retardation in which case the value of a would be negative). They can be used to solve problems instead of using a v–t graph or in conjunction with it. Solutions will often involve solving simultaneous equations.

Example 4

A particle travelling in a straight line with constant acceleration 4 m s^{-2} passes a point O when its velocity is 12 m s^{-1}. It passes another point P after a further 3 s. Find the velocity of the particle at P and the distance OP.

Let us consider the motion of the particle from the moment it passes O. Suppose the velocity of the particle at P is v m s^{-1} and the distance OP is s m.

Using the equation $v = u + at$,

we have
$$v = 12 + 4(3) = 24$$

Hence the velocity of the particle at P is 24 m s^{-1}.

Using the equation $s = \frac{1}{2}(u + v) \times t$

we have $s = \frac{1}{2}(12 + 24) \times 3 = 54$

Hence the distance OP is 54 m.

Example 5

A particle, moving in a straight line with constant acceleration, travels 10 m in the first second and 15 m in the second second. Find

(a) its initial velocity,

(b) its acceleration,

(c) the distance travelled in the third second.

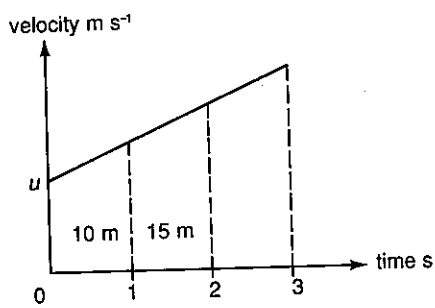


Fig. 20.8

The v–t graph is shown in Fig. 20.8.

Let the initial velocity be u m s^{-1} and the acceleration be a m s^{-2}.

Consider motion in the first second.

Using the equation $s = ut + \frac{1}{2}at^2$,

we have $10 = u + \frac{1}{2}a$

or $2u + a = 20$ (i)

Similarly, consider motion in the first two seconds.

We have $25 = u \times 2 + \frac{1}{2} \times a \times 2^2$

or $2u + 2a = 25$ (ii)

Solving equations (i) and (ii), we obtain
$u = 7.5$ and $a = 5$

Hence (a) the initial velocity is 7.5 m s^{-1}
and (b) the acceleration is 5 m s^{-2}.

(c) The distance travelled in the first three seconds is s m given by

$s = 7.5 \times 3 + \frac{1}{2} \times 5 \times 3^2 = 45$

Hence the distance travelled in the third second
= distance travelled in 1st 3 seconds – distance travelled in 1st 2 seconds
= (45 – 25) m
= 20 m

Example 6

A bus leaves a bus stop and accelerates uniformly for 10 s over a distance of 100 m. It then moves uniformly with the speed it has attained for 30 s and finally retards uniformly to rest at the next stop. If the two bus stops are 1 km apart, find
(a) the maximum velocity,
(b) the acceleration,
(c) the total time taken between the two stops.

The v–t graph is shown in Fig. 20.9. Consider motion in the first 10 s.

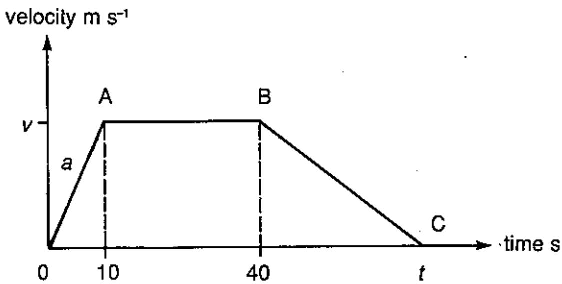


Fig.20.9

(a) Using the equation $s = \frac{1}{2}(u + v) \times t$, the maximum speed v m s^{-1} is given by $100 = \frac{1}{2}(0 + v) \times 10$ giving $v = 20$.

Hence the maximum speed is 20 m s^{-1}.

(b) Using the equation $v = u + at$, the acceleration a m s^{-2} is given by $20 = 0 + a \times 10$ giving $a = 2$.
Hence the acceleration is 2 m s^{-2}.

(c) Let the total time taken be t s.
Area of OABC represents the distance travelled which is 1000 m.
i.e. $\frac{1}{2}(30 + t) \times 20 = 1000$ giving $t = 70$.
Hence the total time taken is 70 s.

Example 7

Two particles P and Q are moving on the same horizontal line towards each other. P passes a point A on the line with speed 8 m s^{-1} and constant acceleration $\frac{3}{4}$ m s^{-2}. Simultaneously, Q passes a point B on the line with speed 4 m s^{-1} and constant acceleration $\frac{5}{4}$ m s^{-2}. Given that the distance AB is 64 m, calculate
(a) the time taken for the particles to collide,
(b) the distance from A of the point of collision.

(a) Let the time taken be t seconds.

Using the formula $s = ut + \frac{1}{2}at^2$, P will travel a distance of $8t + \frac{1}{2} \times \frac{3}{4}t^2$ m and Q a distance of $4t + \frac{1}{2} \times \frac{5}{4}t^2$ m.

These distances total 64 m.

Then $8t + \frac{3}{8}t^2 + 4t + \frac{5}{8}t^2 = 64$ i.e. $t^2 + 12t - 64 = 0$.

Hence $(t + 16)(t - 4) = 0$ giving $t = -16$ (not admissible) or $t = 4$.
So they collide after 4 seconds.

(b) The distance from A of the point of collision is then

$8 \times 4 + \frac{3}{8} \times 16 = 38$ m.

Exercise 20.2 *(Answers on page 648.)*

(All accelerations are to be taken as uniform and in a straight line.)

1 A particle starts with velocity 3 m s^{-1} and accelerates at 0.5 m s^{-2}. What is its velocity after (a) 3 s, (b) 10 s, (c) t s? How far has it travelled in these times?

2 A body, decelerating at 0.8 m s^{-2}, passes a certain point with a speed of 30 m s^{-1}. Find its velocity after 10 s, the distance covered in that time and how much further the body will go until it stops.

3 A particle travelling with an acceleration of 0.75 m s^{-2} passes a point O with speed 5 m s^{-1}. How long will it take to cover a distance of 250 m from O? What will its speed be at that time?

4 If a particle passes a certain point with speed 5 m s^{-1} and is accelerating at 3 m s^{-2}, how far will it travel in the next 2 s? How long will it take (from the start) to travel 44 m?

5 A car, retarding uniformly, passes over three cables, P, Q and R set at right angles to the path of the car and 11 m apart. It takes 1 s between P and Q and 1.2 s between Q and R. Find
(a) its retardation, (b) its velocity when it crosses P,
(c) its distance beyond R when it comes to rest. (C)

6 A car, slowing down with uniform retardation, passes a stationary man. 20 s after passing the man the car is 40 m from him and 30 s after passing him it is 50 m away. Find the speed of the car as it passed the man. Find the distance from the man where the car comes to rest. Draw a sketch of the v–t graph.

7 A body passes a certain point A of the straight line on which it is moving with uniform acceleration. One second afterwards it is 11 m beyond A and in the next second it travels a further 13 m. Find the velocity it had when passing A, and how far it travels in the third second after passing A.

8 A particle starting from rest moves with constant acceleration $x \text{ m s}^{-2}$ for 10 s; travels with constant velocity for a further 10 s, and then retards at $2x \text{ m s}^{-2}$ to come to rest 300 m from its starting point. Find the value of x. (C)

9 A body is travelling along a straight line with constant acceleration $a \text{ m s}^{-2}$. The body passes a fixed point O on the line with velocity $u \text{ m s}^{-1}$. Between 4 s and 5 s after passing O, the body travels 10 m. Between 6 s and 7 s after passing O, the body travels 12 m. Calculate the values of u and a. (C)

10 A body travelling in a straight line with a uniform acceleration of $a \text{ m s}^{-2}$ passes a point O with a velocity of $u \text{ m s}^{-1}$. During the first 5 seconds after passing O, it travels 45 m and during the next second it travels a further 15 m. Calculate the value of a and of u.
Calculate the velocity of the body when it is 140 m from O and the time taken to reach this point. (C)

11 Two particles A and B are moving in the same direction on parallel horizontal lines. They pass a certain point O at the same time, A moving with a speed of 3 m s^{-1} and constant acceleration 0.5 m s^{-2} and B moving with a speed of 4 m s^{-1} and constant acceleration 0.2 m s^{-2}.
If the speeds of A and B are equal after t seconds, calculate
(a) the value of t,
(b) the distance between the particles at that time.

12 Two particles P and Q are moving on the same horizontal straight line towards each other. P passes a point A on the line with speed 4 m s^{-1} and uniform acceleration 1 m s^{-2}. At the same instant, Q passes a point B on the line with speed 6 m s^{-1} and uniform deceleration 0.5 m s^{-2}. If the distance AB is 44 m, calculate
(a) the time taken for the particles to collide,
(b) the distance from A of the point of collision.

13 When $t = 0$ (where t denotes time in seconds), a particle A moves from a point O along a straight line with initial velocity u m s^{-1} and constant acceleration a m s^{-2}. When $t = 4$, a particle B moves from O along the same straight line with initial velocity $\frac{1}{2}u$ m s^{-1} and constant acceleration $2a$ m s^{-2}.
Given that, when $t = 16$, A is ahead of B, obtain, in terms of u and a, an expression for the distance between the particles at that time.
Given also that this distance is 12 m, and that the velocity of A when $t = 16$ is 10 m s^{-1}, calculate (i) the value of u and of a. Hence calculate, when $t = 18$, (ii) the distance between the particles, (iii) the difference between their velocities. (C)

14 Particles A and B are moving along horizontal straight lines which lie in the west–east direction. At a certain time, A passes a point P moving to the east with speed 5 m s^{-1} and constant acceleration 1 m s^{-2}.
Simultaneously, B passes a point Q east of P with speed 4 m s^{-1} towards the west but with constant acceleration 2 m s^{-2} towards the east.
Given that the distance PQ is 28 m, calculate
(a) the distance of A from P when B reverses direction,
(b) the time taken after the start for A to overtake B,
(c) the further time taken for B to overtake A.

15 A particle A moves along a horizontal straight line with a steady speed of 10 m s^{-1}. At a certain instant, another particle B is at rest 100 m ahead of A and starts moving away from A with a constant acceleration of a m s^{-2}.
(a) Show that the shortest distance between A and B occurs when they have the same speed.
(b) Hence find an expression for the shortest distance between A and B in terms of a and find a if the shortest distance is 25 m.

VERTICAL MOTION UNDER GRAVITY

An important case of motion in a straight line with uniform acceleration is that of a body moving in space under the influence of a gravitational force. This will be discussed further in Chapter 24. For the present, the following facts may be stated.

1 The earth (or moon or any planet) attracts bodies near it. This force of attraction is called the gravitational force. It produces an acceleration towards the centre of the earth in any body free to move.

2 This acceleration due to gravity depends on the distance from the centre of the earth. Near the surface, it is about 9.8 m s^{-2} ($\approx$ 10 m s^{-2}). It is slightly less near the equator than near the poles, as the earth is not a perfect sphere. (*Note:* On the surface of the moon, the acceleration due to the gravity of the moon is about 1.62 m s^{-2}.)

3 The symbol g is used to denote the acceleration due to gravity. At the same place, g is the same for all bodies. Ignoring air resistance, a piece of paper and a stone would each fall with the same acceleration g, and theoretically, if they are released simultaneously from the same height, they would hit the ground at the same time.

In our work, we will ignore the effect of air resistance. We will also take g as 10 m s^{-2} unless otherwise stated.

The positive direction of motion must first be stated very clearly so that all vector quantities can take their proper signs. In Example 9, since the upward direction is positive, g is negative and so is the displacement s below 0.

Example 8

A stone is dropped from the top of a building of height 20 m. Find the time it takes to reach the ground and the velocity with which it hits the ground.

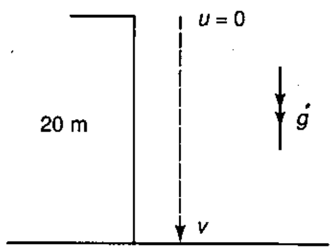


Fig. 20.10

The initial velocity is $u = 0$.

Let the time taken be t s and the velocity with which it hits the ground be v m s^{-1}.

The positive direction is taken to be vertically downwards (Fig. 20.10) and the acceleration is 10 m s^{-2}.

Using the equation $s = ut + \frac{1}{2}at^2$,

we have $20 = 0 + \frac{1}{2} \times 10 \times t^2$

giving $t^2 = 4$

i.e. $t = 2$ (only the positive value is taken, as the negative value is meaningless)

Hence the time taken is 2 s.

Using the equation $v = u + at$,

we have $v = 0 + 10 \times 2 = 20$

Hence the velocity with which it strikes the ground is 20 m s^{-1}.

Example 9

A particle is projected vertically upwards with a velocity of 30 m s⁻¹ from a point O. Find

(a) the maximum height reached,

(b) the time taken for it to return to O,

(c) the time taken for it to be 35 m below 0.

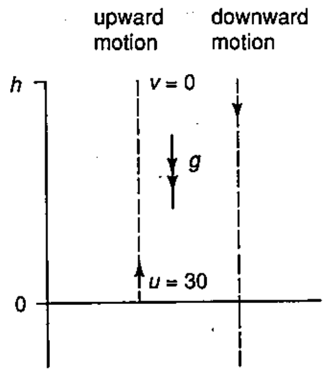


Fig. 20.11

Take the positive direction as vertically upwards (Fig. 20.11).
The initial velocity u is 30 m s^{-1}.

(a) At the maximum height, h m, the velocity v is 0.
The acceleration a is -10 m s^{-2}.

Using the equation $v^2 = u^2 + 2as$,

we have $0 = 30^2 + 2 \times (-10) \times h$

giving $h = 45$

Hence the maximum height reached is 45 m.

(b) When the particle returns to O, the displacement s is 0.

Using the equation $s = ut + \frac{1}{2}at^2$,

we have $0 = 30 \times t + \frac{1}{2} \times (-10) \times t^2$

i.e. $5t^2 - 30t = 0$

giving $t = 0$ (at start)

and $t = 6$ (when particle returns to O)

Hence the time taken for it to return to O is 6 s.

(c) When the particle is 35 m below O, the displacement s is -35 m.

Again using the equation $s = ut + \frac{1}{2}at^2$,

we have $-35 = 30 \times t + \frac{1}{2} \times (-10) \times t^2$

Simplifying, $t^2 - 6t - 7 = 0$

or $(t + 1)(t - 7) = 0$

giving $t = -1$ (not admissible) and $t = 7$.

Hence the particle will be 35 m below O after 7 s.

Example 10

A particle P is projected vertically upwards from O with velocity 40 m s⁻¹. One second later, another particle Q is projected from O with the same vertical velocity. After what time and at what height will the two particles collide?

Let the time of collision be t s after the projection of P.
This will be the same as $(t - 1)$s after the projection of Q as Q is projected 1 s later.

Let P and Q collide at height h m above O.
Take the positive direction as vertically upwards.

Consider the motion of P.

At the time of collision, the displacement is h m.

Using the equation $s = ut + \frac{1}{2}at^2$,

we have $h = 40 \times t + \frac{1}{2} \times (-10) \times t^2$

which gives $h = 40t - 5t^2$ (i)

Similarly, for the motion of Q,

we have $h = 40(t - 1) + \frac{1}{2} \times (-10) \times (t - 1)^2$

$= 40(t - 1) - 5(t - 1)^2$

or $h = 50t - 5t^2 - 45$ (ii)

Solving equations (i) and (ii), we obtain $t = 4.5$ and $h = 78.75$.

Hence the particles collide 4.5 s after the projection of P at a height of 78.75 m.

Exercise 20.3 *(Answers on page 648.)*

1 A ball is thrown upwards with a velocity of 12 m s⁻¹. How high does it reach and how long does it take to get there?

2 A particle, thrown vertically upwards, reaches a height of 5 m. What was its initial velocity?

3 From the top of a cliff 50 m above sea level, a stone is thrown vertically upwards with a velocity of 15 m s⁻¹. After how many seconds will it hit the sea and with what velocity?

4 A ball is dropped from a height of 40 m. Simultaneously and vertically below it, a second ball is thrown vertically upwards with a velocity of 20 m s^{-1}. At what distance from the ground do they collide?

5 A stone is thrown vertically upwards with a speed of 25 m s^{-1}. At what times is it 5 m above the ground?

6 A ball is dropped from a height of 5 m onto a concrete floor and rebounds with a speed 0.8 times the downward speed on arrival. Find the height reached from the rebound.

7 A stone is thrown vertically upwards with a speed of 10 m s^{-1}. One second later a second stone is thrown vertically upwards with a speed of 20 m s^{-1}. At what height above the ground do they collide?

8 Two rockets are fired vertically from launching pads side by side. The first rocket moves vertically upwards with an acceleration of $6g$ and the second with an acceleration of $8g$. If the second rocket is fired 1 s after the first, find how long after its launching the second rocket overtakes the first. (C)

9 From the top of a vertical tower 246 m high, a stone is projected vertically downwards with a speed of 8 m s^{-1}. After x seconds, another stone is projected vertically upwards from the level of the base of the tower with a speed of 25 m s^{-1}. Given that the two stones are first at the same height 6 seconds after the projection of the first stone, calculate

(a) the value of x;
(b) the velocity of the second stone at this instant. (C)

10 A particle is projected vertically upwards and is at a height of 20 m after 4 seconds. Calculate

(a) its initial velocity,
(b) the maximum height reached.
(c) Find in which direction the particle is moving after 4 seconds.

11 A rocket is fired vertically upwards from the ground. After 1 second it has reached a height of 15 m. Calculate

(a) its initial velocity,
(b) the time taken to reach its greatest height,
(c) the greatest height reached,
(d) the length of time during which the rocket is higher than 15 m.

12 When a balloon is at a height of 20 m above the ground, it is rising vertically with a speed of 5 m s^{-1} and constant retardation. It comes to rest 60 seconds later. Calculate its maximum height above the ground.
When the balloon is at rest, a stone is dropped from it and falls freely to the ground. Find the speed with which it reaches the ground.

13 Two missiles A and B are each fired vertically upwards from ground level. The initial speed of A is 3 times that of B. After 2 seconds, both missiles are moving upwards and A is 80 m higher than B. Calculate

(a) the initial speeds of A and B,
(b) how much higher A will rise than B,
(c) the height of A when B reaches its maximum height.

14 A stone is dropped from the top of a tower 125 m high. When it has fallen 20 m, a second stone is thrown vertically downwards with speed V m s^{-1} from the top of the tower. If the two stones reach the ground at the same time, calculate the value of V.

15 A helicopter, initially at rest on the ground, rises vertically with constant acceleration. When it is at a height of 60 m, its upward speed is 5 m s^{-1}. When it is at a height of 240 m, and still rising, an object A is released from the helicopter.
Calculate (i) the initial velocity of A, (ii) the time that A takes to reach the ground. After A is released, the helicopter continues to rise with a different constant acceleration. When it is at a height of 350 m and rising with a speed of 15 m s^{-1}, a second object B is released. (iii) Show that B takes 10 seconds to reach the ground. (iv) Find the time that elapses between the impacts of A and B on the ground. (C)

SUMMARY

- For an s–t graph, gradient of graph represents velocity.
- For a v–t graph, gradient of graph represents acceleration;
  area under graph represents displacement.
- Equations of motion with constant acceleration:

$$v = u + at$$
$$s = \frac{1}{2}(u + v)t$$
$$s = ut + \frac{1}{2}at^2$$
$$v^2 = u^2 + 2as$$

where s = displacement, u = initial velocity, v = velocity at time t, a = acceleration.

- For motion under gravity, acceleration is g towards the centre of the earth ($g \approx 10$ m s^{-2}); and the 4 equations of motion are applicable.

REVISION EXERCISE 20 *(Answers on page 648.)*

1 From a point 120 m above ground level, a stone is projected vertically upwards with a speed of u m s^{-1}. If the stone rises to a height of 5 m above the point of projection, calculate
(a) the value of u,
(b) the time the stone takes from projection until it reaches ground level,
(c) the speed of the stone at ground level. (C)

2 Two particles A and B are moving in the same direction on parallel horizontal tracks. At a certain point the particle A, travelling with a speed of 7 m s^{-1} and accelerating uniformly at 1.5 m s^{-2} overtakes B travelling at 3 m s^{-1} and accelerating uniformly at 2.5 m s^{-2}. Calculate the period of time which elapses before B overtakes A.
If, after this time, B then ceases to accelerate and continues at constant speed, calculate the time taken for A to overtake B again.

3 A particle is projected vertically upwards from the ground with a speed of 36 m s^{-1}. Calculate
(a) the time for which it is above a height of 63 m,
(b) the speed which it has at this height on its way down,
(c) the total time of flight. (C)

4 From the foot of a vertical cliff 28.8 m high, a stone was projected vertically upwards so as just to reach the top. Find its velocity of projection.
One second after the first stone was projected, another stone was allowed to fall from rest from the top of the cliff. The stones passed one another after a further t seconds at a height h m above the ground. Calculate the value of t and of h. (C)

5 Two particles, X and Y, are moving in the same direction on parallel horizontal tracks. At a certain point O, the particle X, travelling with a speed of 16 m s^{-1} and retarding uniformly at 6 m s^{-2}, overtakes Y, which is travelling at 8 m s^{-1} and accelerating uniformly at 2 m s^{-2}.
Calculate
(a) the distance of Y from O when the velocities of X and Y are equal,
(b) the velocity of X when Y overtakes X. (C)

6 A particle X is projected vertically upwards from the ground with a velocity of 80 m s^{-1}. Calculate the maximum height reached by X.
A particle Y is held at a height of 300 m above the ground. At the moment when X has dropped 80 m from its maximum height, Y is projected downwards with a velocity of v m s^{-1}. The particles reach the ground at the same time. Calculate the value of v. (C)

7 A man running a 100 m race accelerates uniformly from rest for the first T seconds and reaches a velocity of 10 m s^{-1}. He maintains this velocity for the rest of the race. His time for the race is 12 s. Sketch a velocity–time graph.
Calculate
(a) the value of T,
(b) the acceleration,
(c) the distance the man runs before he reaches his maximum speed. (C)

8 A car moves along a straight level road, accelerating from rest at a constant rate for 9.6 s over a distance of S_1 m until it reaches a speed of V m s^{-1}. Express S_1 in terms of V.
It then accelerates at a constant rate of 2.5 m s^{-2} over a distance of S_2 m until it reaches a speed of 25 m s^{-1}. Express S_2 in terms of V.
Given that the car has now travelled a total distance of 152 m, calculate the possible values of V. Using the smaller of these values, calculate the time taken to attain a speed of 25 m s^{-1} from rest. (C)

9 A motorcyclist travelling along a straight road passes a fixed point O with a speed of 20 m s^{-1} and continues at this speed for t_1 seconds. Over the next t_2 seconds he accelerates at a constant rate to a speed of 30 m s^{-1}. He then brings the motorcycle to rest in a further t_3 seconds by retarding at a constant rate. His acceleration and retardation are of equal magnitude.

(a) Sketch a velocity–time graph to illustrate the motion of the motorcyclist after passing O.

(b) Obtain an equation connecting t_2 and t_3.

(c) Given that the total distance and the total time represented by the graph are 748 m and 40 s respectively, calculate t_1, t_2 and t_3. (C)

10 During a certain stage of its journey, a train decelerates uniformly from a speed of 25 m s^{-1} to a speed of 15 m s^{-1} which it maintains for a time before accelerating uniformly to its former speed of 25 m s^{-1}. Sketch a velocity-time graph to illustrate this stage.

Given that, for this stage of the journey, the total distance travelled is 12 000 m, the total time taken is 720 seconds and the magnitude of the acceleration is twice that of the deceleration, calculate

(a) the time during which the train is accelerating,

(b) the speed of the train 180 seconds after the start of this stage,

(c) the speed of the train 660 seconds after the start of this stage. (C)

11 A ball is projected vertically upwards with speed u m s^{-1} from ground level. Between 3 and 4 seconds after leaving the ground, it rises 10 m.

Calculate

(a) the value of u,

(b) the maximum height reached by the ball.

12 A stone is projected vertically upwards from the top of a building 80 m high with a speed of 30 m s^{-1}. Calculate

(a) the height it reaches above the top of the building,

(b) the time it takes to reach the ground.

A second stone is dropped from the top of the building t seconds after the first one was projected. Both stones reach the ground at the same time.

(c) Find the value of t.

13 A car starts from rest and accelerates at 1 m s^{-2} to reach a speed of 90 km h^{-1}. It maintains this speed for 5 minutes and then slows down uniformly to a speed of 36 km h^{-1} while covering a distance of 350 m. It then continues at this speed for 10 minutes and is brought to rest with uniform deceleration in a further 15 seconds. Sketch a v–t graph of the motion and hence or otherwise calculate

(a) the total time in minutes taken over the journey,

(b) the total distance travelled in km correct to 3 significant figures.

14 A stone is dropped from a height of 40 m and at the same time another stone is thrown vertically upwards with speed u m s^{-1} from a point on the ground directly below the first stone. The stones collide at the moment when the second stone has reached its maximum height. Calculate

(a) the value of u,

(b) the height above the ground where the stones collide.

15 A straight horizontal track joins two stations A and B which are a distance D km apart.
A train starts from rest at A, accelerates at a constant rate over a distance $\frac{D}{4}$ km before reaching a velocity V km h^{-1} which it maintains for a further distance $\frac{D}{2}$ km before retarding at a constant rate and coming to rest at B.
A second train completes the journey from A to B in T hours by starting from rest at A, accelerating at a constant rate for $\frac{T}{4}$ hours reaching the velocity V km h^{-1} which it maintains for $\frac{T}{2}$ hours before retarding at a constant rate and coming to rest at B.
For each train, calculate the average velocity for the journey and state which train completes the journey in the shorter time. (C)

16 A car travels a distance of 46.8 km in 32 minutes. First it accelerates uniformly from rest to reach a speed of v m s^{-1}, then continues at this speed for 20 minutes and finally retards uniformly to rest. Calculate the value of v in m s^{-1}.

Given also that the magnitude of the retardation is half that of the acceleration, find the acceleration in m s^{-2}.

17 Two particles P and Q are 45 m apart on a smooth horizontal plane. P starts moving towards Q with speed 3 m s^{-1} and constant acceleration 0.5 m s^{-2}. At the same instant, Q starts moving towards P in the same straight line with speed 6 m s^{-1} and constant retardation a m s^{-2}. If they meet when Q has just come to momentary rest, calculate the time taken for the particles to meet.

18 A stone is dropped from a height of h m and falls freely to the ground. It rebounds with $\frac{2}{3}$ of the speed at impact. If it hits the ground again after 2 seconds, calculate the value of h.

19 Particles A and B are moving in the same direction along parallel horizontal lines. At time $t = 0$, A, travelling with speed 6 m s^{-1} and constant acceleration 1.6 m s^{-2}, overtakes B travelling with speed 4 m s^{-1} and constant acceleration 2 m s^{-2}. Calculate

(a) the value of t when B overtakes A,

(b) the speeds of A and B at that instant.

(c) If at that time, the acceleration of B is reduced to 1 m s^{-2}, after what further time will A overtake B?

20 A helicopter, initially at rest on the ground, rises vertically with constant acceleration. When it is at a height of 60 m its upward speed is 5 m s^{-1}. When it is at a height of 240 m, and still rising, an object A is released from the helicopter. Calculate

(a) the initial velocity of A,

(b) the time that A takes to reach the ground.

After A is released the helicopter continues to rise with a different constant acceleration. When it is at a height of 350 m and rising with a speed of 15 m s^{-1}, a second object B is released.

(c) Show that B takes 10 seconds to reach the ground.

(d) Find the time that elapses between the impacts of A and of B on the ground. (C)

Velocity and Relative Velocity | 21

In Chapter 20, we discussed motion of a body in a straight line. Displacement, velocity and acceleration (all vectors) were represented by directed numbers. This was possible as the directed number gave the magnitude and direction, the latter being defined by the straight line in which motion took place. We will now consider motion of a body in a plane.

COMPOSITION OF VELOCITIES

A moving body can have two (or more) velocities at the same time. A simple example of a body having two velocities is the case of a man rowing a boat on a river. Suppose the man is on one side of a river bank at A (Fig. 21.1) and attempts to row in a direction perpendicular to the bank to head for B on the opposite bank. If there is a current, the boat will not move along AB at a right angle to the bank but will move along the path AC at an θ to AB. Thus the resultant path of the boat is determined by two velocities; one as a result of the man's rowing in the direction AB and the other as a result of the current of

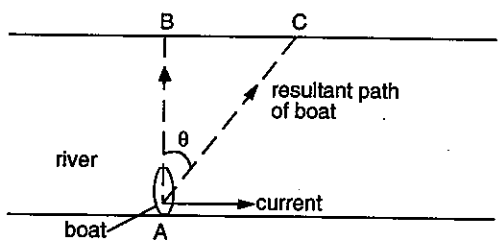


Fig. 21.1

the river downstream. As velocities are vectors, they obey the laws of vector algebra (see Chapter 8). The vector sum of the velocities is the **resultant velocity**. If a body has 2 velocities **a** and **b** as shown in Fig. 21.2, then the resultant velocity **c** = **a** + **b**.

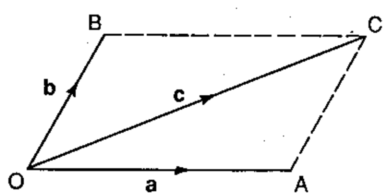


Fig. 21.2

c is represented by the diagonal of the parallelogram OACB. The velocities **a** and **b** are the **components** of the resultant velocity **c**. Hence a body that has velocities **a** and **b** will move as though it has the single velocity **c**.

Let us consider another case of a man rowing a boat on a river. Suppose he rows the boat at a velocity **a** and the river is flowing at velocity **b**. Then the resultant velocity of the boat is **c** = **a** + **b** (Fig. 21.3). Hence the actual path of the boat will be along OC.

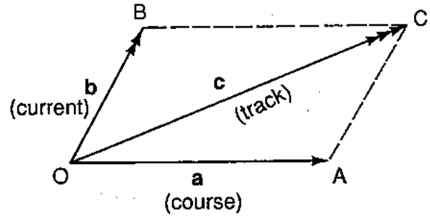


Fig. 21.3

This path is called the **track** of the boat and the actual speed along the track is called the **ground speed**. The direction of **a** is the **course** (which is the direction the boat would have moved if there were no current). In diagrams, it is helpful to mark vectors with different numbers of arrows. Thus in Fig. 21.3, we use the following notations:

course vector marked with 1 arrow $\longrightarrow$

current vector marked with 2 arrows $\longrightarrow\!\!>$

track vector marked with 3 arrows $\longrightarrow\!\!>\!\!>$

Then we have:

$$\underset{(1\text{ arrow})}{\overrightarrow{\text{course vector}}} + \underset{(2\text{ arrows})}{\overrightarrow{\text{current vector}}} = \underset{(3\text{ arrows})}{\overrightarrow{\text{track vector}}}$$

The track vector is the resultant of the other two vectors.

Another practical example is the case of an aircraft flying in a wind. The aircraft has a velocity **a** through the air (Fig. 21.4). Its direction is the course (like in the case of the boat) and its speed along the course is the **airspeed**. The aircraft is also affected by the wind with velocity **b**. As a result, the actual velocity **c** is the resultant of the velocity of the aircraft and that of the wind. Like the case of the boat, the actual path is called the **track** and the speed along the track is the **ground speed**. The same arrow notations are used, with the current velocity replaced by the wind velocity. It should be noted however that a current is described as moving *towards* a certain direction whereas a wind is described as coming *from* a certain direction so a current in the direction NE is moving in the direction 045° whereas a NE wind is actually moving in the direction 225°.

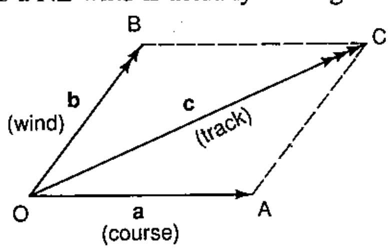


Fig. 21.4

Resolution of Velocities

Just as we can combine component velocities to obtain a resultant velocity, we can also resolve ('split') a velocity into its components. Particularly useful is the resolution of a velocity into 2 perpendicular components, usually in the directions of the x– and y–axes. In practical problems, it is convenient to resolve along the east and north directions respectively (in place of the x– and y–axes). In Fig. 21.5, the velocity **r** is resolved into its components **p** and **q** along the x– and y–directions respectively.

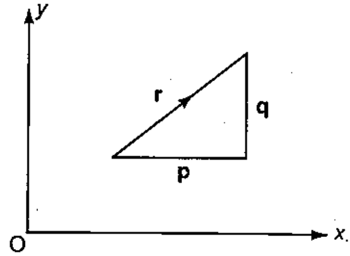


Fig. 21.5

We can also use **unit coordinate vectors i and j**, where **i** is the unit vector along the x-axis and **j** the unit vector along the y-axis. Thus we can represent **r** as $p\mathbf{i} + q\mathbf{j}$ where p and q are the magnitudes of **p** and **q** respectively (Fig. 21.6). So $\mathbf{r} = p\mathbf{i} + q\mathbf{j}$.

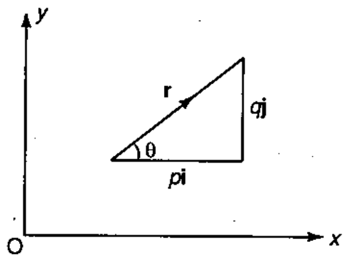


Fig. 21.6

Note that the magnitude of **r** is $r = \sqrt{p^2 + q^2}$. Also, if **r** makes an angle θ with the x-axis, then we have

$$p = r\cos\theta, \quad q = r\sin\theta$$

and $\tan\theta = \frac{q}{p}$.

Example 1

A river is flowing at 0.9 m s⁻¹. A man sets out, at right angles to the banks, to row across. He can row at 1.2 m s⁻¹ in still water.

(a) What is the actual velocity across the river?

(b) If the river is 600 m wide, how long does he take to cross the river?

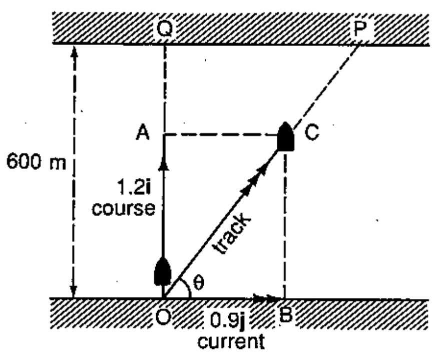


Fig. 21.7

(a) Relative to the water, the boat will move at 1.2 m s^{-1} perpendicular to the banks (Fig. 21.7); but the water, moving at 0.9 m s^{-1}, simultaneously carries the boat along parallel to the banks. The boat therefore moves crabwise across pointing at right angles to the bank. The resultant of the course vector ($\overrightarrow{OA}$) and the current vector ($\overrightarrow{OB}$) is the track vector ($\overrightarrow{OC}$). Hence the path of the boat is along OC. The actual velocity of the boat is represented by the resultant (track) vector and it has magnitude $\sqrt{0.9^2 + 1.2^2} = 1.5$ m s^{-1} at an angle θ to the bank where $\tan\theta = \frac{1.2}{0.9} = \frac{4}{3}$ ($\theta \approx 53°$).

(b) The boat moves along the track OC and will reach the opposite bank at P.

$$\text{The time taken} = \frac{\text{distance of OP}}{\text{speed along the track}}$$
$$= \frac{750}{1.5} \left(\text{using similar triangles, } \frac{OP}{1.5} = \frac{600}{1.2}\right)$$
$$= 500 \text{ s or 8 min 20 s.}$$

(*Note:* The time taken can also be found more simply by considering the course component of velocity, thus time taken = $\dfrac{\text{distance of OQ}}{\text{speed along course}} = \dfrac{600}{1.2} = 500$ s.)

Example 2

An aeroplane has an airspeed of 200 km h^{-1} and flies on a course of N30°E. A wind is blowing steadily at 50 km h^{-1} from the NW. Find, by drawing or calculation, the track and ground speed.

1 *By trigonometry*

In Fig. 21.8, $\overrightarrow{OC}$ is the course vector (1 arrow), $\overrightarrow{OW}$ the wind vector (2 arrows).
$\overrightarrow{OC} + \overrightarrow{OW} = \overrightarrow{OT}$, the track vector (3 arrows) which has speed v and $\angle COT = \theta$.
From $\triangle OCT$, using the cosine rule,

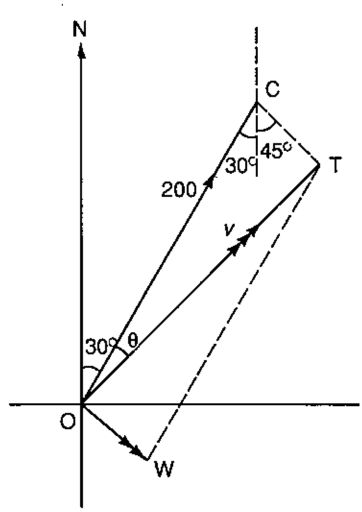
$v^2 = 200^2 + 50^2 - 2 \times 200 \times 50 \times \cos 75°$
giving $v \approx 193$.

Hence the ground speed is 193 km h^{-1}.

By the sine rule, $\frac{\sin\theta}{50} = \frac{\sin 75°}{193}$ giving $\theta \approx 14.5°$.

Hence the track is N44.5°E.

Fig. 21.8



2 *Using unit vectors*

Here we resolve the course vector $\overrightarrow{OC}$ and wind vector $\overrightarrow{OW}$ into components along perpendicular directions OC and OD respectively.

If **i** and **j** are unit vectors along $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively,

then $\overrightarrow{OC} = 200\mathbf{i}$

and $\overrightarrow{OW} = -(50 \sin 15°)\mathbf{i} + (50 \cos 15°)\mathbf{j}$
$= -12.94\mathbf{i} + 48.30\mathbf{j}$

Now $\overrightarrow{OT} = \overrightarrow{OC} + \overrightarrow{OW}$
$= 200\mathbf{i} - 12.94\mathbf{i} + 48.30\mathbf{j}$
$= 187.1\mathbf{i} + 48.30\mathbf{j}$

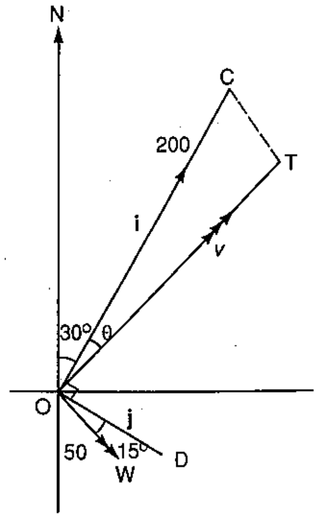
Therefore $v = |\overrightarrow{OT}| \approx 193$

Hence the ground speed is 193 km h^{-1}.

Also $\tan\theta = \frac{48.30}{187.1}$
$\theta \approx 14.5°$

Hence the track is N44.5°E.

Fig. 21.9



3 *By drawing*

This method can only be used if the question specifically allows it.
To be satisfactory, drawings must be made carefully.
Draw a north line ON as in Fig. 21.8.
Now select a suitable scale — as large as possible to ensure reasonably accurate results, say 1 cm for 10 km h^{-1}.
From O, draw OC 20 cm long with $\angle NOC = 30°$.
Draw OW 5 cm long with $\angle NOW = 135°$.
Complete the parallelogram OWTC.
Measure OT and $\angle NOT$ to obtain the results and compare with the calculated values above.

Example 3

A helicopter leaves an airfield A to fly to another airfield B 500 km away on a bearing of 140°. There is a steady wind of 30 km h^{-1} from the NE. The helicopter has an airspeed of 150 km h^{-1}.

(a) Find the course the pilot must take and the time taken for him to reach B.

(b) What course should be taken for the return journey, if the wind continues as before?

(a) 1 *By trigonometry*

The facts are shown in Fig. 21.10; v is the ground speed.

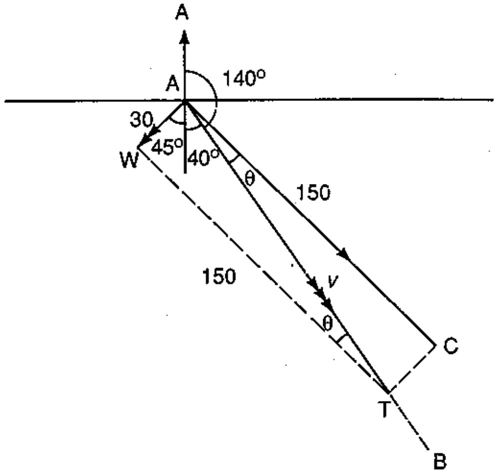


Fig. 21.10

From $\triangle$AWT, using the sine rule,

$\frac{\sin\theta}{30} = \frac{\sin 85°}{150}$ giving $\theta \approx 11.5°$

Hence the course to take is 128.5°.

$\angle AWT = 180° - (85° + 11.5°) = 83.5°$

Using the sine rule again,

$\frac{v}{\sin 83.5°} = \frac{150}{\sin 85°}$ giving $v \approx 149.6$.

Hence the time taken $= \frac{500}{149.6} \approx 3.34$ h $\approx$ 3 h 21 min.

2 *Using unit vectors*

Take **i** along the track $\overrightarrow{AT}$ and **j** along $\overrightarrow{AU}$ (Fig. 21.11).

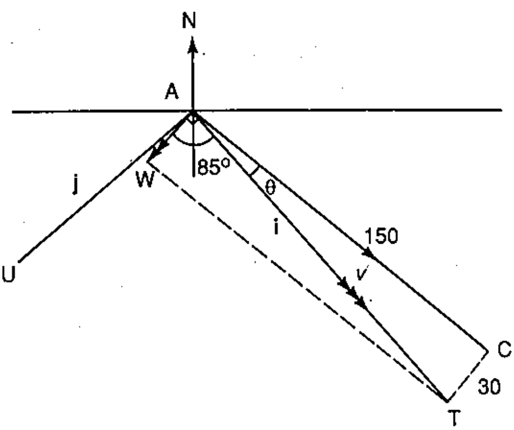


Fig. 21.11

$\overrightarrow{AT} = v\mathbf{i}$ (no $\mathbf{j}$ component)

$\overrightarrow{AC} = (150 \cos \theta)\mathbf{i} - (150 \sin \theta)\mathbf{j}$

$\overrightarrow{AW} = (30 \cos 85°)\mathbf{i} + (30 \sin 85°)\mathbf{j}$

$\overrightarrow{AT} = \overrightarrow{AC} + \overrightarrow{AW}$

Therefore $v\mathbf{i} = (150 \cos \theta + 30 \cos 85°)\mathbf{i} + (30 \sin 85° - 150 \sin \theta)\mathbf{j}$

Equating the $\mathbf{j}$ components,

$0 = 30 \sin 85° - 150 \sin \theta$ giving $\theta \approx 11.5°$ as before.

Equating the $\mathbf{i}$ components,

$v = 150 \cos \theta + 30 \cos 85°$ giving $v \approx 149.6$

and the time taken can be obtained as before.

(b) *By trigonometry*

The starting point is now airfield B (Fig. 21.12). $\overrightarrow{BW}$ is the wind vector, $\overrightarrow{BC}$ the course vector (airspeed 150) and $\overrightarrow{BT}$ is the track vector.

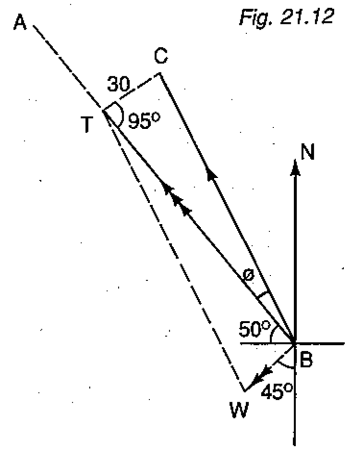
From $\triangle$BCT, using the sine rule,

$\dfrac{\sin \phi}{30} = \dfrac{\sin 95°}{150}$ giving $\phi \approx 11.5°$.

Hence the course for the return journey should be 331.5°.

Verify that this can also be solved using unit vectors as in (a).

Fig. 21.12



Exercise 21.1 *(Answers on page 649.)*

1 Find the resultants (giving both magnitude and direction) of the following velocities by trigonometry or using unit vectors:
(a) 5 km h^{-1} due north and 3 km h^{-1} due east
(b) 8 m s^{-1} north–east and 5 m s^{-1} north-west
(c) 10 m s^{-1} north 60° east and 4 m s^{-1} east
(d) 40 km h^{-1} on a bearing 050° and 40 km h^{-1} on a bearing 110°

2 A steamship is sailing due north at 20 km h^{-1}. There is a wind of 15 km h^{-1} from the west. Find the direction in which the smoke from the funnel is drifting.

3 A ship is travelling at 10 m s^{-1}. A ball is rolled across the deck (at right angles to the motion of the ship) at 4 m s^{-1}. Find the actual velocity of the ball relative to the ground.

4 An aeroplane pilot sets a course due north. The airspeed is 150 km h^{-1}. There is a wind of 40 km h^{-1} from the west. Find the track and the ground speed of the plane.

5 A man sets out to swim across a river 0.4 km wide at right angles to its parallel banks. He can swim at a steady speed of 2 km h^{-1} in still water but there is a current flowing at 1 km h^{-1}. Find
(a) his track and the ground speed across the river,
(b) how far downstream he is carried when he lands.

6 An aeroplane has an airspeed of 200 km h^{-1} and is headed on a course of 330°. There is wind of 50 km h^{-1} from a bearing of 210°. Find the track and the ground speed of the plane.

7 A river flows at 3 km h^{-1} and its parallel banks are 400 m apart. A man wishes to cross in a motorboat which can make 5 km h^{-1} in still water to reach the point directly opposite. Find the direction in which he must steer and the time taken to cross the river.

8 An aeroplane pilot wishes to fly from an airfield P to another airfield Q, 300 km due east. The airspeed is 200 km h^{-1}. There is a steady wind of 40 km h^{-1} from the north-east. Find
(a) the course he must take,
(b) the actual ground speed,
(c) the time taken to reach Q,
(d) the course required for the return journey, the wind being the same,
(e) the time taken over the return journey.

9 An aeroplane, flying at a steady airspeed of 200 km h^{-1}, flies in a straight line from A to B, where B is due east of A and then returns along the same route. There is a constant wind of 60 km h^{-1} blowing from the north-east throughout the double flight. Find the ground speed of the aeroplane on both trips.

10 A motorboat, capable of travelling at 40 km h^{-1} is to be sailed from a point A to a point B where B lies 25 km on a bearing 070° from A. There is a steady current of 10 km h^{-1} running due S. Find the course to be taken and the time required for the journey.

11 An aeroplane starts from A intending to fly to B, 300 km due N of A and the pilot sets a course due N. Due to a wind from the west, the aeroplane reaches a point 30 km due E of B after flying for $1\frac{1}{2}$ hours.
Calculate
(a) the speed of the wind,
(b) the airspeed of the aeroplane.
Find also
(c) the course that should have been taken to reach B directly, and
(d) the time, to the nearest minute, that the flight would then have taken.

12 A stretch of river 300 m wide has straight parallel banks and the speed of the current is 4 m s^{-1}. A man starts to cross the river in a boat from a point A on one bank. The boat can sail at 5 m s^{-1} in calm water. If he takes a course making an angle of 60° up-

stream with the bank, find
(a) the distance where he lands from the point directly opposite A,
(b) the time taken to cross the river.

13 An aeroplane can fly at 300 km h^{-1} in still air and there is a wind blowing from the direction 060° at a constant rate of 40 km h^{-1}.
(a) What course should the pilot take to reach an airfield 200 km due N of his starting point?
(b) If he takes this course, how long will the flight last?
(c) What course should be taken for the return flight (the wind being as before) and how long will this flight last?

14 The speed of a helicopter in still air is v km h^{-1}. The pilot leaves A and flies on a course 067°. There is a wind of 50 km h^{-1} blowing from the direction 020°. After 45 minutes, the helicopter is above a point B which is due E of A. Find
(a) the value of v to the nearest km h^{-1},
(b) the distance AB.

RELATIVE VELOCITY

Consider two aeroplanes A and B flying at the same height. A is flying at 200 km h^{-1} due N and B at 400 km h^{-1} on a course 030°. These are their velocities **relative to the ground** and are their **true** velocities. To the pilot of A, however, B will seem to have a different velocity because he himself is moving. We call this apparent velocity, the **velocity of B relative to A** and we write this as $\overrightarrow{BvA}$. Similarly B will see A moving with the velocity of A relative to B or $\overrightarrow{AvB}$.

In general, suppose **a** represents the velocity of A and **b** the velocity of B (Fig. 21.13). Then $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

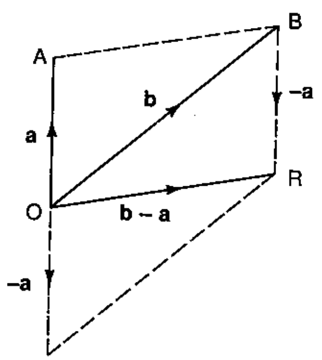


Fig. 21.13

Let us reduce A to rest (theoretically) by introducing a velocity $-\mathbf{a}$. A is now not moving. In order to preserve their relative position, we must also give B the same velocity $-\mathbf{a}$. B now has 2 velocities, **b** and $-\mathbf{a}$ and the resultant of these is $\overrightarrow{OR}$. $\overrightarrow{OR}$ represents the velocity of B as seen from A (as A is not moving). But $\overrightarrow{OR} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. Hence the velocity of B relative to A is $\mathbf{b} - \mathbf{a}$ or $\overrightarrow{BvA} = \mathbf{b} - \mathbf{a}$. Similarly, $\overrightarrow{AvB} = \mathbf{a} - \mathbf{b}$.

We can now find $\overrightarrow{BvA}$ for the two aeroplane. From Fig. 21.14,

$$|\overrightarrow{BvA}|^2 = 200^2 + 400^2 - 2 \times 200 \times 400 \times \cos 30°$$

giving $|\overrightarrow{BvA}| \approx 248$.

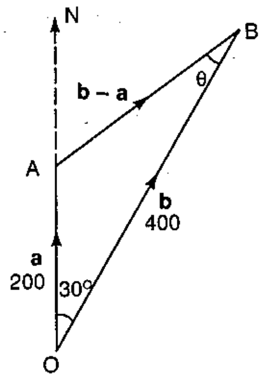


Fig. 21.14

Also $\frac{\sin\theta}{200} = \frac{\sin 30°}{248}$ giving $\theta \approx 23.8°$.

Therefore $\angle NAB \approx 30° + 23.8° = 53.8°$.

Hence the velocity of B relative to A is 248 km h^{-1} in the direction N53.8°E.

Note: Problems on relative velocity may also be solved by drawing if the rubric of the question does not forbid it (see Example 6).

[We can also work with unit vectors as follows:

Take **i** along $\overrightarrow{OE}$ and **j** along $\overrightarrow{ON}$ (Fig. 21.15)

$$\mathbf{a} = 200\mathbf{j}$$

and $\mathbf{b} = (400 \sin 30°)\mathbf{i} + (400 \cos 30°)\mathbf{j}$

$$= 200\mathbf{i} + (200\sqrt{3})\mathbf{j}$$

So $\overrightarrow{BvA} = \mathbf{b} - \mathbf{a}$

$$= 200\mathbf{i} + (200\sqrt{3} - 200)\mathbf{j}$$

$$= 200\mathbf{i} + 146.4\mathbf{j}$$

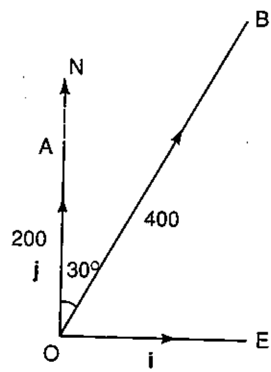


Fig. 21.15

Hence $|\overrightarrow{BvA}|^2 = 200^2 + 146.4^2$ (Fig. 21.16)

giving $|\overrightarrow{BvA}|^2 \approx 248$

and $\tan \phi = \frac{200}{146.4}$ giving $\phi \approx 53.8°$.

Hence $\overrightarrow{BvA}$ is 248 km h^{-1} in the direction N53.8°E as before.]

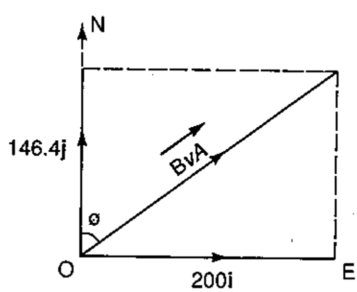


Fig. 21.16

Example 4

Rain is falling vertically at 5 km h^{-1}. A man is sitting by the window of a train travelling at 40 km h^{-1}. In what direction do the raindrops appear to cross the windows of the train?

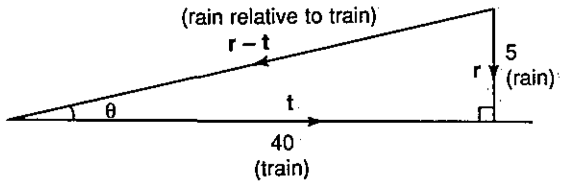


Fig. 21.17

The vectors are shown in Fig. 21.17.

The velocity of the rain relative to the train = **r** − **t**.

So $\tan\theta = \frac{5}{40}$ giving $\theta \approx 7.1°$.

Thus the rain drops appear to cross the windows at 7.1° to the horizontal.

Example 5

A ship is sailing due north at a constant speed of 12 knots. A destroyer sailing at 36 knots is 30 nautical miles due east of the ship. At this moment, the destroyer is ordered to intercept the ship. Find

(a) the course which the destroyer should take,

(b) the velocity of the destroyer relative to the ship,

(c) the time taken for the destroyer to reach the ship.

[It is assumed that both the ship and the destroyer do not change their velocities.]

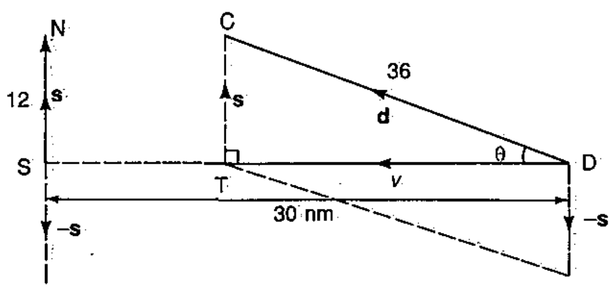


Fig. 21.18

Fig. 21.18 shows the positions of the ship S and the destroyer D. We reduce S to rest by introducing a velocity of 12 knots due south to both S and D. Then the course of D is $\overrightarrow{DC}$ and, to intercept S, its track ($\overrightarrow{DT}$) must lie along DS.

By the cosine rule,

$$|\overrightarrow{OW}|^2 = 40^2 + 10^2 - 2 \times 40 \times 10 \times \cos 30° \text{ giving } |\overrightarrow{OW}| \approx 31.7.$$

Also, $\frac{\sin\theta}{10} = \frac{\sin 30°}{31.7}$ giving $\theta \approx 9.1°$.

Hence the true velocity of the wind is 31.7 km h^{-1} towards the direction (60° – 9.1°) = 50.9° or from the direction 230.9°.

2 *By calculation (using vectors)*

[By vectors, take **i** along $\overrightarrow{OE}$ and **j** along $\overrightarrow{ON}$.

$\overrightarrow{OC} = (40 \sin 60°)\mathbf{i} + (40 \cos 60°)\mathbf{j} = (20\sqrt{3})\mathbf{i} + 20\mathbf{j}$

$\overrightarrow{WvC} = -10\mathbf{i}$

$\overrightarrow{OW} = \overrightarrow{OC} + \overrightarrow{WvC} = (20\sqrt{3} - 10)\mathbf{i} + 20\mathbf{j}$

$|\overrightarrow{OW}|^2 = (20\sqrt{3} - 10)^2 + 20^2$ giving $|\overrightarrow{OW}| \approx 31.7$.

Also, $\tan \phi = \frac{20\sqrt{3} - 10}{20}$ giving $\phi \approx 50.9°$.

Thus we obtain the same results as before.

3 First draw a sketch and label it with all the information given (it should be a rough version of Fig. 21.19). The actual drawing must be done carefully. Choose a suitable scale to ensure reasonably accurate results, say 1 cm for 4 km h^{-1}.
Draw a north line ON as in Fig. 21.19.
From O, draw OC 10 cm long with $\angle NOC = 60°$.
From C, draw CW 2.5 cm long parallel to OE.
Join OW.
Measure OW (and convert to km h^{-1}) and $\angle NOW$.
Compare with the calculated values above.

Exercise 21.2 *(Answers on page 649.)*

1 Aeroplane A is flying due N at 150 km h^{-1}. Aeroplane B is flying due E at 200 km h^{-1}. Find the velocity of B relative to A.

2 Two cars A and B are travelling on roads which cross at right angles. Car A is travelling due east at 60 km h^{-1}, car B is travelling at 40 km h^{-1} due north, both going towards the crossing. Find the velocity of B relative to A. [The magnitude *and* direction must be given].

3 A passenger is on the deck of a ship sailing due east at 25 km h^{-1}. The wind is blowing from the north-east at 10 km h^{-1}. What is the velocity of the wind relative to the passenger?

4 A road (running north-south) crosses a railway line at right angles. A passenger in a car travelling north at 60 km h^{-1} and 600 m south of the bridge, sees a train, travelling west at 90 km h^{-1}, which is 800 m east of the bridge. Find the velocity of the train relative to the car.

5 To a man in a car travelling at 20 km h^{-1} north-east, the wind appears to blow from the west with speed 16 km h^{-1}. Find the true velocity of the wind.

6 Aeroplane A is flying at 400 km h^{-1} north-east and sees aeroplane B which is apparently flying north at 500 km h^{-1}. What is the true velocity of B?

7 An unidentified aircraft is reported as flying due north at 500 km h^{-1}. A fighter plane, which is 100 km on a bearing of 225° from the unknown plane is ordered to contact it. If the fighter can fly at 800 km h^{-1}, what course should it take? After how long will it be in contact?

8 A man on a ship steaming due south at 12 km h^{-1} sees a balloon apparently travelling due west at 15 km h^{-1}. Find the true velocity of the balloon.

9 Two ships A and B are 30 km apart with B due south of A. A is sailing at 10 km h^{-1} in the direction 120° while B is sailing at 15 km h^{-1} in the direction 045°.
(a) Find the velocity of A relative to B.
(b) Calculate the time taken for B to be due west of A.

10 A helicopter is flying on a course 060° with speed 100 km h^{-1}. An aeroplane, which is 50 km due east of the helicopter, can fly at 200 km h^{-1}. What course should the aeroplane take to intercept the helicopter?

11 To a ship sailing due N at 20 km h^{-1} another ship appears to be moving with a velocity of 12 km h^{-1} in the direction 120°. Find the true velocity of this ship.

12 A ship A is heading north at 20 km h^{-1} and at 1200 h is 50 km south-east of a ship B. If B steers at 25 km h^{-1} so as to *just* intercept A, find
(a) the direction in which B must travel,
(b) the time when the interception takes place.

13 A man is walking along a horizontal road at 1.2 m s^{-1}. The rain is coming towards him and appears to be falling with a speed of 4 m s^{-1} in the direction which makes an angle of 60° with the horizontal. Find the actual speed of the rain and the angle this speed makes with the horizontal.

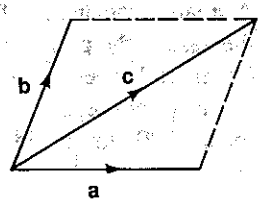
Find the speed and the angle with the horizontal which the rain would appear to make if he walked at the same speed in the opposite direction.

SUMMARY

- *Composition of velocities*

If a body has 2 velocities **a** and **b**, the resultant velocity **c** is given by **c** = **a** + **b**. If **a** and **b** are represented by the sides of a parallelogram, then **c** is represented by the diagonal as shown in the diagram.

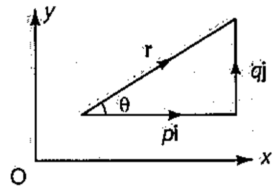
Fig. 21.20



- *Resolution of velocities*

  It is useful to resolve a velocity **r** into 2 perpendicular components, usually along the x– and y-axes. Unit coordinate vectors **i** and **j** are often used as shown in the diagram.

Fig. 21.21



  Also $|\mathbf{r}| = r = \sqrt{p^2 + q^2}$,

  $$p = r\cos\theta,\ q = r\sin\theta \text{ and } \tan\theta = \frac{q}{p}$$

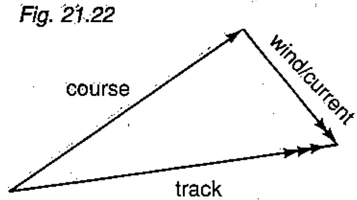
- *Notation*

  ——→ course vector

  ——→→ wind/current vector

  ——→→→ track vector

Fig. 21.22



  The track vector is the resultant of the other 2 vectors.

  ——→ (course) + ——→→ (wind/current) = ——→→→ (track)

- *Relative velocity*

  $\overrightarrow{BvA}$ means the velocity of B relative to A.

  $\overrightarrow{BvA} = \mathbf{b} - \mathbf{a}$

  where **b** = velocity of B

  and **a** = velocity of A.

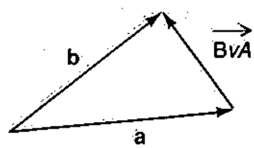


Fig. 21.23

REVISION EXERCISE 21 *(Answers on page 649.)*

1 A river is 160 m wide and runs at 1.2 m s^{-1} between straight parallel banks. A man can row at 2 m s^{-1} in still water.

(a) If he rows in a direction perpendicular to the banks, how far downstream will he land?

(b) At what acute angle (to the nearest degree) to the bank should he now row to return to his starting point?

2 A small aeroplane can fly at 200 km h^{-1} in still air. There is a wind of 50 km h^{-1} from the east. If the pilot wishes to fly due south, what course should he take and what is his ground speed?

The pilot keeps this course but the wind changes and the pilot finds that his ground speed is 200 km h^{-1} in the direction 190°. Calculate the new velocity of the wind.

3 An aeroplane can fly at 300 km h^{-1} relative to the air. If there is a wind of 60 km h^{-1} from the east, what is the least time in which the aeroplane can reach a point 600 km south-west of its starting point?

4 An aeroplane has a speed of V km h^{-1} in still air and there is a wind of $\frac{V}{4}$ km h^{-1} blowing from the NE. Find the course that must be taken if the pilot wishes to reach a point due east.

5 A destroyer detects the presence of a vessel at a range of 30 nautical miles on a bearing of 060°. The vessel is steaming on a course of 150° at a speed of 15 knots. If the destroyer steams at 22 knots determine either by drawing or by calculation the course the destroyer must steer so that its velocity relative to the vessel is in a direction 060°. Hence determine the time taken for the destroyer to intercept the vessel if neither changes course. (C)

6 A helicopter whose speed in still air is 40 km h^{-1}, flies to an oil platform 60 km away on a bearing 060°. The wind velocity is 15 km h^{-1} from due north. Sketch a suitable triangle of velocities and find
(i) the bearing on which the helicopter must fly,
(ii) the time taken to reach the platform. (C)

7 Two canoeists, A and B, can paddle in still water at 6 m s^{-1} and 5 m s^{-1} respectively. They both set off at the same time from the same point on one bank of a river which has straight parallel banks, 240 m apart, and which flows at 3 m s^{-1}. A paddles in the direction that will take him across the river by the shortest distance whilst B paddles in the direction that will take him across the river in the shortest time. Determine
(i) the direction in which A must paddle,
(ii) the direction in which B must paddle,
(iii) the time taken by each canoeist,
(iv) the distance between the points at which they land. (C)

8 A plane flies in a straight line from London to Rome, a distance of 1400 km on a bearing of 135°. Given that the plane's speed in still air is 380 km h^{-1}, that the wind direction is 225° and that the journey takes 4 hours, determine
(i) the wind speed,
(ii) the direction the pilot should set for the flight.
Find also the direction the pilot should set for the return flight, assuming that the speed and direction of the wind remain unchanged. (C)

9 A tanker is sailing on a fixed course due west at 30 km h^{-1}. At a time of 0900 a destroyer wishing to refuel is 160 km away on a bearing of 225° from the tanker. If the destroyer travels at 60 km h^{-1}, determine
(i) the direction in which the destroyer should travel in order to reach the tanker,
(ii) the time at which the destroyer reaches the tanker. (C)

10 A cyclist is travelling due north at 20 km h^{-1} and finds that the wind relative to him appears to be blowing from a direction 040° with a speed of 30 km h^{-1}. Find the true speed and direction of the wind.

Assuming that the speed and direction of the wind remain unchanged, find the speed of the wind relative to the cyclist when the cyclist is travelling due south at 20 km h^{-1}. (C)

11 An aircraft A is flying due east at 600 km h^{-1} and its bearing from aircraft B is 030°. If aircraft B has a speed of 1000 km h^{-1} find the direction in which B must fly in order to intercept A. If the aircraft are initially 50 km apart find the time taken, in minutes, for the interception to occur. (C)

12 An aircraft whose speed in still air is 300 km h^{-1} flies from A due north to B and back, where AB = 500 km. The wind velocity is 120 km h^{-1} from 240°. Find
(i) the bearing on which the aircraft must fly for the outward journey,
(ii) the time of flight of the outward journey,
(iii) the bearing on which the aircraft must fly for the return journey. (C)

13 An aircraft is flying from a point A to a point B 400 km north-east of A, and a wind is blowing from the west at 50 km h^{-1}. The speed of the aircraft in still air is 300 km h^{-1}. Find
(i) the direction in which the aircraft must be headed,
(ii) the distance of the aircraft from B one hour after leaving A. (C)

14 (a) An aircraft flies in a straight line from a point A to a point B 200 km east of A. There is a wind blowing at 40 km h^{-1} from the direction 240° and the aircraft travels at 300 km h^{-1} in still air. Find
(i) the direction in which the pilot must steer the aircraft,
(ii) the time, to the nearest minute, for the journey from A to B.
(b) To an observer in a ship sailing due north at 10 km h^{-1}, a second ship appears to be sailing due east at 24 km h^{-1}. Find the magnitude and direction of the actual velocity of the second ship. (C)

15 When a man walks at 4 km h^{-1} due west, the wind appears to blow from the south. When he walks at 8 km h^{-1} due west, the wind now appears to blow from the south-west. Find, by drawing or calculation, the true velocity of the wind.

16 Two ships A and B leave a harbour at the same time, A sailing due N at 20 km h^{-1}, B sailing at 25 km h^{-1} in the direction 060°. Calculate
(a) the velocity of B relative to A (stating the magnitude and direction),
(b) the bearing of B from A in the subsequent motion,
(c) the distance between the ships after 2 hours of sailing.

17 An aeroplane has a speed of V km h^{-1} in still air and flies in a straight line from A to B. There is a wind of speed $\frac{3}{4}V$ km h^{-1} blowing from a direction making an angle θ with AB where $\sin\theta = \frac{4}{5}$. Find
(a) the angle which the course to be taken makes with AB,
(b) the ground speed in terms of V.

18 A plane is scheduled to fly from London to Rome in $2\frac{1}{2}$ hours. Rome is 1400 km from London and on a bearing of 135° from London. Given that there is a wind blowing from the north at 120 km h^{-1} find, by calculation or drawing,

(i) the speed of the plane in still air,

(ii) the course which the pilot should set.

Assuming that the velocity of the wind and the speed of the plane in still air remain unchanged, find

(iii) the course which the pilot should set for the return journey.

(iv) the time taken, to the nearest minute. (C)

19 (a) An aircraft is flying due south at 350 km h^{-1}. The wind is blowing at 70 km h^{-1} from the direction θ°, where θ is acute. Given that the pilot is steering the aircraft in the direction 170°, find

(i) the value of θ,

(ii) the speed of the aircraft in still air.

(b) A man who swims at 1·2 m s^{-1} in still water wishes to cross a river which is flowing between straight parallel banks at 2 m s^{-1}. He aims downstream in a direction making an angle of 60° with the bank.

Find

(i) the speed at which he travels,

(ii) the angle which his resultant velocity makes with the bank. (C)

20 (a) Two particles, A and B, are 60 m apart with B due west of A. Particle A is travelling at 9 m s^{-1} in a direction 300° and B is travelling at 12 m s^{-1} in a direction 030°. Find

(i) the magnitude and direction of the velocity of B relative to A,

(ii) the time taken for B to be due north of A.

(b) A wind is blowing from the direction 320° at 30 km h^{-1}. Find, by drawing or by calculation, the magnitude and direction of the velocity of the wind relative to a man who is cycling due east at 18 km h^{-1}.

Projectiles

22

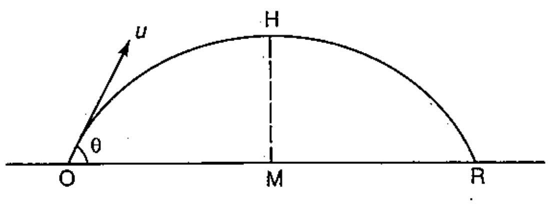
In the previous Chapter, we dealt with motion of a body in a plane where the body has 2 or more velocities. No acceleration was involved.

We now take the discussion of motion in a plane further by considering the effect of gravity on the moving body. As in earlier work, we shall ignore resistance in our discussion.

PROJECTILES

If we throw a ball at an angle to the horizontal (not vertically — vertical motion under gravity was discussed in Chapter 20), the path of the ball will look like the curve in Fig. 22.1. The ball is a **projectile** and the path is a **parabola.**

Fig. 22.1



The ball is projected from O, with speed u, at an angle θ (called the **angle of projection**) to the horizontal. H is the highest point reached and HM is the axis of the parabola. The ball reaches the ground again at R where OM = MR. OR is the **range** and the time taken from O to R is the **time of flight**.

VELOCITY COMPONENTS

The principle used in dealing with the motion is that *the horizontal and vertical components of velocity are independent*. The initial horizontal and vertical components of u are $u \cos \theta$ and $u \sin \theta$ respectively (Fig. 22.2).

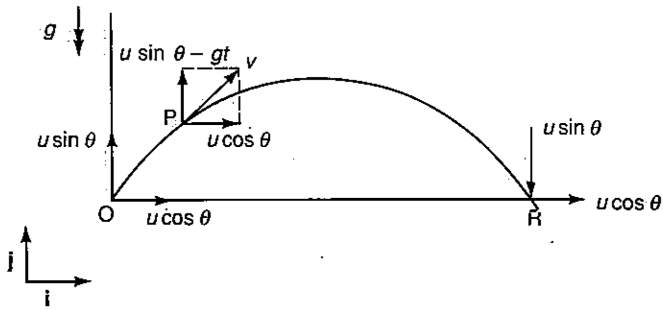


Fig. 22.2

The horizontal component $u \cos \theta$ is not affected by gravity and thus remains constant. The vertical component is affected by gravity and changes continually as the projectile has an acceleration g vertically downwards. At any point P of the path, reached after time t, the vertical component will be $u \sin \theta - gt$ (from $v = u + at$).

So the components of velocity at time t are:

horizontal component: $V_x = u \cos \theta$ (i)

vertical component : $V_y = u \sin \theta - gt$ (ii)

The velocity v at P will actually lie along the tangent at P.

In usual vector notation where **i** and **j** are unit vectors along the x– and y-axes respectively,

$$\mathbf{v} = (u \cos \theta)\mathbf{i} + (u \sin \theta - gt)\mathbf{j}$$

At R, by symmetry, the vertical component of velocity will be $u \sin \theta$ downwards.

Coordinates

The coordinates of P at time t referred to O (Fig. 22.3) will be

$$x = (u \cos \theta)t \quad \text{(iii)}$$

and $$y = (u \sin \theta)t - \tfrac{1}{2}gt^2 \quad \text{(iv)}$$

$\left(\text{from } s = ut + \tfrac{1}{2}at^2\right)$.

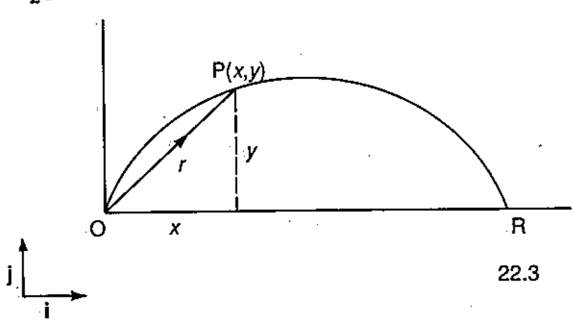


Fig. 22.3

Hence if $\mathbf{r}$ is the position vector of P,

$$\mathbf{r} = [u \cos \theta]t\mathbf{i} + [(u \sin \theta)t - \tfrac{1}{2}gt^2]\mathbf{j}$$

Greatest Height

At the greatest height (h_{max}) at H (see Fig.22.1), the vertical component of velocity $V_y = 0$

i.e. $u \sin \theta - gt = 0$ (from (ii))

giving $t = \dfrac{u \sin \theta}{g}$ (time taken to reach the maximum height).

Substituting this into equation (iv) gives

$$y = u \sin \theta\left(\frac{u \sin \theta}{g}\right) - \tfrac{1}{2}g\left(\frac{u \sin \theta}{g}\right)^2 = \frac{u^2 \sin^2 \theta}{2g}$$

i.e. greatest height, $h_{max} = \dfrac{u^2 \sin^2 \theta}{2g}$ (v)

Time of Flight

The time of flight, T, is obtained at R when the vertical displacement is zero (see Fig. 22.1), i.e. $y = 0$

or $(u \sin \theta)t - \tfrac{1}{2}gt^2 = 0$ (from (iv))

giving $t = 0$ (at O) or $t = \dfrac{2u \sin \theta}{g}$ (at R)

i.e. time of flight, $T = \dfrac{2u \sin \theta}{g}$ (vi)

Note that this is twice the time taken from O to H (see previous section on greatest height).

Horizontal Range

The horizontal range R_x is obtained by substituting the time of flight, T, into equation (iii), thus:

$$x = (u \cos \theta)\left(\frac{2u \sin \theta}{g}\right)$$
$$= \frac{2u^2}{g} \sin \theta \cos \theta$$
$$= \frac{u^2}{g} \sin 2\theta \quad \text{(since } 2 \sin \theta \cos \theta = \sin 2\theta\text{)}$$

i.e. the horizontal range $R_x = \dfrac{u^2}{g} \sin 2\theta$ (vii)

From this equation, we can find the angle of projection for which the range is a maximum.

Since the maximum value of $\sin 2\theta = 1$, $2\theta = 90°$

i.e. $\theta = 45°$.

Hence the range is maximum when the angle of projection $\theta = 45°$.

The maximum range is then $R_{max} = \frac{u^2}{g}$ (viii)

In solving problems, you may make use of the equations (i) to (viii), but these must be memorized. Alternatively, this can be avoided by using the principle of independence and working from the beginning with the values given. In the first example that follows, we will use both alternatives. As in previous work, we will take g as 10 m s^{-2} unless stated otherwise.

Example 1

A particle is projected from O with speed 30 m s^{-1} at an angle of 30° to the horizontal. Find

(a) the time of flight,
(b) the range on the horizontal ground,
(c) the maximum height reached.

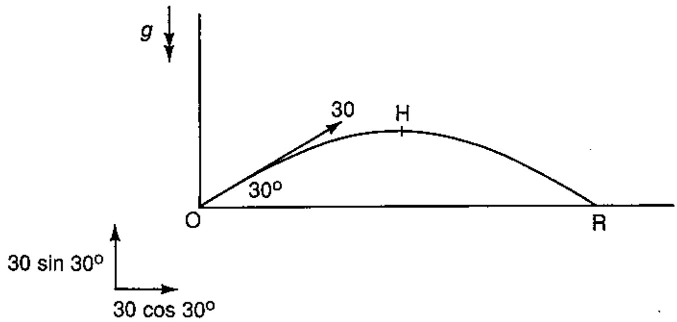


Fig. 22.4

(a) Using equation (vi), the time of flight is

$$T = \frac{2 \times 30 \times \sin 30°}{10} = 3 \text{ s}$$

(b) Using equation (vii), the range is

$$R_x = \frac{30^2 \times \sin(2 \times 30°)}{10} = 77.9 \text{ m (to 3 sig. fig.)}$$

(c) Using equation (v), the maximum height is

$$h_{max} = \frac{30^2 \times \sin^2 30°}{2 \times 10} = 11.25 \text{ m}$$

Alternative solution

The initial components of velocity are

$30 \cos 30° = 15\sqrt{3}$ m s^{-1} horizontally

and $30 \sin 30° = 15$ m s^{-1} vertically.

(a) Consider the vertical motion.

The time of flight is obtained when the particle is at R. At R, the vertical displacement from O is zero.

Using $s = ut + \frac{1}{2}at^2$,

we have $0 = 15t - \frac{1}{2} \times 10 \times t^2$

giving $t = 0$ (at O) or $t = 3$ (at R).

Therefore, time of flight is 3 s.

(b) Now consider the horizontal motion.
The horizontal range is obtained when $t = 3$ s (time of flight).

Therefore, $R_x = 15\sqrt{3} \times 3 = 77.9$ m (to 3 sig. fig.).

(c) For maximum height, we consider the vertical motion again. At maximum height, the vertical velocity is zero.

Using $2as = v^2 - u^2$,

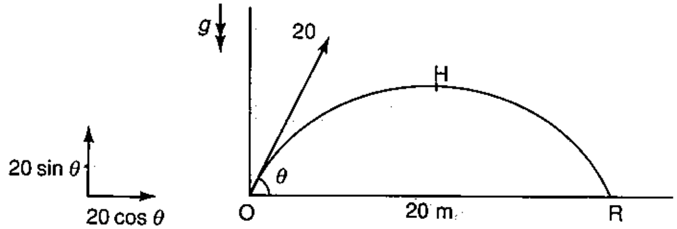
we have $2(-10)s = 0^2 - 15^2$ giving $s = 11.25$ m.

Therefore the maximum height is 11.25 m.

Example 2

A ball is thrown with a speed of 20 m s⁻¹ and hits the horizontal ground 20 m away. Find the angle of projection θ.

Fig. 22.5



The horizontal range = 20 m

Using result (vii), $\frac{20^2}{10}\sin 2\theta = 20$ giving $\sin 2\theta = \frac{1}{2}$.

Therefore $2\theta = 30°$ or $150°$ i.e. $\theta = 15°$ or $75°$.

So there are 2 possible angles of projection (15° and 75°) for a given range. (Note that these add up to 90°. This is generally true except when the range is a maximum, in which case there is only one angle of projection, viz. 45° (Fig. 22.6).)

$\theta + \phi = 90°$

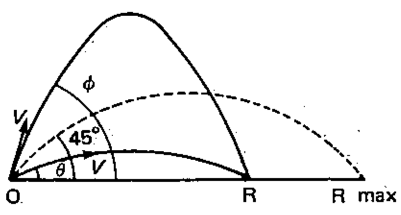


Fig.22.6

Example 3

A golf ball is struck at an angle θ where $\tan\theta = \frac{4}{3}$, with an initial speed of 25 m s^{-1}. Find

(a) when the ball is at a height of 15 m above the ground,

(b) the distance between the positions of the ball when it is 15 m above the ground.

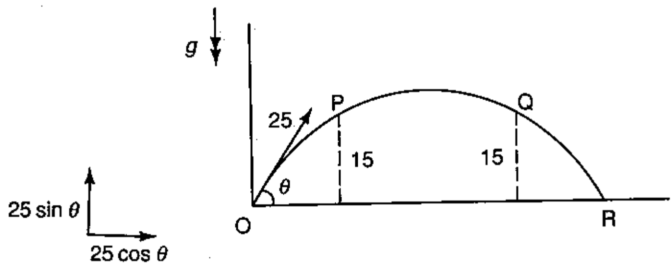


Fig. 22.7

If $\tan\theta = \frac{4}{3}$, then $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

(a) Consider the vertical motion.

When the ball is 15 m above the ground (at P and Q) at time t, we have $\left(\text{using } s = ut + \frac{1}{2}at^2\right)$,

$$15 = (25\sin\theta)t + \tfrac{1}{2}(-10)t^2$$

i.e. $15 = 25\left(\frac{4}{5}\right)t - 5t^2$

or $5t^2 - 20t + 15 = 0$

$$t^2 - 4t + 3 = 0$$

$$(t-1)(t-3) = 0$$

giving $t = 1, 3$

Hence the ball is 15 m above the ground 1 s and 3 s after being struck.

(b) The horizontal displacement after 1 s,

$s_1 = (25\cos\theta)1 = 25\left(\frac{3}{5}\right) = 15$

The horizontal displacement after 3 s,

$s_3 = (25\cos\theta)3 = 25\left(\frac{3}{5}\right)3 = 45$

Hence the distance between P and Q is $s_3 - s_1 = 30$ m.

Example 4

From the edge of a cliff 60 m high, a stone is thrown into the air with speed 10 m s^{-1} at an angle of 30° to the horizontal (Fig. 22.8). Find

(a) the time of flight,

(b) how far from the foot of the cliff it strikes the sea,

(c) the velocity with which it strikes the sea.

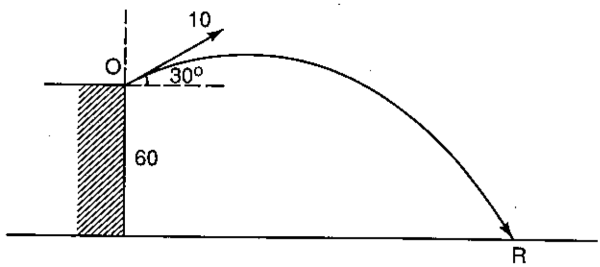


Fig. 22.8

The initial components of velocity are 10 cos 30° (= $5\sqrt{3}$) horizontally and 10 sin 30° (= 5) vertically (upwards).
At time t, the coordinates (referred to O) are

$$x = (5\sqrt{3})t \text{ and } y = 5t - 5t^2.$$

(a) When the stone reaches the sea, $y = -60$.
So $5t - 5t^2 = -60$ i.e. $t^2 - t - 12 = 0$
or $(t-4)(t+3) = 0$ giving $t = 4$ or -3.
Hence the time of flight is 4 s (−3 is not valid).

(b) When $t = 4$, $x = (5\sqrt{3}) \times 4 = 20\sqrt{3} \approx 34.6$
Hence the stone strikes the sea 34.6 m from the foot of the cliff.

(c) When it strikes the sea at R, the horizontal component of the velocity remains unchanged at $5\sqrt{3}$ m s^{-1}. The vertical component v m s^{-1} is given by

$$v = 5 - 10(4) = -35$$

Hence the vertical component is 35 m s^{-1} downwards (Fig.22.9).

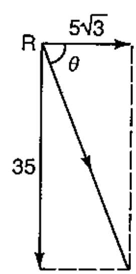


Fig. 22.9

Therefore the magnitude of the velocity = $\sqrt{(5\sqrt{3})^2 + 35^2} \approx 36$ and the angle θ it makes with the horizontal is given by

$$\tan\theta = \frac{35}{5\sqrt{3}} \text{ giving } \theta \approx 76°.$$

Hence the stone strikes the sea with a velocity of 36 m s^{-1} at 76° to the horizontal.

Exercise 22.1 *(Answers on page 650.)*

[Take $g = 10$ m s^{-2} in numerical questions.]

1 For each of the following projectiles with an initial speed V at an angle θ to the horizontal as given below, find (a) the time of flight, (b) the range and (c) the maximum height reached.

(i) $V = 10\sqrt{2}$ m s^{-1}, $\theta = 45°$
(ii) $V = 12$ m s^{-1}, $\theta = 30°$
(iii) $V = 30$ m s^{-1}, $\tan\theta = \frac{3}{4}$
(iv) $V = 26$ m s^{-1}, $\tan\theta = \frac{5}{12}$

2 A particle is projected from a point O on horizontal ground with a speed of 30 m s^{-1} at an angle of θ to the horizontal where $\tan\theta = \frac{4}{3}$.
Find (a) the time of flight, (b) the magnitude and direction of the velocity of the particle after 4 seconds.

3 If a ball is thrown upwards with speed 12 m s^{-1} at an angle of 30° to the horizontal, at what times after the start is it 1 m above the ground? What is the maximum height reached?

4 A stone is thrown with speed 12 m s^{-1} at an angle of 60° to the horizontal but hits a vertical wall 9 m away. How high above the ground does it hit the wall?

5 A ball is thrown at an angle of 60° to the horizontal with a speed of 15 m s^{-1}. At what times after the start is it moving in a direction making an angle of 45° with the horizontal?

6 A rifle is held horizontally and aimed at the centre of a target 100 m away. If the bullet emerges with speed 400 m s^{-1}, how far below the centre of the target does it hit the target?

7 A ball is thrown from a height of 5 m above level ground with a speed of 10 m s^{-1} at an angle of 30° to the horizontal. After what time does it strike the ground and how far horizontally is this? At what angle does it strike the ground?

8 A stone is thrown at an angle $\theta\left(\tan\theta = \frac{4}{3}\right)$ to the horizontal so that it just passes over the top of a wall 3 m away horizontally. If it takes $\frac{1}{2}$ s to reach the top of the wall, find
(a) the speed of projection,
(b) the height of the wall.

9 A particle is projected with a speed of 20 m s^{-1} at 60° to the horizontal. One second later, a second particle is also projected from O with the same speed at angle θ to the horizontal. After 2 s the particles collide. Find

(a) the horizontal and vertical distances of the point of collision,

(b) the value of θ.

10 Two golf balls are struck simultaneously from points very close together with the same speed, 20 m s^{-1}. The angles of projection are θ and ϕ where $\tan\theta = \frac{3}{4}$ and $\tan\phi = \frac{4}{3}$.

(a) Which ball reaches the ground first?

(b) Which ball travels further horizontally?

11 A stone is kicked horizontally off the edge of a roof 20 m high so as to give it a speed of 8 m s^{-1}. How far away horizontally does it land on the level ground?

12 A and B are two points on level ground 30 m apart. One ball is thrown from A towards B with speed 30 m s^{-1} at an angle of 45° to the horizontal while simultaneously a second ball is thrown from B towards A with speed 20 m s^{-1} at an angle of 60°. How far horizontally from A is the point of collision and after what time does this happen?

13 The sloping roof of a house is 13 m long and its two ends are 20 m and 15 m above the ground. A stone slides down the roof and leaves it with a speed of 6.5 m^{-1}. How far horizontally from the end of the roof does it strike the ground?

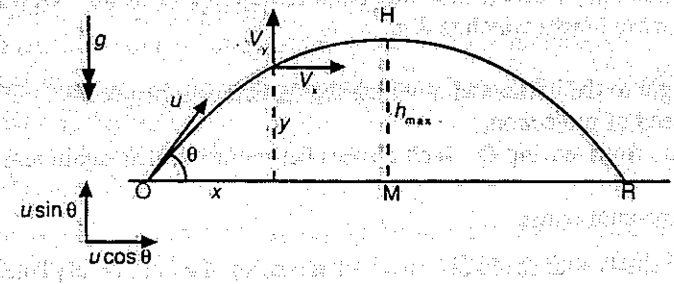
14 A particle is fired at a speed of 40 m s^{-1} at an angle of θ where $\tan\theta = \frac{4}{3}$. Find the horizontal distance travelled when the particle is at a height of 35 m and coming down. Show that the direction of motion at this point is perpendicular to the direction at the start.

15 A particle is projected from O on level ground with speed 25 m s^{-1}. It just passes over a vertical wall 4 m high placed 10 m horizontally from O. If θ is the angle of projection and t is the time taken to reach the wall, show that

(a) $t\cos\theta = \frac{2}{5}$,

(b) $2\sec^2\theta = 25\tan\theta - 10$, and solve this equation for $\tan\theta$.

SUMMARY



u = initial velocity
θ = angle of projection
OR = horizontal range
H = highest point reached
h_{max} = maximum height

Initial components of velocity: $U_x = u\cos\theta$
$U_y = u\sin\theta$

Components of velocity at time t: $V_x = u\cos\theta$
$V_y = u\sin\theta - gt$

Coordinates at time t: $x = (u\cos\theta)t$
$y = (u\sin\theta)t - \frac{1}{2}gt^2$

Time of flight: $T = \dfrac{2u\sin\theta}{g}$

Horizontal range (OR): $R_x = \dfrac{u^2}{g}\sin 2\theta$

Maximum range: $R_{max} = \dfrac{u^2}{g}$ (when $\theta = 45°$)

Maximum height: $h_{max} = \dfrac{u^2\sin^2\theta}{2g}$

(*Note*: The time taken to reach maximum height = $\frac{1}{2}$ the time of flight = $\dfrac{u\sin\theta}{g}$.)

REVISION EXERCISE 22 *(Answers on page 650.)*

1 A particle is projected from a point O on horizontal ground and moves freely under gravity. When the particle is at a horizontal distance of 12 m from O, it achieves its greatest vertical height which is 8 m.
Calculate
(a) the angle to the horizontal at which the particle was projected,
(b) the speed of projection,
(c) the time from leaving O which elapses before the particle again reaches ground level,
(d) the horizontal range. (L)

2 A gun fires shells with speed 280 m s^{-1} from a point of a horizontal plane. The shells move freely under gravity. Prove that the greatest range on the plane is 7840 m and state the angle of elevation at which a shell must be fired for this maximum range.
A target on the plane is at a distance of 3920 m from the gun. Find the smaller angle of elevation at which a shell should be fired in order to hit the target. Find also the greatest vertical height attained by this shell during its flight. (L)

3 A particle is projected horizontally at 35 m s^{-1} from the top of a cliff which is 80 m above sea level.
(a) Show that the particle strikes the water after 4 seconds.
(b) Find the horizontal distance travelled by the particle before it strikes the water.
(c) Find the vertical component of the velocity of the particle when it strikes the water.
(d) Hence find the angle made by the path of the particle with the horizontal when the particle strikes the water. (L)

4 An aircraft A is flying with constant velocity 100 m s^{-1} in a straight line and at a constant height of 980 m over the sea. At the instant when A is vertically above an anchored observation ship S, a bomb is released. The bomb falls freely under gravity and hits the sea at a point T.
(a) Explain why A is vertically above T at the instant when the bomb hits the sea.
Calculate
(b) the time taken by the bomb to reach the sea from the instant of release from the aircraft,
(c) the distance of T from S,
(d) the vertical component of the velocity of the bomb at the instant when the bomb hits the sea. (L)

5 A projectile has an initial speed of 84 m s^{-1} and rises to a maximum height of 40 m above the level horizontal ground from which it was projected. Calculate
(a) the angle of projection,
(b) the time of flight,
(c) the horizontal range. (L)

6 A particle is projected from a point O on horizontal ground with velocity 40 m s^{-1} at an angle of 30° to the horizontal. Given that, 0.7 seconds after projection, the particle is at a point P, calculate

(i) the height of P above ground,
(ii) the magnitude and direction of the velocity of the particle at P.

The particle reaches the ground again at a point Q. Calculate

(iii) the time of flight,
(iv) the greatest height reached,
(v) the distance OQ. (C)

7 A particle is projected from a point O on a horizontal plane with velocity 40 m s^{-1} at 30° to the horizontal. It reaches the plane again at the point A. Calculate

(i) the time of flight,
(ii) the greatest height reached,
(iii) the distance OA,
(iv) the magnitude and direction of the velocity of the particle 2.5 seconds after leaving O.

A second particle is projected from O with velocity V m s^{-1} at 30° to the horizontal. Given that this particle meets the horizontal plane at the point B, where OB = $180\sqrt{3}$ m, calculate the value of V. (C)

8 (a) Fig. 22.10 shows part of the path of a projectile fired from A with a velocity of 8 m s^{-1} at 30° to the horizontal. The projectile passes through a point B, 0.6 m above the horizontal plane through A. Find

(i) the maximum height above AN attained by the projectile,
(ii) the distance AN.

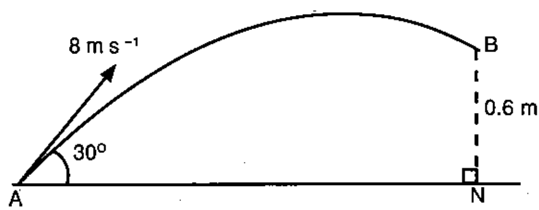


Fig.22.10

(b) A particle is projected from a point O and arrives, 4 seconds later, at a point P on the horizontal plane through O, where OP = 60 m. Find the magnitude and direction of the velocity of projection. (C)

9 (a) An aircraft is flying horizontally in a straight line at a height of 80 m. Its speed is 50 m s^{-1}. The pilot drops an object in order to hit a target ahead which is directly below his flight path. Calculate his horizontal distance from the target at the moment when he releases the object.

(b) A projectile is fired with an initial speed of 160 m s^{-1} at an angle of 72° to the horizontal. Calculate

(i) the height of the projectile 4 seconds after firing,

(ii) the magnitude and direction of the velocity of the projectile 4 seconds after firing,

(iii) the range of the projectile on the horizontal plane through the point of projection. (C)

10 From the top of a building 45 m high a stone is projected upwards with speed V m s^{-1} at an angle of 30° to the horizontal. Two seconds later another stone is let to fall from rest. If the stones reach the ground at the same time, find the value of V.

11 Fig. 22.11 shows an end view of a house. A particle is projected from the topmost point T horizontally with speed V m s^{-1}. If it is not to hit the roof again, what is the minimum value of V?

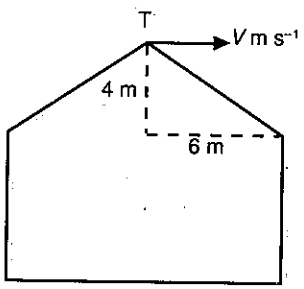


Fig. 22.11

12 A target is set on a hillside and is 50 m horizontally from a point O on ground level. A man tries to hit the target by firing from O at an angle of 60° to the horizontal, the speed of the projectile being 40 m s^{-1}. If he hits the target, what is its vertical height above O?

To knock the target over, the projectile must have a speed greater than 25 m s^{-1}. Can he do this?

13 A golf ball is struck from a point A on level ground with speed 35 m s^{-1} at an angle of θ to the horizontal where $\sin\theta = \frac{4}{5}$. Two seconds later, another ball is struck from a point very close to A and both balls hit the ground at the same time and at the same place. Find the initial speed and direction of the second ball.

14 A stone attached to a string is swung in a vertical circle of radius 1 m whose centre is 1.5 m above the ground. When the stone has completed $\frac{5}{6}$ of a revolution starting from the highest point of the circle, the string breaks and the stone flies off along the tangent at that point with speed $2\sqrt{3}$ m s^{-1}. How far horizontally from the centre of the circle does it hit the ground?

15 A ball is projected from a point O at an angle θ to the horizontal. Simultaneously, a stone is dropped from a point P which is 36 m horizontally and 48 m vertically from O. If the ball and stone collide, show that $\tan\theta = \frac{4}{3}$.

16 A particle is projected from a point A on level horizontal ground with a speed of 40 m s^{-1} at an angle θ to the horizontal where $\tan\theta = \frac{4}{3}$. Find
(i) the maximum height reached,
(ii) the time taken first to reach a height of 44 m above the ground,
(iii) the speed of the particle after 1 second.
Determine, with full working, whether or not the particle will clear a vertical wall, 10 m high, the base of which is 144 m away from A. (C)

17 A particle is projected from a point A on level horizontal ground with a velocity which has a horizontal component of $X \text{ m s}^{-1}$ and a vertical component of $Y \text{ m s}^{-1}$. After 4 s the particle passes through a point that is 40 m higher than A and at a horizontal distance of 180 m from A. Calculate
(i) the value of X and of Y,
(ii) the speed of projection,
(iii) the angle of projection,
(iv) the greatest height reached.
Given that the particle hits the ground at B, calculate
(v) the time taken to reach B,
(vi) the horizontal distance AB. (C)

18 A particle is released from rest at the highest point A of a smooth plane which is inclined at θ to the horizontal, where $\sin\theta = \frac{3}{5}$. The particle slides down a line of steepest slope and leaves the plane at its lowest point B, where AB is 3 m. Show that the particle reaches B with a speed of 6 m s^{-1} and hence find the vertical and horizontal components of the velocity of the particle at B.

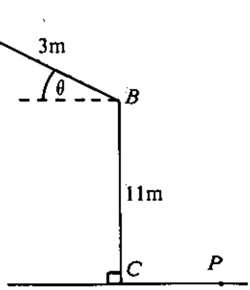


Fig 22.12

On leaving B the particle travels as a projectile under the action of gravity, striking the horizontal plane through C at the point P. Given that B is 11 m vertically above C, calculate
(i) the vertical component of the velocity of the particle at P,
(ii) the magnitude of the velocity of the particle at P,
(iii) the inclination to the horizontal of the path of the particle at P,
(iv) the time taken for the particle to travel from B to P,
(v) the distance PC. (C)

19 A particle is projected from a point O on a horizontal plane with a speed of 50 m s^{-1} at an angle θ to the horizontal, where $\sin\theta = \frac{4}{5}$. Calculate the maximum height reached by the particle and the time taken to reach this height.
During its flight the particle passes through two points P and Q both of which are at a height of 60 m above the horizontal plane. Calculate
(i) the time taken to travel from P to Q,
(ii) the distance PQ,
(iii) the magnitude and direction of the velocity of the particle at Q. (C)

20

Fig. 22.13

A particle is projected from A with a velocity of $V \text{ m s}^{-1}$ at angle $\alpha°$ to the horizontal. After 0·6 s it is at B, where $BN = 9$ m and $AN = 12$ m. Calculate
(i) the value of V and of α,
(ii) the time taken to reach the maximum height.
Given that the particle passes through C, where C is at the same height as B, calculate
(iii) the time taken to travel from A to C,
(iv) the distance AM.

Force, Friction

23

FORCES

We all have some idea of a **force** and are familiar with many varieties of force. We push some object with a force or pull on a rope which transmits our pull. These forces act directly on a body or in contact with it. We have already mentioned gravitational force which attracts bodies towards the earth. Magnets exert a magnetic force of attraction. These forces act at a distance without any apparent contact. It is also fairly obvious that a force is a vector quantity. The direction in which I push a box is just as important as the strength of my push. Hence we can represent a force by a straight line (Fig.23.1). The length of the line represents the magnitude of the force **F** and the direction of AB is the direction of the force. A third important feature is the point or line of application. It will make some difference if I push the box in Fig.23.2 at P or Q in the directions shown. A force is an example of a *tied* or bound vector where the position of the vector is important.

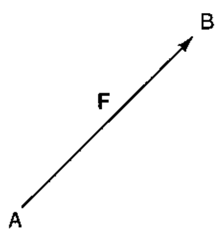


Fig. 23.1

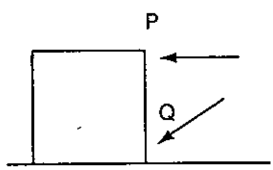


Fig. 23.2

Unit of Force

The unit of force is 1 **Newton** (1 N) which we shall discuss in the next chapter.

TYPES OF FORCE

Certain types of force have particular names and we introduce them here.

Weight

If I hold a brick, I feel it exerting a downward force on my hand. If I let go, the brick will fall to the ground. The force I felt on my hand is now free to pull the brick towards the ground. This force, the gravitational attraction of the earth on the brick is called the **weight** of the brick. It always acts vertically downwards (towards the centre of the earth). The weight of a brick of mass 2 kg is about 20 N (Fig.23.3). This will give some idea of the size of the Newton. In fact, a mass of m kg will have a weight of mg N ($\approx 10m$ N). We shall assume this for the rest of this chapter and discuss it further in the next chapter. Note that weight is the one inescapable force on earth. All bodies have weight.

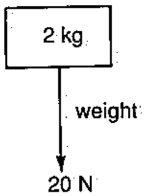


Fig.23.3

The weight of a body acts through a point called the **centre of gravity (CG)**. For symmetrical bodies, the CG is at the centre of symmetry. For example, the CG of a cube is at its centre; the CG of a long *uniform* rod (same cross section throughout) is at the geometrical centre of the rod. The CG of a thin rectangular plate (called a **lamina**) is at the intersection of the diagonals and similarly for a parallelogram.

Reaction

If I place the brick, of weight 20 N, on a horizontal table, it does not move. We say that the brick is in *equilibrium*. Now it would be unrealistic to assume that the gravitational attraction has ceased, so there must be an opposing force exactly equal to 20 N, acting vertically upwards and also through the CG (Fig.23.4(a)).

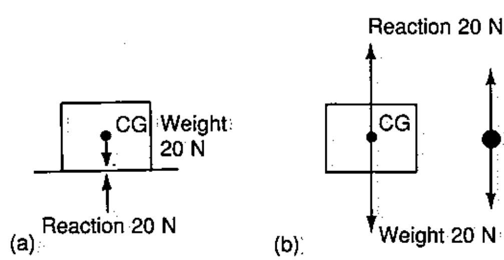


Fig. 23.4

This is the **normal reaction** on the brick from the table, normal in the geometrical sense of being at right angles to the common surface. Sometimes this force is called the normal contact force. In diagrams we show only the forces acting *on* a body, so Fig.23.4(b) shows the forces on the brick. Note also that the dimensions of the body are not relevant in this context. The brick is treated as a **particle**.

Tension

If I suspend the brick by a string (or spring) and the string does not break, the brick is again in equilibrium (Fig.23.5).

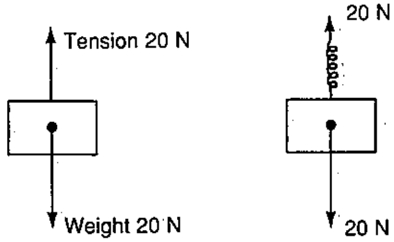


Fig. 23.5

In this case the force which equalizes the weight passes along the string and is called the **tension** in the string. If using a spring, the weight will also stretch the spring until the equilibrium position is reached. (We regard the string as being inextensible, i.e. any stretch is too small to be noted.)

Friction

I place the brick on a horizontal table and try to push it along (Fig.23.6).

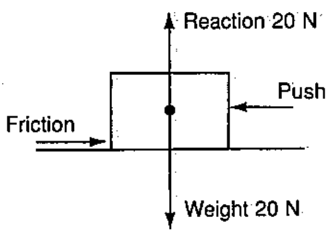


Fig. 23.6

I have to exert a force to do so. There is resistance to my push, called the **frictional resistance** or simply friction. We study this in more detail later. In this situation there are four forces acting on the brick. Note that the weight and normal reaction are still equal and opposite and the brick is in vertical equilibrium. If I push harder, eventually I overcome the frictional force and the horizontal equilibrium will be broken.

Thrust

If the brick is placed on a large strong spring, the spring will resist the weight with a force called a **thrust** (Fig.23.7). This is a similar force to a tension, but acts in the opposite direction.

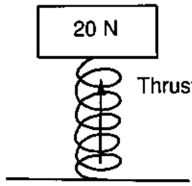


Fig.23.7

All of the above are examples of forces, but they need not *all* operate together in any given problem.

COMPOSITION OF 2 FORCES

Like velocities, forces as vectors can be combined to produce a single **resultant force** (Fig.23.8).

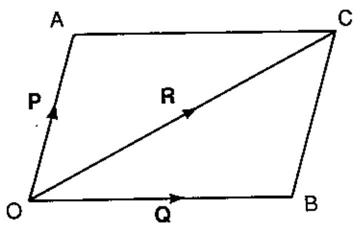


Fig. 23.8

If 2 forces **P** and **Q** are represented by the adjacent sides OA and OB of the parallelogram, then the resultant force **R** is represented by the diagonal OC. The two forces **P** and **Q** could be replaced by the single force **R** which would have the same effect.

Example 1

$\mathbf{F}_1$ and $\mathbf{F}_2$ are 2 forces acting on a particle O. $\mathbf{F}_1$ has magnitude 5 N in direction N30°E and $\mathbf{F}_2$ has magnitude 8 N in direction E. Find the resultant of the 2 forces.

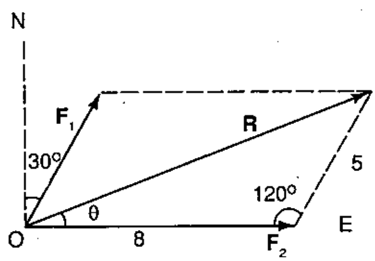


Fig. 23.9

We use trigonometry to solve this problem. Using the cosine rule, the magnitude of the resultant $|\mathbf{R}|$ is given by

$$|\mathbf{R}|^2 = 8^2 + 5^2 - 2 \times 8 \times 5 \cos 120° \text{ giving } |\mathbf{R}| = 11.36 \text{ N}.$$

Using the sine rule, we have

$$\frac{\sin\theta}{5} = \frac{\sin 120°}{11.36} \quad \text{giving } \theta = 22.4°.$$

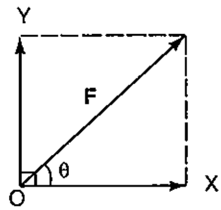
Therefore the resultant is 11.36 N in the direction N67.6°E.

(*Note:* The resultant can also be obtained by a scale drawing.)

RESOLUTION OF A FORCE

Again, like velocity, a single force can be resolved into 2 component forces. Particularly useful is the resolution of a force into two perpendicular components (Fig.23.10).

Fig. 23.10



It is then simpler to deal with these components than with the force itself. The directions of the perpendicular components can be chosen as we please to suit the problem.

The components of **F** are $\overrightarrow{OX} = (F\cos\theta)\mathbf{i}$ where $F = |\mathbf{F}|$

and $\overrightarrow{OY} = (F\sin\theta)\mathbf{j}$

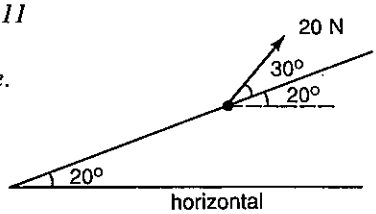
where **i** and **j** are the unit force vectors along the x– and y-axes respectively.

Note that $|\mathbf{F}|^2 = |\overrightarrow{OX}|^2 + |\overrightarrow{OY}|^2$ and $\tan\theta = \dfrac{|\overrightarrow{OY}|}{|\overrightarrow{OX}|}$.

Example 2

Resolve the force shown in Fig.23.11
(a) horizontally and vertically,
(b) along and normal to the plane.

Fig. 23.11



(a) We use unit vectors, taking **i** along the horizontal direction and **j** the vertical direction as shown in Fig. 23.12.

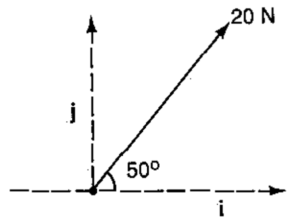


Fig. 23.12

The horizontal component is $(20 \cos 50°)\mathbf{i} \approx 12.9\mathbf{i}$.
The vertical component is $(20 \sin 50°)\mathbf{j} \approx 15.3\mathbf{j}$.

Hence the horizontal and vertical components are 12.9 N and 15.3 N respectively.

(b) We take **i** along the plane and **j** normal to the plane as shown in Fig.23.13.

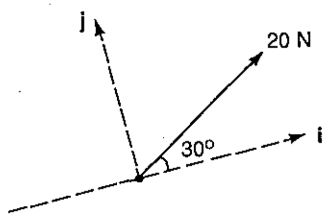


Fig. 23.13

The component along the plane is $(20 \cos 30°)\mathbf{i} \approx 17.3\mathbf{i}$.
The component normal to the plane is $(20 \sin 30°)\mathbf{j} = 10\mathbf{j}$.

Hence the components along and normal to the plane are 17.3 N and 10 N respectively.

COPLANAR FORCES ACTING ON A PARTICLE

Coplanar forces are forces acting in the same plane. We can find the resultant of any number of coplanar forces acting on a particle by resolving the forces in 2 perpendicular directions of our choice and then recombining the components to obtain the resultant.

Example 3

Forces of magnitudes 1 N, 2 N, 3 N, 4 N and 5 N act along the lines OA, OB, OC, OD and OE respectively, where OABCDE is a regular hexagon. Find the resultant of the forces.

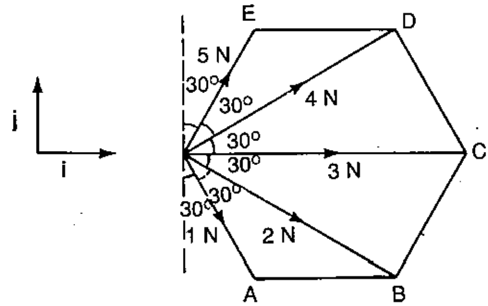


Fig. 23.14

The forces acting are shown in Fig.23.14.
Take **i** and **j** along and perpendicular to OC respectively.

Sum of components along OC

$= (1 \cos 60° + 2 \cos 30° + 3 + 4 \cos 30° + 5 \cos 60°)\mathbf{i} \approx 11.2\mathbf{i}$

Sum of components perpendicular to OC

$= (-1 \sin 60° - 2 \sin 30° + 0 + 4 \sin 30° + 5 \sin 60°)\mathbf{j} \approx 4.5\mathbf{j}$

Hence the resultant force $\mathbf{r} = 11.2\mathbf{i} + 4.5\mathbf{j}$ (Fig.23.15).

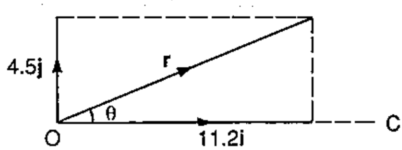


Fig. 23.15

$|\mathbf{r}|^2 = 11.2^2 + 4.5^2$ giving $|\mathbf{r}| \approx 12.1$.

The angle θ which the resultant makes with OC is given by

$\tan\theta = \frac{4.5}{11.2}$ giving $\theta = 21.9°$.

Hence the resultant is 12.1 N acting at 21.9° to OC.

We can also obtain the resultant graphically.

Let the forces along OA, OB, OC, OD, OE be **a**, **b**, **c**, **d**, **e**, respectively. A suitable scale is chosen, say 1 cm : 1 N. We start by drawing a line to represent **a** in magnitude and direction. (We can start with any force, the order is immaterial). Then we add the other forces, one at a time, each force starting from the end point of the one before.

Fig.23.16 shows the construction where **a**, **b**, **c**, **d**, **e** are the forces given above. The resultant **r** is then represented by the line OR in magnitude and direction. By measurement, we find that $|\mathbf{r}| = 12.1$ and $\theta \approx 22°$.

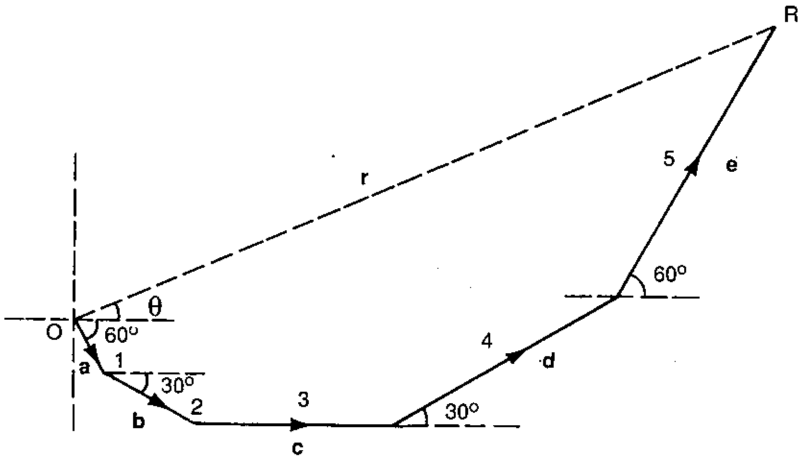


Fig. 23.16

Hence the resultant is 12.1 N acting at 22° to OC (approximately the same as before).

Exercise 23.1 *(Answers on page 650.)*

1 Resolve the forces shown in each diagram (Fig.23.17) in the directions of the dotted lines.

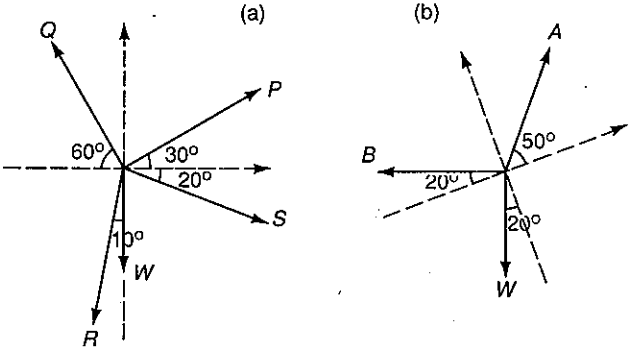


Fig. 23.17

2 Find the resultant of the following forces, in magnitude and direction (Fig.23.18).

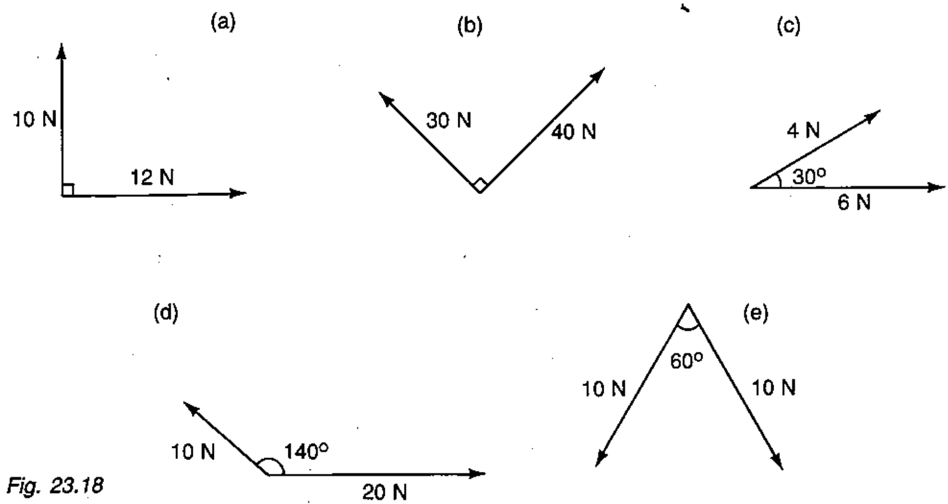


Fig. 23.18

3 Find the sum of the components of the forces acting as shown in Fig.23.19 in the direction of the dotted lines and hence find their resultants.

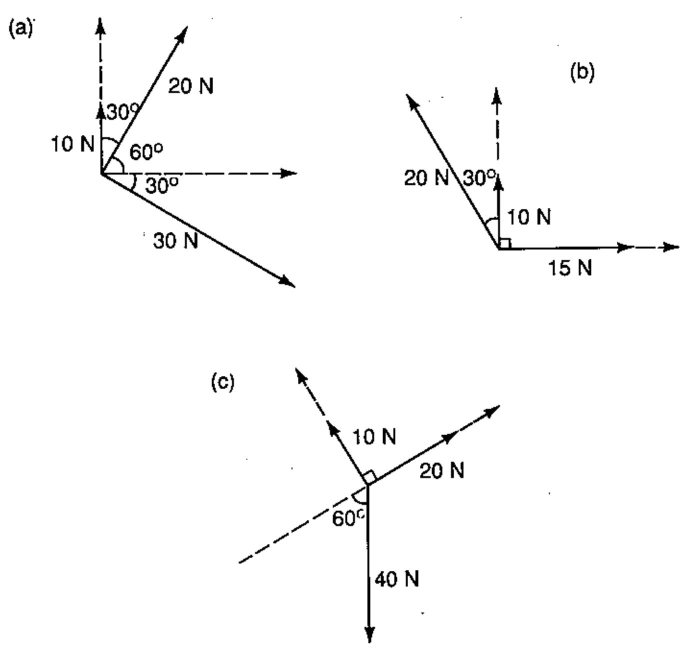


Fig. 23.19

4 Two forces P N and Q N include an angle of 120° and their resultant is $\sqrt{19}$ N. If the included angle between the forces were 60°, their resultant would be 7 N. Find P and Q. (C)

5 Find the magnitude and direction of the resultant of the following coplanar forces acting at a point O: 10 N in direction 000°; 5 N in direction 090°; 20 N in direction 135°; 10 N in direction 225°.

6 The resultant of two forces X N and 3 N is 7 N. If the 3 N force is reversed, the resultant is $\sqrt{19}$ N. Find the value of X and the angle between the two forces.

7 Two forces of 13 N and 5 N act at a point. Find the angle between the forces when their resultant makes the largest possible angle with the 13 N force. Find also the magnitude of the resultant when the angle between the forces has this value. (C)

8 The resultant of a force $2F$ N in a direction 090° and a force F N in a direction 330° is a force of 12 N. Calculate the value of F.

It is required to add a third force X N in a direction 270° so that the resultant of the system is in a direction 000°. Calculate the value of X. (C)

9 Two concurrent forces of equal magnitude have a resultant of 12 N. When one force is reversed the resultant becomes 6 N. Calculate the magnitude of each force and the angle between them. (C)

10 Fig.23.20 shows four forces in a plane. Given that $\cos\theta = \frac{3}{5}$ and that the resultant of the forces is $6\sqrt{2}$ N in a direction 225°, calculate the values of P and Q. (C)

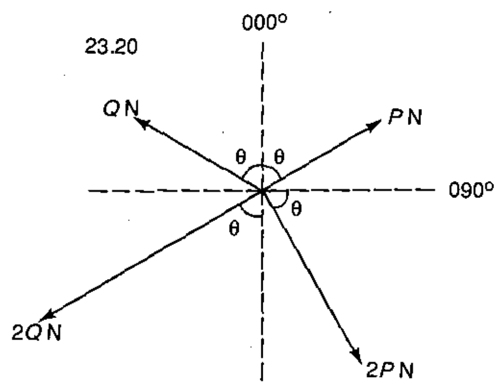


Fig.23.20

11 Four horizontal forces of magnitudes 1 N, 2 N, 3 N, and 4 N act at a point in the directions whose bearings are 000°, 060°, 120° and 270° respectively. Calculate the magnitude of their resultant.

A fifth horizontal force of magnitude 3 N now acts at the same point so that the resultant of all five forces has a bearing of 090°. Find the bearing of this fifth force. (C)

EQUILIBRIUM OF A PARTICLE

A particle under the action of 2 or more forces is in equilibrium if the resultant of the forces (i.e. the vector sum of all applied forces) is zero. If a particle under the action of 2 forces is in equilibrium, then the 2 forces **must be equal in magnitude but opposite in direction.**

TRIANGLE OF FORCES

Consider a particle acted upon by three forces **P**, **Q** and **R** as shown in Fig.23.21(a).

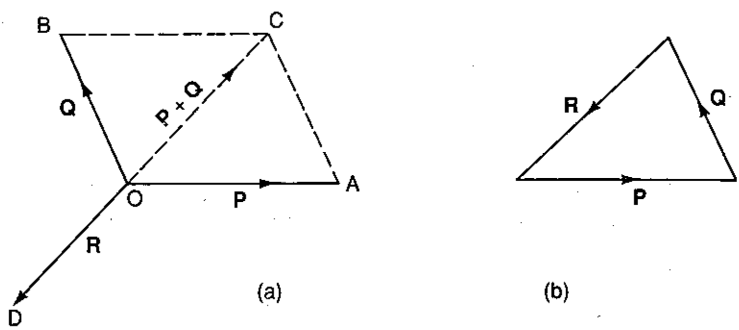


Fig. 23.21

The resultant of **P** and **Q** is represented by the diagonal $\overrightarrow{OC}$ of the parallelogram. The particle will be in equilibrium if **R** is equal in magnitude but opposite in direction to the resultant of **P** and **Q**.

This means that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OD} = 0$. This may be illustrated by the triangle shown in Fig.23.21(b), where the three forces **P**, **Q** and **R** are represented in magnitude and direction by the sides of the triangle taken in order (following on one after the other). This important result is known as the **triangle of forces**. We restate this result more formally thus:

If three forces act at a point and are in equilibrium, then they can be represented in magnitude and direction by the three sides of a triangle taken in order.

The converse is also true:

If three forces acting at a point can be represented by the sides of a triangle taken in order, then they are in equilibrium.

Example 4

A particle of weight 4 N is supported by a string attached to a fixed point and is pulled from the vertical by a horizontal force of 3 N. Find the tension in the string and the angle which the string makes with the vertical.

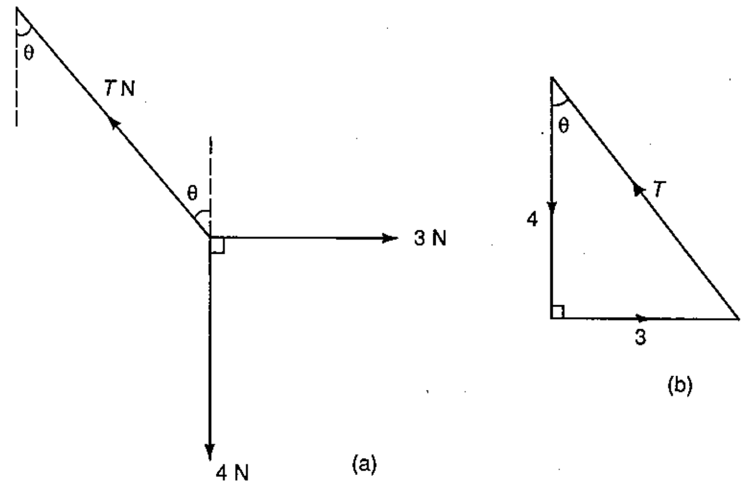


Fig. 23.22

Let the tension in the string be T N. The forces acting on the particle are shown in Fig.23.22(a). We now draw the triangle of forces (Fig.23.22(b)).

We then have $T^2 = 3^2 + 4^2$, giving $T = 5$.

Also $\tan\theta = \frac{3}{4}$, giving $\theta \approx 36.9°$.

Hence the tension in the string is 5 N and the string makes an angle of 36.9° with the vertical.

LAMI'S THEOREM

If three forces are in equilibrium, **Lami's theorem** relates them to the angles between their directions. (It is in fact a version of the sine rule.)

P, Q, R are three forces in equilibrium (Fig.23.23(a)) and they form a triangle of forces (Fig.23.23(b)).

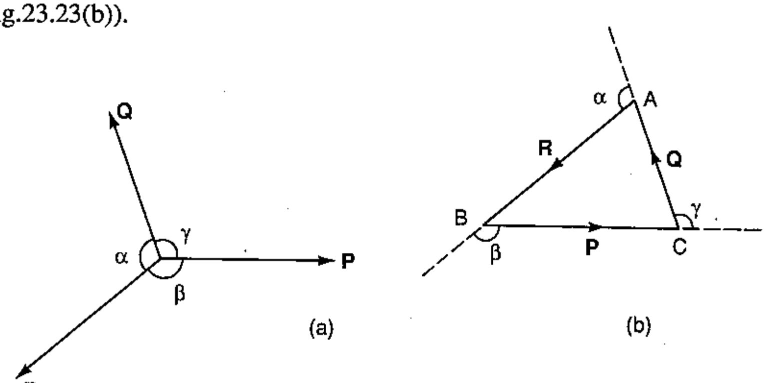


Fig. 23.23

If the angle between **P** and **Q** in Fig.23.23(a) is γ, then $\angle ACB = 180° - \gamma$ in Fig.23.23(b) and similarly for the other angles.

By the sine rule,

$$\frac{BC}{\sin(180° - \alpha)} = \frac{AC}{\sin(180° - \beta)} = \frac{AB}{\sin(180° - \gamma)}$$

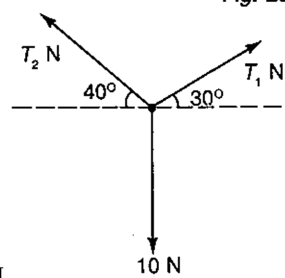
i.e. $\dfrac{P}{\sin \alpha} = \dfrac{Q}{\sin \beta} = \dfrac{R}{\sin \gamma}$ (where $P = |\mathbf{P}|$, $Q = |\mathbf{Q}|$ and $R = |\mathbf{R}|$)

as P, Q, R are proportional to BC, AC and AB respectively.

Example 5

A particle of weight 10 N is suspended by 2 strings. If these strings make angles of 30° and 40° to the horizontal, find the tensions in the string.

Fig. 23.24



Let the tensions in the strings be T_1 N and T_2 N respectively (Fig.23.24).

Using Lami's theorem,

$$\frac{T_1}{\sin(90° + 40°)} = \frac{T_2}{\sin(90° + 30°)} = \frac{10}{\sin 110°}$$

giving $T_1 \approx 8.2$ and $T_2 \approx 9.2$.

Hence the tensions in the strings are 8.2 N and 9.2 N.

POLYGON OF FORCES

We can extend the idea of the triangle of forces to more than three forces acting at a point in equilibrium if the forces are coplanar. This extension is called the **polygon of forces.** If a number of forces acting at a point are in equilibrium then they can be represented in magnitude and direction by the sides of a closed polygon taken in order. Fig.23.25(a) shows forces **P, Q, R, S, T** which can be represented by the sides of the polygon ABCDE in Fig.23.25(b).

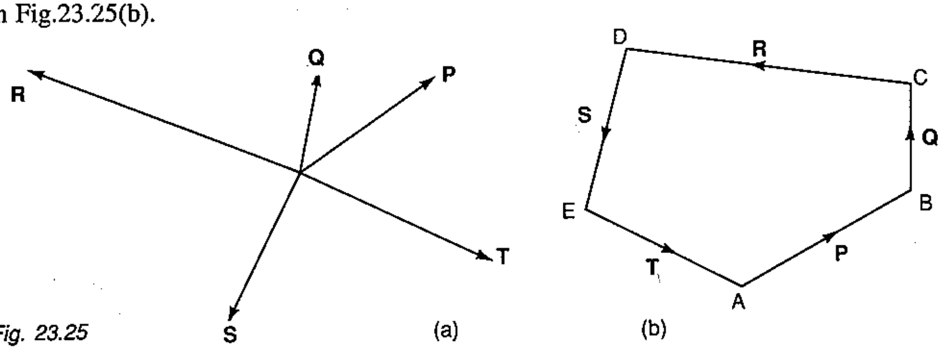


Fig. 23.25 (a) (b)

The sides of the polygon may cross each other if this is necessary by the layout of the forces, but must follow each other in order, that is, one force must begin where the previous force ends. We may start with any force. Constructing a polygon of forces can be used as a graphical method to solve problems involving more than three forces acting at a point.

Example 6

Five forces act as shown in Fig.23.26 and are in equilibrium. Find the magnitude and direction of force **P**.

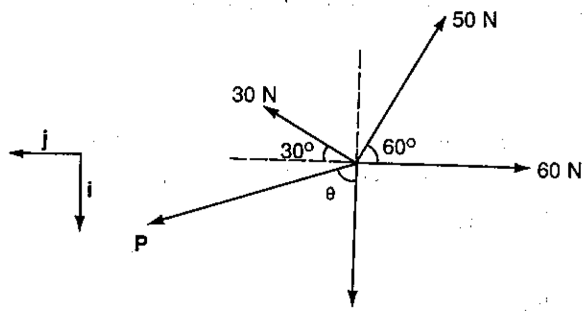


Fig. 23.26

We will solve this problem using 2 methods.

1 *Graphical method*

Construct the polygon of forces to a suitable scale (say 1 cm : 10 N). ABCDE is drawn and then E is joined to A. $\overrightarrow{EA}$ would then represent the force P in magnitude and direction. From the drawing, we find that $\mathbf{P} = 62$ N at an angle 73° to the 40 N force.

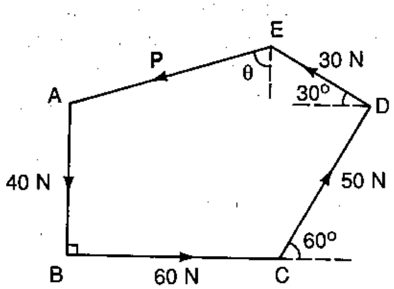


Fig. 23.27

2 *Resolution of forces*

Since the forces are in equilibrium, the resultant is zero, and hence its component in any direction is zero. Take $\mathbf{i}$ in the direction of the 40 N force and $\mathbf{j}$ in the negative direction of the 60 N force (Fig.23.26).

Resolving along the $\mathbf{i}$ direction, we have
$(P \cos \theta + 40 - 50 \sin 60° - 30 \sin 30°)\mathbf{i} = 0$ (where $P = |\mathbf{P}|$)
giving $P \cos \theta = 18.30$. (i)

Resolving along the **j** direction, we have
$(P \sin\theta - 60 - 50 \cos 60° + 30 \cos 30°)\mathbf{j} = 0$
giving $P \sin\theta = 59.02$. (ii)

The components of **P** are shown in Fig. 23.28.

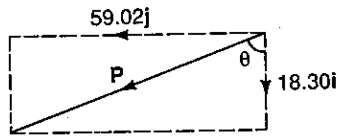


Fig. 23.28

The magnitude of $\mathbf{P} = \sqrt{18.30^2 + 59.02^2} \approx 61.8$.

The angle θ which **P** makes with the 40 N force is given by

$$\tan\theta = \frac{59.02}{18.30} \text{ giving } \theta \approx 72.8°.$$

Hence **P** is 61.8 N making an angle 72.8° with the 40 N force.

Exercise 23.2 *(Answers on page 650.)*

1 Find the unknown forces and angles in Fig.23.29. The forces in each set are in equilibrium.

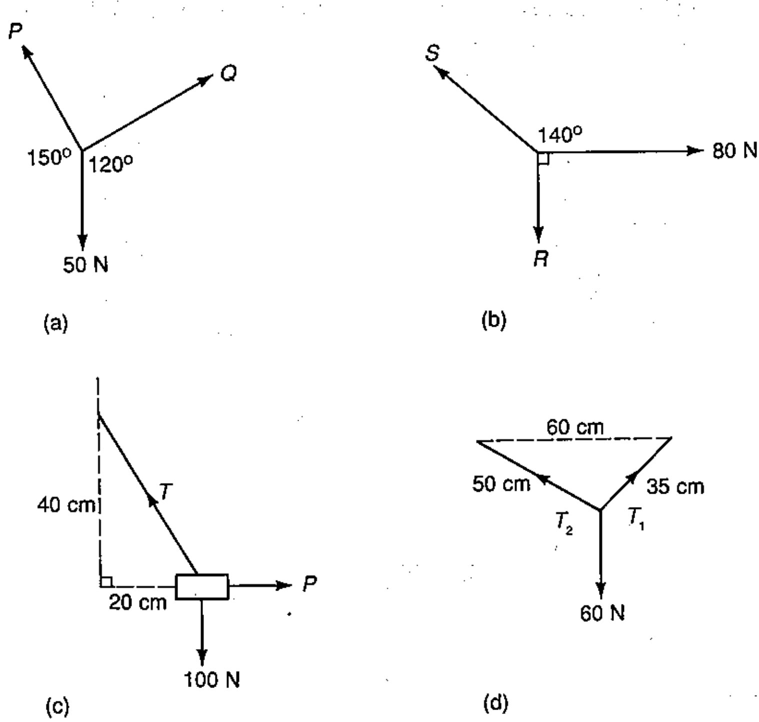


Fig. 23.29

2 A particle M of weight 20 N is supported by two strings, MA and MB making angles of 30° and 45° on opposite sides with the vertical through M. Find, by drawing or calculation, the tensions in the strings.

3 A body of weight 40 N hangs from a string attached to a point on a vertical wall. The string will break when its tension exceeds 50 N. If the body is pulled away from the wall by a horizontal force P N, what angle does the string make with the vertical when it breaks and what is the value of P?

4 Three forces acting at the origin O can be represented by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ where A, B, C have coordinates (5,2), (–3,8), (–2,–10) respectively. Show that the forces are in equilibrium. (L)

5 The three coplanar forces P N, $2P$ N and 20 N are in equilibrium as shown in Fig.23.30. Calculate the value of P and of θ. (C)

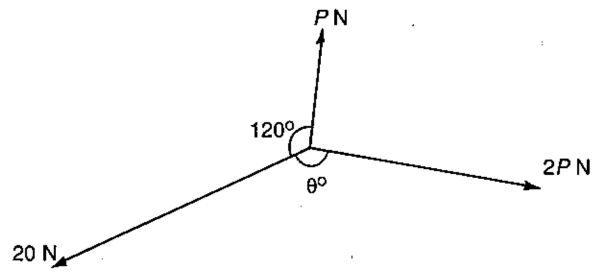


Fig. 23.30

6 Find the unknown forces in magnitude and direction in the following systems of forces which are in equilibrium.

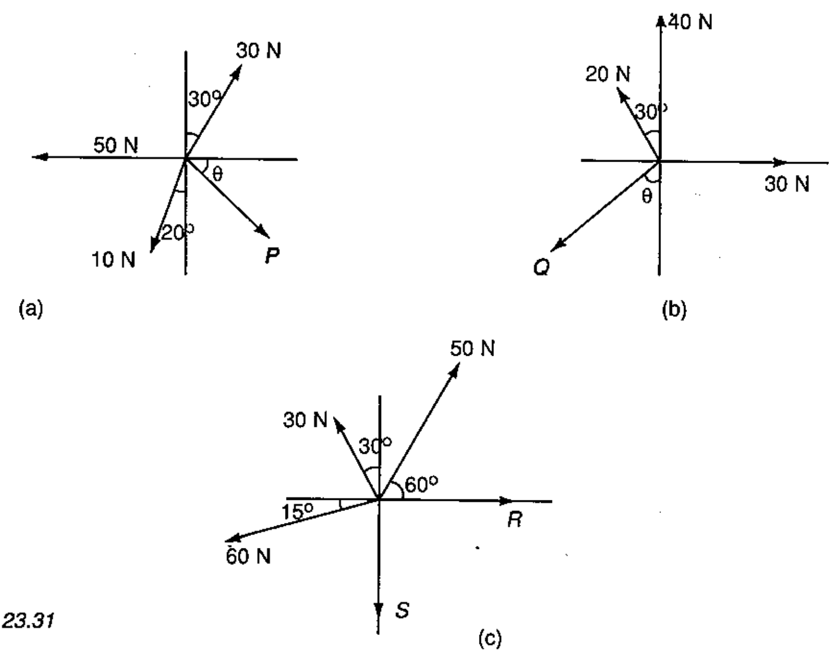


Fig. 23.31

7 Find the resultant in magnitude and direction of forces 10, 20, 30, 40 N acting in directions 060°, 120°, 180°, 270° respectively.

8 Referred to rectangular axes with O as the origin, A is the point (4,3), B is the point (12,5), C is the point (0,–6) and the same scale is used for the x– and the y-coordinates. Forces of magnitudes 75 N, 65 N and 35 N act at O towards A, B and C respectively. Find, by calculation or drawing, the magnitude of the resultant force and the angle it makes with the x-axis. (L)

9 Five strings are attached to a point in equilibrium and radiate horizontally from this point. The tensions and directions of four of the strings are: 50 N, 060°; 40 N, 090°; 100 N, 270°; 20 N, 330°. Find the tension in the fifth string and its direction.

10 Fig.23.32 shows a particle in equilibrium under the action of five horizontal forces. Calculate the values of θ and P.

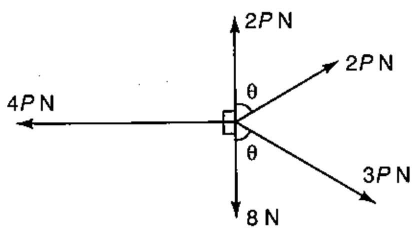


Fig. 23.32

11 The four coplanar forces in Fig.23.33(a) have the single force shown in Fig.23.33(b) as their resultant. If $\tan \phi = \frac{3}{4}$, find the value of P and of ϕ.

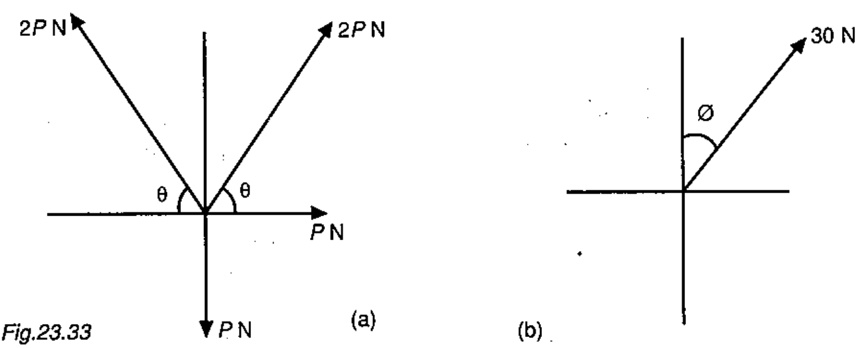


Fig.23.33

12 Forces of 8 N and P N act along the lines OA and OB where ∠AOB is an obtuse angle.

(a) If $P = 4$, the resultant is perpendicular to OB. Find ∠AOB.

(b) Find the value of P if the resultant is to bisect ∠AOB.

13 When the angle between two forces P and Q is 60°, the magnitude of their resultant is $\sqrt{19}$ N. When the angle is 120°, the magnitude is $\sqrt{7}$ N. Calculate the value of P and of Q.

14 The following four coplanar forces acting at a point O are in equilibrium: 10 N in direction 000°, 20 N in direction 090°, Q N in direction $(180 + \theta)°$ $(\theta < 90)$ and P N in direction 300°.

(a) By drawing a polygon of forces, find the possible values of P and of θ when $Q = 20$.

(b) Deduce the values of P and θ if Q is to be the minimum force to maintain the equilibrium and find the value of Q in this case.

FRICTION

'Smooth' surfaces are often specified in problems. A smooth surface is a convenient approximation in Applied Mathematics. In real life, however, it is not possible to obtain such a surface. When one body moves over another, there is always some resistance to motion, the frictional resistance between the two surfaces. The size of this resistance will depend on the nature of the two surfaces in contact.

Consider a particle of weight W at rest on a rough horizontal plane (Fig.23.34).

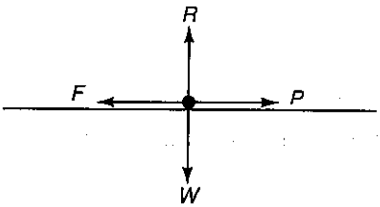


Fig. 23.34

The normal reaction R of the plane on the particle has the same magnitude as W.

We now apply a horizontal force P, slowly increasing its magnitude from zero, to the particle. The frictional force comes into operation when an attempt is made to move the particle. This frictional force always acts to oppose the tendency to move. Initially the particle does not move because the frictional force is equal (but opposite) to the applied force. As the force P is increased, the frictional force also increases to equal the applied force. Eventually, however, the frictional force reaches its maximum and a slight increase in P will make the particle slide.

This maximum value of the frictional force F is called the **limiting friction**. From experiments, we obtain the ratio

$$\frac{\text{limiting frictional force}}{\text{normal reaction}} = \mu$$

where the Greek letter μ (read *mu*) is a constant quantity and is called the **coefficient of friction**. The value of μ depends only on the nature of the surfaces in contact and is independent of the areas in contact or the forces present. We can also write limiting frictional force $F = \mu R$ (where R = normal reaction).

Hence until sliding takes place, the friction F is less than μR. In general then we have $F \leq \mu R$ with the equality holding only at the point of sliding.

After the limiting friction has been reached and the particle starts to slide, the frictional force will continue to act at this maximum value, μR to oppose the motion of the particle. (This is not strictly true but we shall ignore any variation.)

In many machines, where metal parts move against each other, friction is produced. Friction is usually reduced by lubrication so that machines can function more effectively. For example, the piston in a car cylinder is lubricated by the oil pumped up from the sump. On the other hand we need friction to move about. We rely on the friction between our shoes and the ground to push us forward. On smooth ice or a polished floor, we tend to slip as the frictional force is small. Similarly the frictional force between the tyres of a car and the road is essential to the motion of the car.

In solving problems involving friction, we first mark and find the normal reaction between the particle and the surface. We put $F = \mu R$ if sliding is taking place or about to take place. Otherwise $F < \mu R$.

As mentioned earlier in this chapter, we take the weight of a particle of mass m kg to be $10m$ N.

Example 7

A particle of mass 1 kg rests on a horizontal floor. The coefficient of friction between the particle and the floor is $\frac{1}{2}$. What force is required just to make the particle move when

(a) pulling horizontally,

(b) pulling at an angle of 30° to the horizontal?

(a) The weight of the particle = 10×1 N as the mass is 1 kg.
The normal reaction R = 10 N (Fig.23.35) as the particle is in equilibrium vertically.

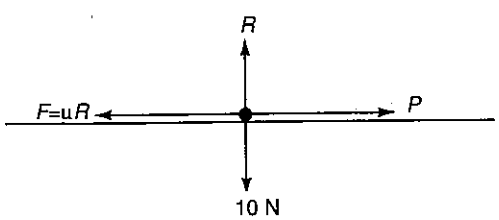


Fig. 23.35

On the point of sliding,

$F = \mu R = \frac{1}{2} \times 10 = 5$

Hence the horizontal force P (= F) required just to make the particle move is 5 N.

(b) In this case, the normal reaction is not 10 N, as the vertical component of P will have to be taken into account (Fig.23.36).

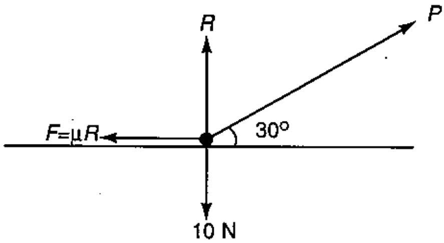


Fig. 23.36

Resolving vertically, we have

$$R + P \sin 30° = 10 \qquad \text{(i)}$$

Since the particle is in limiting equilibrium,

$$P \cos 30° = \mu R$$

i.e. $P \times \frac{\sqrt{3}}{2} = \frac{1}{2}R$, giving $R = \sqrt{3}P$. (ii)

Substituting into (i),

$$\sqrt{3}P + \tfrac{1}{2}P = 10 \text{ giving } P \approx 4.5.$$

Hence the required force is 4.5 N.

Example 8

A particle of mass 1 kg is placed on a rough plane inclined at an angle 30° to the horizontal. The coefficient of friction is $\frac{2}{5}$. Find the least force parallel to the plane that is required

(a) to hold the particle at rest,

(b) to make the particle slide up the plane.

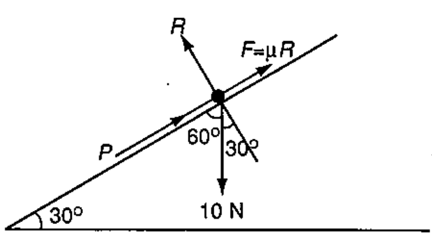


Fig. 23.37

(a) Fig.23.37 shows the forces acting. As the particle is being held at rest, it is on the point of slipping down. Hence the frictional force F ($= \mu R$) acts upwards. We now resolve the forces acting on the particle normal and parallel to the plane respectively to obtain:

$$R = 10 \cos 30°$$

i.e. $R = 5\sqrt{3}$ (i)

and $P + \mu R = 10 \sin 30°$

i.e. $P + \frac{2}{5}R = 5$ (ii)

Substituting R from (i) into (ii), we have

$$P + \tfrac{2}{5}(5\sqrt{3}) = 5 \text{ giving } P \approx 1.54 \text{ N.}$$

Hence the least force (parallel to the plane) required to hold the particle is 1.54 N.

b) In this case, when the particle is about to move upwards, the frictional force F $(= \mu R)$ acts down the plane (Fig.23.38).

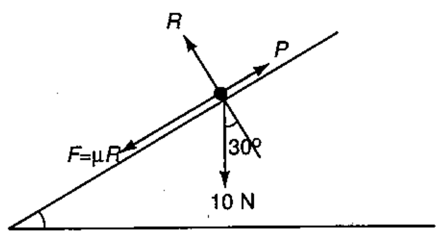


Fig.23.38

Resolving forces as before, we have

$$R = 10 \cos 30°$$

i.e. $R = 5\sqrt{3}$ (iii)

and $P = 10 \sin 30° + \mu R$

i.e. $P = 5 + \frac{2}{5}R$ (iv)

Substituting R from (iii) into (iv), we have

$$P = 5 + \tfrac{2}{5}(5\sqrt{3}) \text{ giving } P \approx 8.46 \text{ N.}$$

Hence the least force (parallel to the plane) required to make the particle move up the plane is 8.46 N.

Note that the force in part (a) is considerably less than that in part (b).

Exercise 23.3 *(Answers on page 651.)*

(Take $g = 10\ m\ s^{-2}$.)

1 A mass of 4 kg rests on a rough table ($\mu = 0.4$). Find the least force required to make it move.

2 If a force of 10 N is just sufficient to move a mass of 2 kg resting on a rough horizontal table, find the coefficient of friction.

3 A block of mass 300 kg is just pulled along rough horizontal ground by two equal forces P N inclined at 30° to the line of motion. If the coefficient of friction is 0.6, find the value of P.

4 A block of mass 1 kg is placed on an inclined plane of angle 60° and is just held there at rest by a horizontal force P. If the coefficient of friction is 0.4, find P.

5 A body of mass 5 kg can just rest on a plane when the plane is inclined at 60° to the horizontal. Find the force parallel to the plane required to push the body up the plane if the inclination is reduced from 60° to 30°.

6 A horizontal force of 10 N just prevents a mass of 2 kg from sliding down a rough plane inclined at 45° to the horizontal. Find the coefficient of friction.

7 A mass of M kg is slowly pulled up a rough inclined plane of angle 30° by a string parallel to the plane. The coefficient of friction is 0.6. If the string will break when its tension exceeds 100 N, what is the largest possible value of M?

8 A particle of mass M kg is placed on a rough inclined plane whose angle with the horizontal can be changed. When this angle is θ, the particle just begins to slip. Show that the coefficient of friction = $\tan\theta$.

9 A block of mass 5 kg is placed on a rough plane inclined at an angle α to the horizontal where $\sin\alpha = \frac{3}{5}$. The coefficient of friction between the block and the plane is $\frac{2}{5}$. A force Q N acts on the block in an upward direction parallel to a line of greatest slope of the plane.
Calculate
(i) the minimum value of Q which will prevent the block from sliding down the plane,
(ii) the value of Q when the block is about to move up the plane. (C)

10 A particle of mass 0.5 kg is at rest on a rough plane inclined at an angle θ to the horizontal where $\sin\theta = \frac{3}{5}$. The particle is just prevented from sliding from the plane by a force of 2 N applied in an upward direction parallel to a line of greatest slope of the plane.
(i) Draw a figure showing all the forces acting on the particle.
(ii) Calculate the coefficient of friction between the particle and the plane.
(iii) Calculate by how much the force of 2 N must be increased so that the particle is about to move up the plane. (C)

11 A particle of mass 0.5 kg rests on a plane inclined at an angle θ to the horizontal where $\tan\theta = \frac{3}{4}$. It is just prevented from sliding down the plane by a horizontal force of 2.5 N. Find the value of the coefficient of friction between the particle and the plane.

When the horizontal force of 2.5 N is replaced by a force of P N acting up a line of greatest slope of the plane, the particle is about to move up the plane. Calculate the value of P. (C)

12 A mass of 10 kg is in equilibrium on a rough inclined plane under the action of a force acting up the plane along a line of greatest slope. The angle of inclination of the plane to the horizontal is α where $\sin\alpha = \frac{3}{5}$.

When the force is P N, the mass is on the point of sliding down the plane. When the force is $2P$ N, the mass is on the point of sliding up the plane.

If the coefficient of friction between the mass and the plane is μ, calculate the value of P and of μ. (C)

13 A block of mass 5.2 kg is placed on a rough plane inclined at an angle α to the horizontal where $\sin\alpha = 0.6$. The coefficient of friction between the block and the plane is 0.4. The block is just prevented from sliding down the plane by a horizontal force P N.

Draw a diagram showing all the forces acting on the block and calculate the value of P. (C)

14 Fig.23.39 shows a small block of mass 5 kg held against a rough vertical wall by a force P N inclined at an angle of 60° to the wall. If the coefficient of friction is 0.4, calculate the value of P when the block is

(a) just prevented from slipping down,

(b) just about to move up the wall.

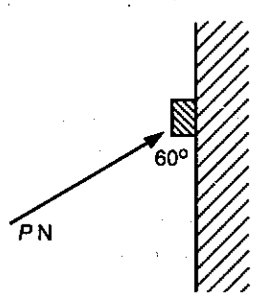


Fig. 23.39

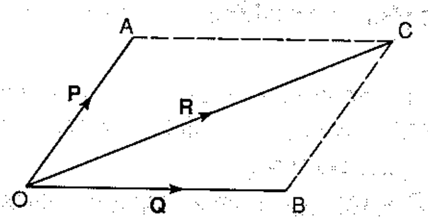
15 Find the resultant in magnitude and direction of the following coplanar forces acting at a point: 10 N in direction 000°, 20 N in direction 060° and 5 N in direction 090°.

If these forces are attached to a particle of mass 8 kg on rough horizontal ground and the body is just about to move, calculate the coefficient of friction.

SUMMARY

- *Composition of 2 forces,* **P** *and* **Q**

  If $\overrightarrow{OA}$ represents **P** and $\overrightarrow{OB}$ represents **Q**, then the diagonal $\overrightarrow{OC}$ of the parallelogram represents the resultant **R**.



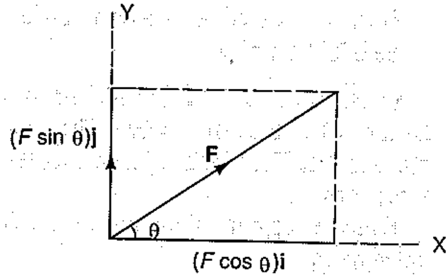
- *Resolution of a force,* **F**

  F can be resolved into 2 perpendicular components:

  x-component $= (F \cos \theta)\mathbf{i}$

  y-component $= (F \sin \theta)\mathbf{j}$

  where $F = |\mathbf{F}|$.



- *Coplanar forces acting at a point*

  The resultant can be found by first resolving all the forces in any 2 perpendicular directions and then recombining the components into one force.

- *Equilibrium of a particle*

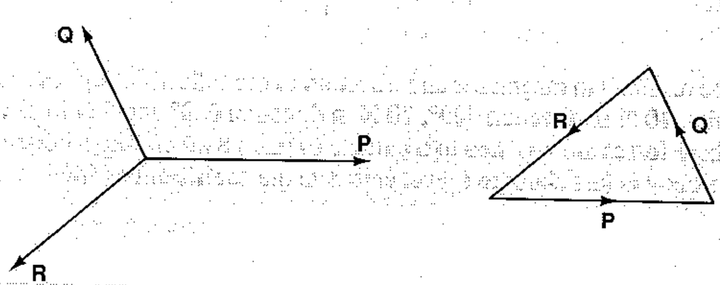
  A particle under the action of 2 or more forces is in equilibrium if the resultant of the forces is zero.

- *Triangle of forces*

  If three forces act at a point and are in equilibrium, then they can be represented in magnitude and direction by the three sides of a triangle taken in order.

  The converse is also true:

  If three forces acting at a point can be represented by the sides of a triangle taken in order, then they are in equilibrium.

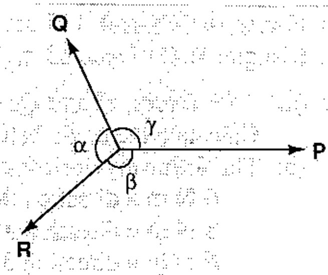


- *Lami's theorem*

  In the figure, if **P**, **Q**, **R** are in equilibrium then

  $\frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}$

  where $P = |\mathbf{P}|$, $Q = |\mathbf{Q}|$ and $R = |\mathbf{R}|$.



- *Polygon of forces*

  If a number of coplanar forces acting at a point are in equilibrium, then they can be represented in magnitude and direction by the sides of a closed polygon.

- *Friction*

  Friction always acts to oppose relative motion.
  The coefficient of friction, μ determines the limiting friction.
  The frictional force F and the normal reaction R are related by $F \leq \mu R$.
  At limiting friction, $F = \mu R$ and the object starts to slide over the surface.

REVISION EXERCISE 23 (*Answers on page 651.*)

1 A particle M of mass 4 kg is suspended by two strings MA and MB of lengths 40 cm and 30 cm respectively. A and B are on the same horizontal level and AB = 50 cm. Find the tensions in the strings.

2 Four horizontal concurrent forces, 20 N acting in a direction 245°, 12 N in a direction 020°, P N in a direction 320° and Q N in a direction 110° are in equilibrium. Determine the value of P and of Q. (C)

3 The following horizontal forces pass through a point O: 5 N in a direction 000°, 1 N in a direction 090°, 4 N in a direction 225° and 6 N in a direction 315°. Find the magnitude and direction of their resultant.

Two further horizontal forces are introduced to act at O: P N in a direction 135° and Q N in a direction 225°. If the complete set of forces is now in equilibrium, calculate the value of P and of Q. (C)

4 A mass of 5 kg is suspended by a light string from a fixed point. A force of P N making an angle of 30° with the horizontal keeps the mass in equilibrium with the string making an angle of 40° with the vertical. By drawing or calculation, find the value of P and the tension in the string.

5 The resultant of a force $2P$ N in a direction 060° and a force 10 N in a direction 180° is a force of $\sqrt{3}P$ N. Calculate the value of P and the direction of the resultant.

A third force of 25 N, concurrent with the other two and in the same plane, is added so that the resultant of the system is in the direction 180°. Find the direction in which the third force is applied and find the magnitude of the resultant. (C)

6 Forces of 5 N and 3 N act along the sides $\overrightarrow{AB}$, $\overrightarrow{AC}$ respectively of an equilateral triangle ABC of side 12 m. Find the magnitude and direction of their resultant. (C)

7 (a) Two forces, P and Q, of 2 N and 6 N respectively have a resultant of 5 N. Calculate the angle between P and Q.
(b) The following horizontal forces pass through a point O:
6 N in a direction 045°,
8 N in a direction 180°,
P N in a direction 330°.
The resultant of these forces is in a direction 000°. Calculate
(i) the value of P.
(ii) the magnitude of the resultant. (C)

8 Two equal concurrent forces, each of F N, have a resultant of 6 N. When the magnitude of one of the forces is doubled, the resultant becomes 11 N. Calculate the value of F and the angle between the forces. (C)

9 A, B, C, D, E, F are the vertices of a regular hexagon. Forces of 10 N, 10 N, 5 N and 20 N act at A in the directions AB, AC, EA and AF respectively. Find, by drawing or calculation, the magnitude of their resultant and the angle it makes with AB.

10 A body of mass 4 kg lies on a rough horizontal plane. It is acted on by an upward force of 26 N inclined at an angle α to the horizontal, where $\sin \alpha = \frac{5}{13}$. If the body is about to move, calculate the coefficient of friction between the body and the plane.
The force is now removed and the plane is tilted to an angle β to the horizontal, where $\sin \beta = \frac{3}{5}$. A force P N acts in an upward direction on the body parallel to a line of greatest slope of the plane. Given that the body is about to move up the plane, calculate the value of P. (C)

11 Three ropes are attached to a heavy block which is about to be pulled due N along rough level ground. The ropes are pulled horizontally in the directions 350°, 000° and 030° with forces P N, 100 N and 160 N respectively. Find the value of P and the resultant force acting on the block.
If the mass of the block is 80 kg, calculate the coefficient of friction.

12 A block of mass 1.5 kg is placed on a rough plane inclined at an angle α to the horizontal where $\sin \alpha = 0.6$. The coefficient of friction between the block and the plane is $\frac{1}{4}$. The block is acted on by a force P N perpendicular to the plane and is on the point of slipping downwards.
Draw a diagram showing all the forces acting on the block and calculate the value of P. (C)

13 A body of mass m kg is held in equilibrium on a rough plane, inclined at an angle θ to the horizontal, by a force of P N, acting up a line of greatest slope.
When $P = 15.6$, the body is about to slide up the plane, and when $P = 8.4$, the body is about to slide down the plane.
Given that $\tan \theta = \frac{3}{4}$, find the value of m and of μ, the coefficient of friction between the body and the plane. (C)

14 A body of mass 4 kg rests on rough horizontal ground. The coefficient of friction between the body and the ground is $\frac{4}{3}$. A light rod attached to the body is pushed towards the body in a downward direction at an angle of θ to the horizontal until the body is on the point of sliding.
Given that $\tan\theta = \frac{1}{4}$, calculate
(i) the force exerted by the rod,
(ii) the frictional force between the body and the ground. (C)

15 A block of mass 20 kg stands on a rough inclined plane of angle 20°. A man pulls on a rope attached to the block and just prevents it from slipping down. If the rope makes an angle of 10° with the plane and the coefficient of friction is $\frac{1}{4}$, with what force does he pull?

16 The least force required to move a mass of 10 kg on a rough horizontal plank is 25 N. If the plank is now tilted to make an angle of 45° with the horizontal, the coefficient of friction being unchanged, what is the least force required to move the mass up the plank?

17 A long handle is attached to a body of mass 15 kg resting on rough horizontal ground. The handle has two positions making angles of θ and ϕ to the horizontal where $\tan\theta = \frac{2}{3}$ and $\tan\phi = \frac{4}{5}$.
If the coefficient of friction between the body and the ground is $\frac{3}{4}$, calculate the difference in the forces in each position needed to just move the body.

18 A body of mass 8 kg is placed on a rough inclined plane of angle 30°. The coefficient of friction is 0.5. Find the least force parallel to the plane required to keep the body in equilibrium.

19 The following coplanar forces act at a point O: 10 N in direction 045°, 15 N in direction 000°, P N in direction 270°.
(a) Find their resultant in magnitude and direction when $P = 20$.
(b) Find also by drawing or calculation, the minimum force which must be added to produce equilibrium and state the value of P in this case.

20 Coplanar forces of 8 N in direction 000°, 5 N in direction 090°, P N in direction 240° and Q N in direction 150° act at a point and are in equilibrium. By drawing or calculation, find the values of P and Q.
If however $P = 10$ and the forces are still in equilibrium, find the new value of Q and its direction.

Newton's Laws of Motion

24

In this chapter, we shall discuss the relation between the force acting on a body and its motion. It was Newton who first understood this relationship and who enunciated the three laws of motion. **Dynamics,** which is the study of the effect of forces acting on a body, is based on these laws.

NEWTON'S LAWS

Newton's laws may be stated as follows:

First law: Every body remains at rest or moves with uniform velocity unless it is made to change this state by external forces.

Second law: If a force acts on a body and produces a certain acceleration, then the force is proportional to the product of the mass of the body (assumed constant) and the acceleration. Also the acceleration takes place in the direction of the force.

Third law: To every action there is an equal and opposite reaction.

The first law disposed of a pre-Newtonian misconception that a force was required to keep a body moving. It explains why a spacecraft once free of the earth's gravitational field will continue to travel in a straight line with steady speed until it is affected by the pull of another planet. It is not possible to attain this state on earth, as friction or air resistance is always acting and gradually slows down any moving body.

The second law involves the concept of mass. The mass of a body is usually defined as the quantity of matter in the body. Though this is not very helpful, the idea is reasonably easy to grasp; that is, a small body requires less force to give it a certain acceleration than a more massive one. Thus, if I push a certain truck with a certain force, an acceleration will be produced. If I push two similar trucks joined together with the same force, the acceleration will be halved as I am pushing an object twice the mass. The standard unit of mass is 1 kg (kilogram) = 1000 g (gram). The second law can be written symbolically as follows:

$$P \propto ma$$
$$\text{or } P = kma$$

where P = force, m = mass, a = acceleration, k = some constant.

For a standard mass of 1 kg having standard unit acceleration 1 m s^{-2}, the force acting will be

$$P = k \times 1 \times 1 = k \text{ units}$$

If we now take the unit of force to be that which will produce an acceleration of 1 m s^{-2} in a mass of 1 kg, then $P = 1$ and $k = 1$. This unit of force is 1 **Newton** (1 N). The equation of motion then simplifies to

$$\begin{array}{l} P = ma \\ \text{or force} = \text{mass} \times \text{acceleration} \\ \quad\ \text{(in N)} \quad \text{(in kg)} \quad\ \ (\text{in m s}^{-2}) \end{array}$$

For large forces, the kilonewton, kN can be used (1 kN = 1000 N).

The third law means that the force a body exerts on another is always accompanied by an equal and opposite force exerted by the second body on the first. We have used this law implicitly when we discussed reaction in the last chapter and we shall dwell on this further when we discuss connected particles.

MASS AND WEIGHT

All bodies are attracted to the earth by a gravitational force. Near the surface of the earth, this force produces an acceleration g m s^{-2}. g is about 9.8, but varies slightly over the surface of the earth. A mass of m kg if free to move would fall with an acceleration g m s^{-2}. (Note that the acceleration is independent of the mass of the body.) The force acting on the body is thus mg N and this is the weight of the body.

The mass of a body is constant but its weight is not. In fact, the weight varies with locality depending on the value of g which we noted earlier varies slightly over the surface of the earth. So the weight of a mass of 1 kg is about 9.8 N. In space where $g = 0$ or nearly so, its weight would be zero (weightless). On the moon where $g = 1.6$, its weight would be about 1.6 N though its mass is still 1 kg.

In everyday life, the distinction between mass and weight is blurred. When we say the 'weight' of a person is 50 kg, we really mean that his 'mass' is 50 kg. His actual weight would be 50g N (or approximately 500 N). Similarly, a packet of washing powder labelled 'net weight: 1 kg' should actually read 'net mass: 1 kg'. In our work the distinction will be carefully noted between mass and weight and the correct units used for each. The relation is that the weight W of a mass m (in kg) is mg (in N) i.e. $W = mg$.

As in Chapter 20, we will continue to take the approximate value of $g = 10$ m s^{-2} in our work. In the worked examples that follow, we will use $\longrightarrow$ to represent a force and $\twoheadrightarrow$ to represent acceleration in our diagrams.

Example 1

A force of 1 N acts on a particle of mass 2 kg which is initially at rest. Find the resulting acceleration. Find also the velocity of the particle after 5 s.

Using $P = ma$

we have $1 = 2a$

giving $a = \frac{1}{2}$

Hence the acceleration is $\frac{1}{2}$ m s^{-2}.

The velocity v m s^{-1} after 5 s is given by

$$v = u + at$$
$$= 0 + \tfrac{1}{2}(5) = 2.5$$

Therefore the velocity after 5 s is 2.5 m s^{-1}.

Example 2

A horizontal force of 0.5 N acts on a body of mass 0.2 kg (Fig. 24.1). There is a frictional force of 0.2 N opposing the first force. What acceleration will be produced?

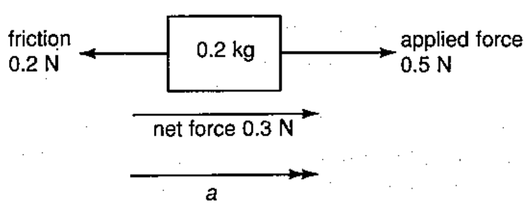


Fig. 24.1

The resultant or net force acting is $0.5 - 0.2 = 0.3$ N.

The acceleration a m s^{-2} will take place in the direction of the resultant and is given by

$P = ma$

i.e. $0.3 = 0.2a$

giving $a = 1.5$

Hence the acceleration produced is 1.5 m s^{-2}.

Example 3

Two forces 5 N and 8 N in the directions 030° and due E respectively act on a body of mass 2 kg. Find the acceleration of the body.

We first find the resultant of the two forces by drawing or calculation (Fig. 24.2).

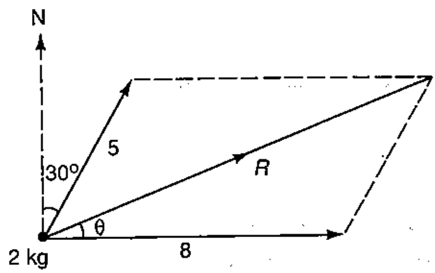


Fig. 24.2

The resultant force is $P = 11.36$ N in the direction 067.6° ($\theta = 22.4°$).

Then using $\quad P = ma,$

we have $\quad 11.36 = 2a$

giving $\quad a = 5.68$

Hence the acceleration is 5.68 m s^{-2} in the direction 067.6°.

Example 4

A parcel of 4 kg is suspended from a spring balance in a lift. What does the balance read if the lift is

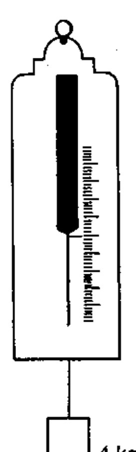
(a) moving with uniform speed,

(b) accelerating upwards at 0.5 m s^{-2},

(c) accelerating downwards at 0.5 m s^{-2}?

A spring balance consists of a strong spring with a pointer and scale attached. Using a result in Physics that the extension of a spring is proportional to the tension in the spring, the scale can be calibrated to show the force extending the spring.

Fig. 24.3



We consider the forces acting on the parcel i.e. its weight of $4g$ N acting vertically downwards and the tension in the spring vertically upwards.

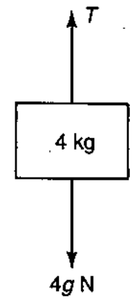


Fig. 24.4

(a) Since the parcel is moving with uniform speed, there is no net force acting on it.

Therefore $T = 4g$

$= 40$ N

which is what the balance should read.

(b) There is an upward acceleration. This means there is a resultant force upwards and this resultant force

$= T - 4g$

Therefore $T - 4g = 4(0.5)$

$T - 40 = 2$

$T = 42$

Hence the new reading on the balance is 42 N.

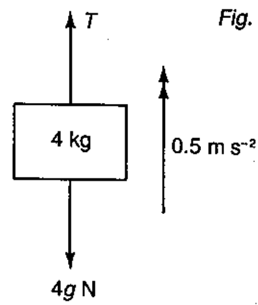


Fig. 24.5

(c) Similarly, when there is a downward acceleration, we have a resultant force downwards and

$4g - T = 4(0.5)$

i.e. $40 - T = 2$

$T = 38$

So the reading on the balance will then be 38 N.

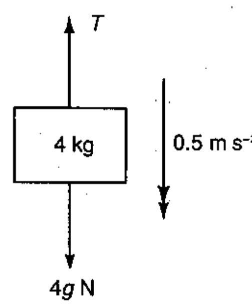


Fig. 24.6

Example 5

A particle initially at rest slides down from the top of a rough plane 1 m long inclined at an angle of 60° to the horizontal. The coefficient of friction between the plane and the particle is $\frac{1}{2}$. Find the velocity of the particle at the bottom of the plane.

Let m = mass of particle in kg,
R = normal reaction in N,
F = frictional force in N
and a = acceleration in m s^{-2}.

Fig. 24.7 shows the forces acting on the particle. We resolve the weight along and normal to the plane (Fig. 24.8).

Fig. 24.7

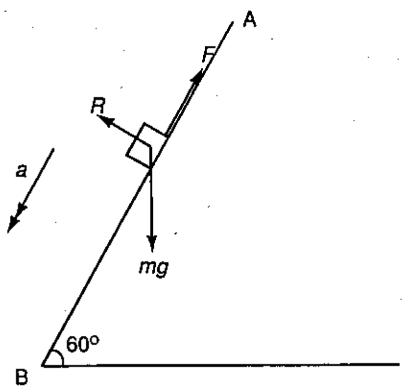
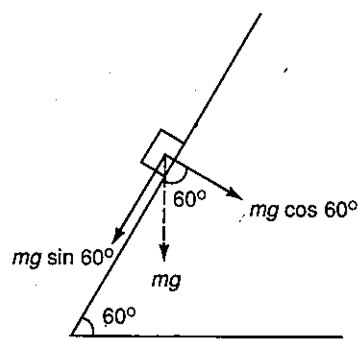


Fig. 24.8



We then consider motion along and normal to the plane respectively. Since the particle slides along the plane, there is acceleration (a m s^{-2}) along the plane only and no acceleration normal to the plane. Hence there is no resultant force normal to the plane. So we have

$$R = mg \cos 60°$$

i.e. $R = \frac{1}{2} mg$

The frictional force F is given by

$$\begin{aligned} F &= \mu R \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}mg\right) \\ &= \tfrac{1}{4}mg \end{aligned}$$

The net force down the slope is then

$$\begin{aligned} mg \sin 60° - F &= mg \times \frac{\sqrt{3}}{2} - \frac{1}{4}mg \\ &= \left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right)mg \end{aligned}$$

Using $P = ma$,

we have $\left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right) mg = ma$

giving $a = \left(\frac{\sqrt{3}}{2} - \frac{1}{4}\right) g$

$\approx 6.16 \text{ m s}^{-2}$

The velocity v m s^{-1} of the particle at the bottom of the plane is given by

$$v^2 - u^2 = 2as$$

i.e. $\quad v^2 - 0 = 2(6.16)(1)$

$$v^2 = 12.32$$

Therefore $\quad v = 3.51$

Hence the velocity of the particle at the bottom of the plane is 3.51 m s^{-1}.

Note: The method of resolving the weight in this solution should be carefully noted. As a general rule, all forces should be resolved into components parallel and normal to any acceleration. The equation of motion is then applied in the direction of the acceleration, the other components being in equilibrium.

Exercise 24.1 *(Answers on page 651.)*

[Take $g = 10$ m s^{-2} where required.]

1 If a force of 20 N acts on a mass of 2 kg, what is the acceleration produced?

2 If a force of 5 N acts on mass of 750 g, what acceleration results?

3 If a force of 4 kN acts on a mass of 2.2 tonne (1 tonne = 10^3 kg), what is the acceleration in m s^{-2}?

4 If $g = 1.6$ m s^{-2} on the moon, what is the weight of a packet of tea labelled: net mass 250 g?

5 A mass of 1.5 kg has an acceleration of 0.8 m s^{-2}. What force is acting on it?

6 If a force of 2 N acts on a mass of 1.5 kg at rest initially, what is its velocity after 6 s?

7 A mass of 5 kg is dragged across a rough surface (frictional force equal to 3 N opposing the motion) by a horizontal force of 20 N. What is the acceleration produced?

8 Two forces, 20 and 10 N, act on a body of mass 0.5 kg at right angles to each other. What is the acceleration of the mass, in magnitude and direction?

9 A force of 20 N is applied at an angle of 60° to the horizontal to a mass of 4 kg on a smooth horizontal table. What is the acceleration of the mass?

10 A particle of mass 2.5 kg is moving at a steady speed of 12 m s^{-1} when it meets with a fixed resistance of 10 N. How long does it take to come to rest?

11 Find the constant force which would give a body of mass 5 kg at rest a velocity of 6.4 m s^{-1} in 4 s.

12 A mass of 2 kg is at rest on a rough horizontal table. A force of 20 N is applied to the mass, the force making an angle of 30° with the table. Frictional resistance is equal to 5 N. What is the acceleration of the mass?

13 A toy engine of mass 350 g exerts a driving force of 0.1 N. With what acceleration could it climb a smooth slope of $\frac{1}{100}$ $\left(\text{i.e. a slope making angle } \theta \text{ with the horizontal where } \sin\theta = \frac{1}{100}\right)$?

14 A man of mass 80 kg stands in a lift. What is the reaction from the floor of the lift if the lift

(a) moves upwards with steady speed,

(b) moves upwards with acceleration 0.5 m s^{-2},

(c) moves downwards with acceleration 0.4 m s^{-2}?

15 A car of mass 750 kg is accelerating up a slope of θ to the horizontal where $\sin\theta = \frac{1}{70}$ at 1.5 m s^{-2}. Ignoring any road resistance, find the tractive force of the engine.

16 A block of mass 10 kg is placed on an inclined plane of angle 30° to the horizontal. The coefficient of friction is 0.5. Find

(a) the acceleration of the block down the plane,

(b) the least force parallel to the plane required to keep the block at rest,

(c) the least force parallel to the plane required to make the block begin to move up the plane,

(d) the force parallel to the plane required to move the block up the plane with an acceleration of 0.5 m s^{-2}.

17 A barge of mass 50 000 kg is being towed by two tugs. The two ropes make an angle of 15° on each side of the line of motion and the tension in each one is 1000 N. If the barge is moving with an acceleration of 0.02 m s^{-2}, find the resistance to its motion.

18 A body of mass 2 kg is pushed up an inclined plane of angle 30° to the horizontal by a horizontal force of 20 N. If the coefficient of friction is $\frac{1}{4}$, find the acceleration of the body.

19 A body of mass 4 kg is pulled from rest to a speed of 4.5 m s^{-1} in a time of 3 seconds on a rough horizontal surface by a force of 20 N which makes an angle of 10° with the horizontal. Find the coefficient of friction.

20 A particle slides down an inclined plane of angle θ to the horizontal, where $\sin\theta = \frac{3}{5}$, with acceleration 2 m s^{-2}. Calculate the coefficient of friction.

CONNECTED PARTICLES

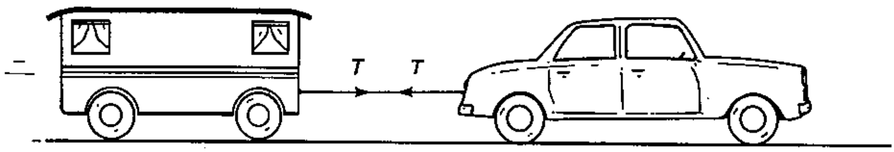


Fig. 24.9

If a car pulls a caravan (Fig. 24.9), the pull of the car is transmitted through the tie rod to the caravan but the caravan equally pulls the car backwards. The two pulls are the same size (T) but opposite in direction, according to Newton's Third Law. If we are considering the car, we must include the backward pull; if we consider the caravan we include the forward pull. If however, we consider the two as *one* body, the two pulls cancel out as internal forces and need not be considered.

Again, if two masses are suspended by a string over a frictionless (smooth) pulley, the string transmits a tension which pulls the mass A (Fig. 24.10) upwards when considering A, but pulls the mass B upwards when considering B. If the two masses are taken as one body, again the two tensions cancel out as internal forces and need not be considered.

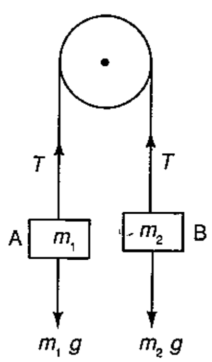


Fig. 24.10

Example 6

A string (assumed to have no weight and not to stretch) passes over a smooth light pulley. To the ends of the string are attached masses of 3 kg (A) and 2 kg (B) and both parts of the string are vertical. With what acceleration does the system move? What is the reaction at the axle of the pulley?

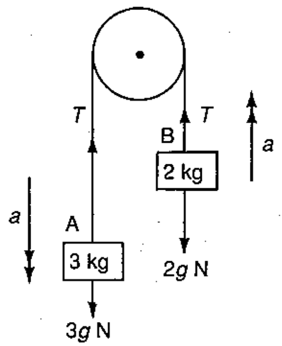


Fig. 24.11

The system is shown in Fig. 24.11 with the weights of the masses.
Let the acceleration of A be a m s^{-2} downwards and hence B will have the same acceleration upwards.
Now consider each mass and the pulley separately (Fig. 24.12).

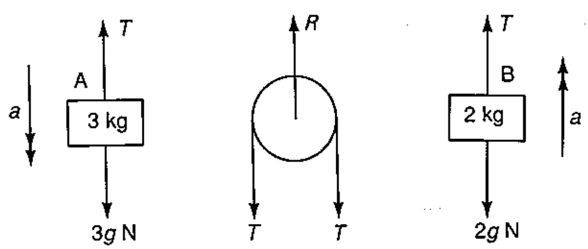


Fig. 24.12

The string transmits a tension T and the reaction at the axle of the pulley is R.

For A, since the acceleration is downwards, $3g - T = 3a$ (i)

For the pulley, since it has no acceleration vertically, $R = 2T$ (ii)

For B, since the acceleration is upwards, $T - 2g = 2a$ (iii)

Now solve these equations for a, T and R.

Adding (i) and (iii), $g = 5a$

giving $a = \frac{g}{5} \approx 2 \text{ m s}^{-2}$

From (iii), $T = 2a + 2g$
$\approx 4 + 20 = 24$ N

From (ii), $R = 2T$
$= 48$ N

(Note that this is less than the total weight of the masses, 50 N.)

Example 7

Fig. 24.13 shows two particles A and B each of mass 0.5 kg, joined by a light inelastic string which passes over a smooth fixed pulley at C. The system is held at rest with A hanging freely while B is on a rough horizontal surface. The coefficient of friction between B and the surface is 0.4. Find the magnitude of the acceleration of each particle and the tension in the string when the system is released.

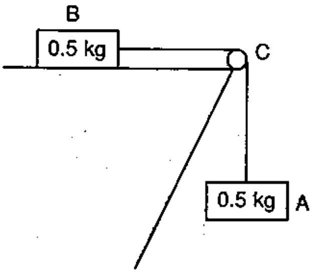


Fig. 24.13

Let the acceleration of A be a m s^{-2} downwards and hence B will move towards C with the same acceleration. The string transmits a tension T. Fig. 24.14 shows the forces acting on A and B.

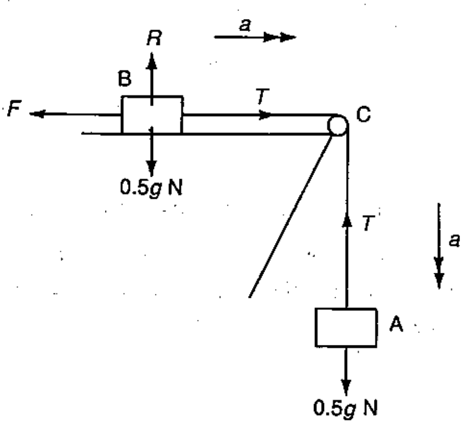


Fig. 24.14

For A, since the acceleration is downwards,

$$0.5g - T = 0.5a \qquad \text{(i)}$$

For B, since the acceleration is towards C,

$$T - F = 0.5a \qquad \text{(ii)}$$

where the frictional force $F = \mu R$

$$= 0.4(0.5g)$$
$$= 0.2g$$

Substituting this into (ii), we have

$$T - 0.2g = 0.5a \qquad \text{(iii)}$$

Now solve (i) and (iii) to obtain a and T.

Adding (i) and (iii), $\quad 0.3g = a$

i.e. $\quad a \approx 3$

From (iii), $\quad T - 0.2(10) = 0.5(3)$

giving $\quad T = 3.5$

Hence the acceleration of each particle is 3 m s^{-2} and the tension in the string is 3.5 N.

Example 8

Fig. 24.15 shows two particles A of mass 1 kg and B of mass 2 kg connected by a light inelastic string which passes over a smooth pulley at C. The system is held at rest with B hanging freely while A is on a rough plane inclined at θ to the horizontal where $\tan\theta = \frac{3}{4}$. The coefficient of friction between A and the plane is 0.2. Find the magnitude of the acceleration of each particle and the tension in the string when the system is released.

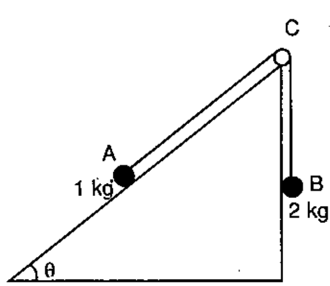


Fig. 24.15

Let the acceleration of B be a m s^{-2} downwards and hence A will move towards C with the same acceleration. The string transmits a tension T. Fig. 24.16 shows the forces acting on A and B.

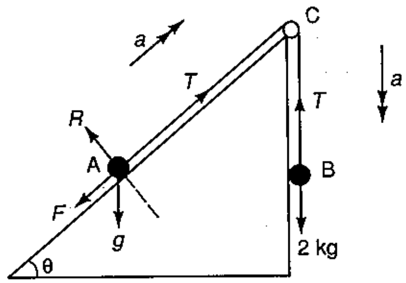


Fig. 24.16

Now consider the motion of each mass separately.

For B, since the acceleration is downwards,

$$2g - T = 2a \qquad \text{(i)}$$

For A, since the acceleration is along AC, and the tension is opposed by the frictional force, F and the component of the weight downslope, $g \sin \theta$, we have

$$T - (F + g \sin \theta) = 1a \qquad \text{(ii)}$$

Resolving forces perpendicular to the plane, the normal reaction is

$$R = g \cos \theta$$

The frictional force, F is given by

$$F = \mu R$$
$$= \mu g \cos \theta$$

Substituting this into (ii),

$$T - (\mu g \cos \theta + g \sin \theta) = a \qquad \text{(iii)}$$

Now solve (i) and (iii) to obtain a and T.

Adding (i) and (iii),

$$2g - (\mu g \cos \theta + g \sin \theta) = 3a$$

i.e. $$2(10) - \left[(0.2)(10)\tfrac{4}{5} + 10\left(\tfrac{3}{5}\right)\right] = 3a$$

giving $$a \approx 4.13$$

From (i) $$2(10) - T = 2(4.13)$$

giving $$T = 11.74$$

Hence the acceleration of each particle is 4.13 m s^{-2} and the tension in the string is 11.74 N.

Example 9

A car of mass 0.75 tonnes is pulling a trailer of mass 200 kg along rough level ground. The resistance to motion on each vehicle is 2 N per kg of mass. Find the force exerted by the engine and the tension in the tow-bar, when the vehicles are
(a) travelling at a constant speed, and
(b) when they are accelerating at 0.8 m s^{-2}.

The resistance to motion of the car = $2 \times 0.75 \times 1000 = 1500$ N and to the trailer 400 N.

(a) First we take the two as one body (Fig. 24.17(a)).

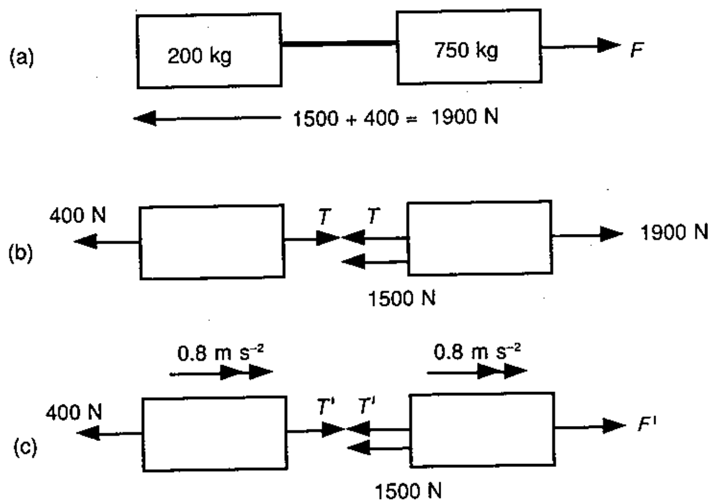


Fig.24.17

The total resistance is 1900 N and so the force exerted by the engine $F = 1900$ N as there is no acceleration.

To find the tension (T) in the tow-bar we must consider either the car or the trailer separately (Fig. 24.17(b)). It is simpler to consider the trailer.

$T = 400$ N as there is no acceleration. (For the car, $1900 = T + 1500$.)

(b) Taking the two as one body (Fig. 24.17(c)), $F' - 1900 = 950 \times 0.8$ giving $F' = 2660$ N.

Now taking the trailer alone, $T' - 400 = 200 \times 0.8$ giving $T' = 560$ N.

Example 10

Particles A and B of masses 3 kg and 1 kg respectively (Fig. 24.18(a)) hang at the ends of a light string passing over a smooth pulley. A is released from rest when it is 2 m above the ground. Find

(a) the common acceleration of the particles,

(b) the speed of B when A reaches the ground,

(c) how much higher B will travel afterwards. (It is assumed that B never reaches the pulley.)

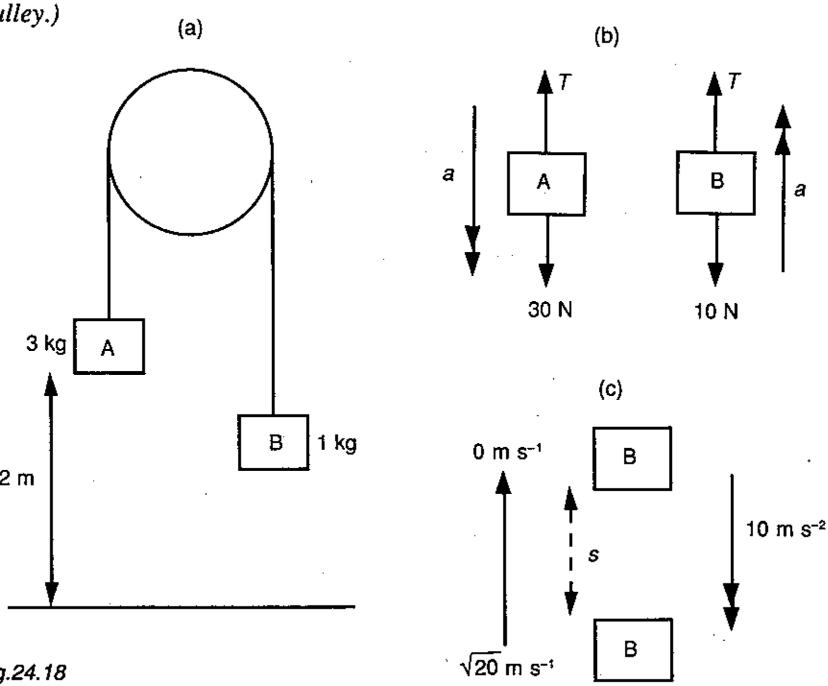


Fig.24.18

(a) From Fig. 24.18(b), $30 - T = 3a$

$$T - 10 = a$$

Solving the two equations gives $a = 5$ m s^{-2}.

(b) When A reaches the ground, B will have travelled 2 m with acceleration 5 m s^{-2}.

Using $v^2 = u^2 + 2as$, $v^2 = 0 + 2 \times 5 \times 2 = 20$

so $v = \sqrt{20}$ m s^{-1}.

(c) The string is now slack. B continues moving upwards but with acceleration g downwards (Fig. 24.18(c)).

At the highest point reached, B's velocity is 0.

Using $v^2 = u^2 + 2as$ again, $0 = 20 - 2 \times 10 \times s$ giving $s = 1$.

B will rise a distance of 1 m after A hits the ground.

Exercise 24.2 *(Answers on page 652.)*

[Take $g = 10\ m\ s^{-2}$ where required.]

1 Masses of 5 kg and 3 kg are connected by a light string over a smooth pulley. Find
(a) the acceleration of the masses,
(b) the tension in the string,
(c) the reaction at the axle of the pulley.

2 A mass of 3 kg rests on a smooth horizontal table connected by a light string passing over a smooth pulley at the edge of the table to another mass of 2 kg hanging vertically. When the system is released from rest, with what acceleration do the masses move and what is the tension in the string?

3 A body of mass 10 kg lies on a smooth inclined plane. A light string attached to this body passes over a smooth pulley at the top of the plane and supports a mass of 2 kg hanging freely. If the inclination of the plane is θ to the horizontal where $\sin\theta = \frac{1}{14}$, find the acceleration of the masses.

4 Masses of 4 kg and m kg are connected by a light string passing over a smooth pulley. When free to move, their acceleration is 0.5 m s^{-2}. Find the possible values of m.

5 A mass of 8 kg is placed on a horizontal table ($\mu = 0.3$) connected by a light inextensible string placed over a smooth pulley at the edge of the table to another mass of 4 kg hanging freely. Find the acceleration of the masses when released from rest.

6 A lorry of mass 1000 kg pulls a trailer of mass 450 kg on level ground. Resistance to motion for either vehicle is 4 N per kg of mass. Find the tension in the tow-bar and the tractive force of the engine when they are
(a) moving at a steady speed,
(b) accelerating at 0.6 m s^{-2}.

7 A car of mass 800 kg is pulling a trailer of mass 300 kg up a slope of angle θ to the horizontal where $\sin\theta = \frac{1}{200}$. Resistance to motion (apart from gravity) is 1.5 N per kg of mass for each vehicle. Calculate the pull of the engine and the tension in the tow-bar when they are
(a) moving with constant speed,
(b) accelerating at 0.2 m s^{-2}.

8 In Fig. 24.19, PQR is a fixed wedge on level ground where PQ = 5 m, QR = 3 m and PRQ is a right angle. Particle A, of mass 1.5 kg, lies at the foot of the smooth slope PQ, attached by a light string passing over the smooth pulley at Q, to the particle B of mass 1 kg. B is released from rest when it is 2 m above ground level. Find

(a) the acceleration of the particles,
(b) how far A will travel up the slope before coming to momentary rest.

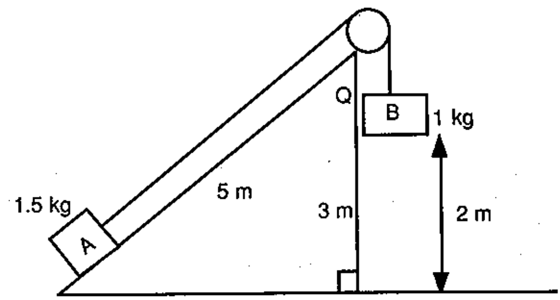


Fig.24.19

9 A wedge has two equally rough faces each inclined at 30° to the horizontal. Masses of 5 kg and 2 kg, one on each face, are connected by a light string passing over a smooth pulley at the top of the wedge. The coefficient of friction μ between each mass and the surface of the wedge is 0.2. Find the acceleration of the masses when they are released.

10 In Fig. 24.20, the particle of mass 2.4 kg is held at rest on the rough horizontal surface AB (the coefficient of friction is 0.5). It is connected by a light string passing over a smooth pulley at B to a particle of mass 3.6 kg. The sloping face BC is smooth and makes an angle of 30° with the horizontal. Find the acceleration of the particles when they are released.

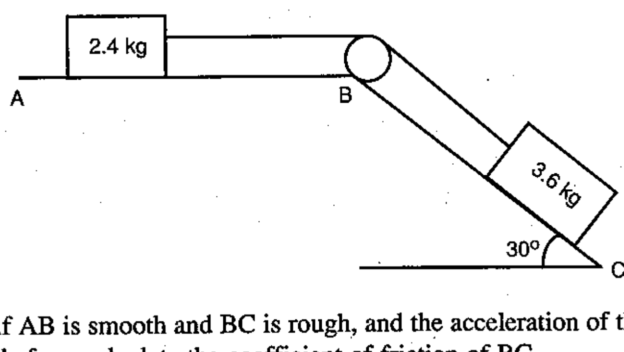


Fig.24.20

11 In Fig. 24.20, if AB is smooth and BC is rough, and the acceleration of the particles is the same as before, calculate the coefficient of friction of BC.

12 Two bodies A and B, joined by a light inextensible string, are placed on a plane which is inclined to the horizontal at an angle whose tangent is 0.75 so that the string is taut and lies along a line of greatest slope and B is higher up the plane than A. The body A is smooth and its mass is 9 kg. The mass of B is 3 kg and the coefficient of sliding friction between B and the plane is 0.5. The system is allowed to slide down the plane. Calculate
(a) the frictional resistance to the motion of B,
(b) the acceleration of the system,
(c) the tension in the string. (L)

13 A model engine of mass 2 kg is pulling two trucks, each of mass 0.5 kg, on a level track. Resistance to motion of the engine is 30 N and 5 N each for the trucks. Calculate the pull of the engine and the tension in each of the couplings when the three are
(a) moving at a steady speed,
(b) accelerating at 0.1 m s^{-2}.

14 Particles A, B and C, of masses 2, 1 and 3 kg respectively, are connected as shown in Fig. 24.21 by two light strings passing over smooth pulleys. The surfaces on which B and C move are smooth. When the particles are free to move, calculate
(a) their acceleration,
(b) the tensions in the strings.

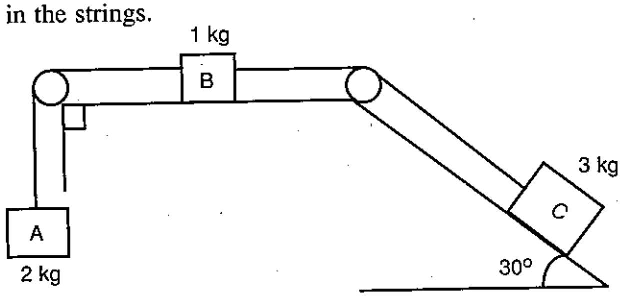


Fig.24.21

SUMMARY

- Newton's Laws:
  1. Every body remains at rest or moves with uniform velocity unless it is made to change this state by external forces.
  2. If a force acts on a body and produces a certain acceleration, then the force is proportional to the product of the mass of the body and the acceleration.
  3. To every action, there is an equal and opposite reaction.
- Force = mass × acceleration
  $P = ma$ (P in N, m in kg, a in m s^{-2})
- Unit of force is 1 N which produces an acceleration of 1 m s^{-2} on a mass of 1 kg.
- Weight = force due to gravity
  $W = mg$ N (mass m in kg, g = acceleration due to gravity ≈ 10 m s^{-2})

REVISION EXERCISE 24 *(Answers on page 652.)*

1 A block of mass 6 kg is placed on a rough plane inclined at an angle α to the horizontal, where $\sin \alpha = 0.6$. The coefficient of friction between the block and the plane is 0.4. A force P N acts on the block in an upward direction parallel to a line of greatest slope of the plane. Calculate
(i) the minimum value of P which will prevent the block sliding down the plane,
(ii) the direction of motion and the acceleration of the block when $P = 12$. (C)

2 A load of 400 kg is lifted by a cable through a vertical distance of 48 m. The load moves upwards from rest with a uniform acceleration of 0.5 m s^{-2} over the first 36 m and then decelerates uniformly to rest. Calculate
(i) the tension in the cable during acceleration,
(ii) the maximum velocity attained by the load,
(iii) the tension in the cable during deceleration. (C)

3 A body of mass 3 kg rests on a rough plane inclined at an angle of 45° to the horizontal. The coefficient of friction between the body and the plane is $\frac{1}{3}$. The body is just prevented from sliding down the plane by a force of P N acting towards, and at right angles to the plane. Calculate the value of P.
If the force P is reduced to 36 N, calculate the acceleration of the body down the plane. (C)

4 A particle of mass 0.5 kg is projected up a line of greatest slope of a rough plane inclined at an angle θ to the horizontal, where $\sin \theta = \frac{3}{5}$.
Given that the speed of projection is 6 m s^{-1} and that the coefficient of friction between the particle and the plane is $\frac{3}{8}$, calculate
(a) the distance travelled up the plane when the speed has fallen to 4 m s^{-1};
(b) the speed of the particle when it returns to its point of projection. (C)

5 An engine of mass 50 000 kg is pulling two trucks each of mass 10 000 kg along a level track at constant speed. Resistances are 50 N per 1000 kg for the engine and 30 N per 1000 kg for each of the trucks. Calculate
(i) the tractive force exerted by the engine,
(ii) the tension in the coupling between the engine and the first truck.
(iii) the tension in the coupling between the two trucks.

Calculate the corresponding forces when the train is accelerating at 0.1 m s^{-2}. (C)

6 A motor-boat of mass 1500 kg is towing a water-skier of mass 80 kg. The boat experiences a constant resistance of 1 200 N while the skier experiences a resistance of 150 N. Assuming that the cable remains taut and horizontal, calculate the tractive force exerted by the motor and the tension in the cable when the boat and skier are
(i) travelling at constant velocity;
(ii) accelerating at 3 m s^{-2}. (C)

7 A particle A of mass 5 kg lies on a rough horizontal table, the coefficient of friction being $\frac{1}{2}$. It is connected by a light inextensible string which passes over a smooth pulley at the end of the table to a particle B of mass 4 kg which hangs freely above the ground. The system is released from rest with A at a distance of 3 m from the edge of the table. Find the acceleration of the particles.
If B reaches the ground after $1\frac{1}{2}$ seconds, calculate
(i) the distance of A from the edge of the table at this instant,
(ii) the subsequent deceleration of A. (C)

8 A particle of mass 3 kg is held at rest on a rough horizontal table, connected by a light inextensible string which passes over a small smooth pulley at the end of the table to a particle of mass 2 kg which hangs freely. The coefficient of friction between the particle and the table is $\frac{1}{3}$. The particles are released from rest. Calculate

(i) their acceleration,

(ii) the tension in the string,

(iii) the force exerted by the string on the pulley. (C)

9 Fig. 24.22 shows two masses of 5 kg and 7 kg respectively connected by a light inextensible string which passes over a smooth fixed pulley. The system is released from rest and the 7 kg mass reaches the ground after 3 s. Calculate

(i) the acceleration of the masses while the string remains taut,

(ii) the total distance moved by the 5 kg mass before it comes instantaneously to rest, assuming that it does not reach the pulley. (C)

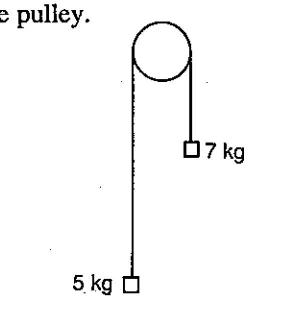


Fig.25.22

10 Two bodies, *A* and *B*, of mass 3 kg and 2 kg respectively, are connected by a light string passing over a smooth pulley. *A* rests on a rough plane inclined at 20° to the horizontal. When the bodies are released from rest, *B* moves downward with an acceleration of 0.5 m s^{-2}. Calculate the value of μ, the coefficient of friction between *A* and the inclined plane. (C)

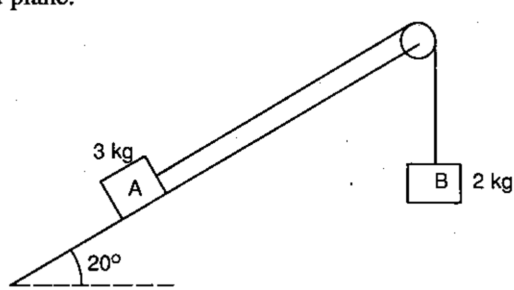


Fig. 24.23

11 Particles A and B, of masses 1 kg and 0.6 kg, are connected by a light string passing over a smooth pulley. The particles are held at rest with B 1.5 m higher than A. B is then given a speed of 2 m s^{-1} *downwards*. Calculate

(a) the acceleration of the particles,

(b) the time taken for B to be at the same level as A,

(c) for how long B will be lower than A.

12 Particle A of mass 0.8 kg is held at rest on a rough horizontal surface and is connected by a light string passing over a smooth pulley at the edge of the surface to a particle B of mass 0.4 kg hanging freely. When A is released, the acceleration of the particles is $\frac{2}{3}$ m s^{-2}.

(a) Calculate the coefficient of friction between A and the surface.

(b) After travelling for 3 seconds, B is stopped. Calculate the total distance travelled by A before it comes to a stop.

13 A car of mass 600 kg is pulling a trailer of mass 200 kg up an incline of angle θ to the horizontal where $\sin\theta = \frac{1}{250}$. The resistance to motion on either vehicle is 0.2 N per kg of mass. Calculate the driving force of the engine and the tension in the tow-bar when the vehicles are accelerating at 0.25 m s^{-2}.

14 Particle A of mass 4 kg lies on a rough plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{4}{5}$. The coefficient of friction between A and the plane is 0.5. A is connected to another particle B of mass m kg by a light string passing over a smooth pulley at the top of the plane and B hangs freely. When the particles are free to move, the acceleration of B is 2 m s^{-2}. Find the possible values of m.

15 In Fig. 24.24, particles P and Q of masses 5 kg and 2 kg respectively, lie on the faces AC and BC of the fixed wedge ABC. $\angle A = \angle B = 45°$ and the coefficients of friction on the faces AC and BC are 0.2 and 0.5 respectively. Find the acceleration of the particles when they are free to move.

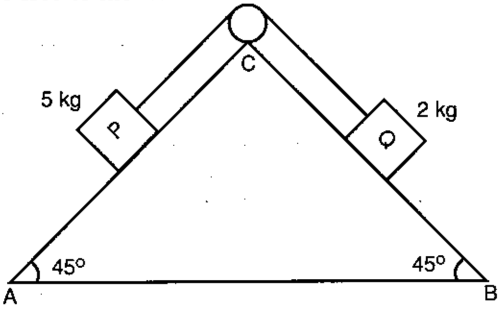


Fig. 24.24

16 Masses of 2.8 kg, 2.2 kg, 1 kg are connected by light inextensible strings, one of which passes over a smooth fixed pulley as shown in the diagram. If the system is released from rest, calculate

(i) the acceleration of the masses,

(ii) the tension in the string joining the 2.2 kg and 1 kg masses.

If after $1\frac{1}{2}$ seconds the string joining the 2.2 kg and 1 kg masses breaks, calculate the further distance the 2.2 kg mass falls before coming instantaneously to rest. (C)

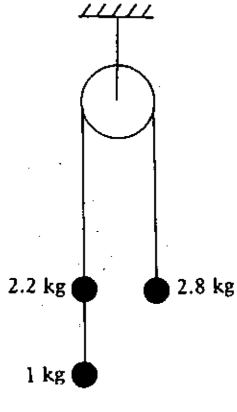


Fig. 24.25

17

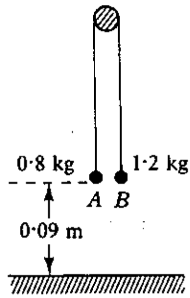


Fig 24.26

The diagram shows two particles A and B, connected by a light inextensible string which passes over a smooth fixed peg. The system is held with the string taut and with A and B each at a height of 0·09 m above a fixed horizontal plane; it is then released from rest. When B reaches the plane it becomes stationary. Calculate

(i) the tension in the string while both particles are in motion,

(ii) the speed of the particles when B reaches the plane,

(iii) the maximum height above the plane attained by A, assuming that A does not reach the height of the fixed peg. (C)

18 The diagram shows two bodies, A and B, connected by a light inextensible string passing over a smooth peg. The body A has a mass of 8 kg and lies on a rough plane inclined at an angle α to the horizontal, where $\cos\theta = \frac{3}{5}$. The body B has a mass of 2 kg and hangs freely.

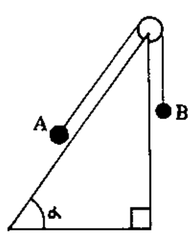


Fig. 24.27

(i) In the case where the bodies are free to move and A accelerates down the plane at 2 m s^{-2}, calculate the tension in the string and the coefficient of friction between A and the plane.

(ii) Find the smallest mass which, when attached to B, would prevent A from sliding down the plane. (C)

19 The diagram shows two particles, A, of mass 0·3 kg, and B, of mass 0·2 kg, joined by a light inelastic string which passes over a smooth fixed pulley at C. The system is held at rest with A on a smooth plane inclined at 30° to the horizontal and B on a rough horizontal surface. The coefficient of friction between B and the surface is 0·4. Show that, when the system is released from rest, the acceleration of each particle has a magnitude of 1·4 m s^{-2} and calculate the tension in the string.

After 2 seconds, and before B reaches C, A hits an obstacle and comes instantaneously to rest. Calculate

(i) the speed of B at this instant,

(ii) the subsequent deceleration of B.

Given that B just reaches C, find the distance of B from C at the start of the motion.

(C)

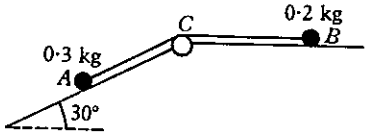


Fig. 24.28

20 A car of mass 560 kg is pulling a caravan of mass 240 kg along a horizontal road. There are constant resistances of 120 N to the motion of the car and 80 N to the motion of the caravan.

Given that the tractive force of the car is 1200 N, calculate

(i) the acceleration of the car and caravan,

(ii) the tension in the tow-bar,

(iii) the power of the car's engine when the speed is 12 m s^{-1}.

The car now pulls the caravan up a road inclined at θ to the horizontal, where $\sin\theta = \frac{1}{16}$. Assuming that the tractive force and the resistance are unchanged,

(iv) calculate the acceleration of the car and caravan,

(v) show that the tension in the tow-bar is unchanged.

(C)

Work, Energy, Power

25

WORK

When a force acts on a body and causes it to move, we say the force does **work** on the body. The amount of work done is defined as the product of the force and the distance moved by the body in the direction of the force.

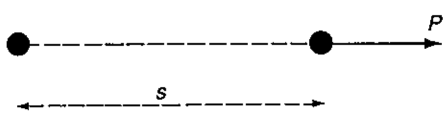


Fig. 25.1

In Fig. 25.1, if the force P moves the body through a distance s in the direction of the force, the work done = Ps.

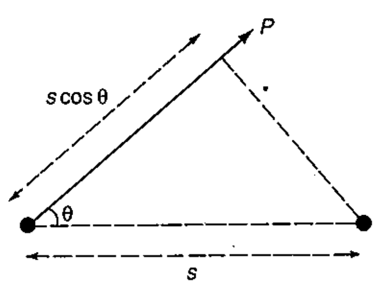


Fig. 25.2

In Fig. 25.2, P acts at an angle θ to the direction in which the body moves. When the body moves a distance s, then the distance moved by the body in the direction of the force is $s \cos \theta$. Hence the work done in this case is $= P \times s \cos \theta$.

Alternatively, we can also find the work done by the component of the force P (Fig. 25.3) in the direction in which the body moves. This component is $P \cos \theta$ and the work done is then $P \cos \theta \times s$ which is the same result as before. The other component of the force ($P \sin \theta$) does no work as the body does not move in the direction of this component.

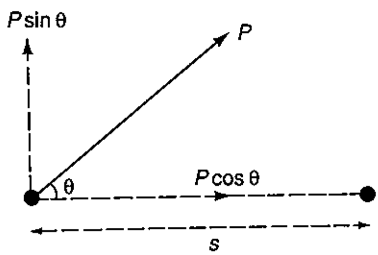


Fig. 25.3

In vector terms, work (W) is the scalar product of the force **P** and the displacement **s**, thus $W = \mathbf{P} . \mathbf{s}$

$$= |\mathbf{P}| \times |\mathbf{s}| \times \cos \theta$$

or simply $Ps \cos \theta$

where P is the magnitude of **P** and s is the magnitude of **s**.

As work is the product of a force (in N) and a distance (in m), the unit of work will be 1 N × 1 m = 1 N m (Newton-metre). This unit is given the special name **joule** in honour of the scientist James Joule who did fundamental work on the conservation of energy in the 19th century. So if a force P N acts on a body and moves it through a distance of s m in the direction of the force, the work done = Ps J. For large amounts of work, the kJ (1 kilojoule = 1000 J) can be used.

Example 1

A force of 10 N acts on a body and moves it 5 m in the direction of the force. What is the work done by the force?

Work done = 10 × 5
= 50 J

Example 2

A trolley is pulled horizontally through 5 m by a force of 70 N at an angle of 60° to the horizontal. What is the work done?

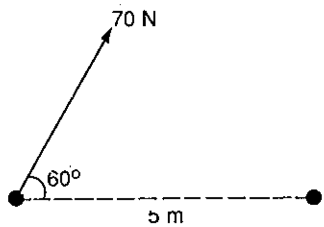


Fig. 25.4

The component of the force in the direction of motion is

$$70 \times \cos 60° = 70 \times \tfrac{1}{2}$$
$$= 35 \text{ N}$$

Hence the work done = $35 \times 5 = 175$ J.

Example 3

The engine of a car exerts a constant pull of magnitude 500 N. Find the work done by this force as the car travels 1 km.

Work done = 500×1000 J
$= 500$ kJ

KINETIC ENERGY (KE)

Suppose a force P N acts on a particle of mass m kg at rest and gives it a velocity v m s^{-1} after moving it a distance of s m in the direction of the force.

If the acceleration is a m s^{-2},

then $P = ma$ (i)

Also, $v^2 = 0^2 + 2as$

or $as = \frac{v^2}{2}$ (ii)

Work done $= P \times s$

$= mas$ (from (i))

$= m\frac{v^2}{2}$ (from (ii))

or $\frac{1}{2}mv^2$

This quantity, which is $\frac{1}{2}$(mass) $\times$ (velocity)^2 is called the **kinetic energy** (KE) of the particle when its velocity is v. The kinetic energy of a particle can thus be regarded as the work done on the particle by a force in giving it the velocity v from rest. The unit of KE is therefore the same as the unit of work, 1 J.

Let us now consider a particle mass m kg travelling at u m s^{-1} in a straight line. A force P N now acts on it in the direction of motion and gives it an acceleration a m s^{-2}. If the particle acquires a velocity v m s^{-1} after travelling a distance s m, then

$$P = ma \qquad \text{(i)}$$

and $$v^2 = u^2 + 2as$$

or $$as = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 \qquad \text{(ii)}$$

therefore work done $= P \times s$

$$= mas \qquad \text{(from (i))}$$
$$= m(\tfrac{1}{2}v^2 - \tfrac{1}{2}u^2) \qquad \text{(from (ii))}$$
$$= \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$

The quantity $\frac{1}{2}mu^2$ is the initial KE and the quantity $\frac{1}{2}mv^2$ is the final KE.

Thus, the above shows that

work done by a force = final KE – initial KE
= increase in KE

The work done by a force in increasing the velocity of the particle from u to v is converted into the increased KE of the particle. Conversely, some or all of the KE possessed by a particle can be converted into work. Hence the loss of KE = work done against a force.

The quantity $\frac{1}{2}mv^2$ is always positive and is not a vector quantity. The KEs of 2 equal particles moving in any 2 directions with the same speed are equal. Also, as work can be converted into KE and vice versa, work is also a scalar quantity.

Example 4

A force acting on a body of mass 2 kg moves it from rest to a velocity of 3 m s^{-1} over a distance of 5 m. What is the magnitude of the force?

Let the magnitude of the force be P N.

Work done by the force = gain in KE
= final KE – initial KE

i.e. $$P \times 5 = \tfrac{1}{2}(2)(3)^2 - 0$$

giving $$P = 1.8 \text{ N}$$

Example 5

A particle of mass 0.5 kg is projected up an incline of angle θ where $\sin\theta = \frac{3}{5}$ with speed 4 m s^{-1}. How far will it travel up the incline if
(a) the surface is smooth,
(b) the coefficient of friction is $\frac{1}{4}$?

(a)

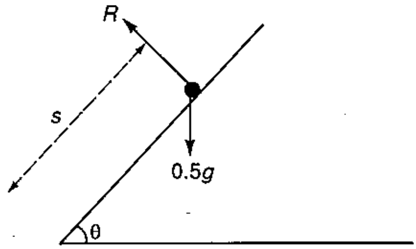


Fig. 25.5

The only resisting force in the direction of motion is the downward component of the weight. This is

$$0.5g \sin\theta = 0.5 \times 10 \times \tfrac{3}{5}$$
$$= 3 \text{ N}$$

Work done against this resistance in travelling s m up the incline

$$= 3 \times s \text{ J}$$

The initial KE $= \frac{1}{2} \times 0.5 \times 4^2 = 4$ J

When the particle reaches the highest point, the velocity is zero and its KE is also zero. The loss in KE is therefore 4 J.

Work done against resistance = loss in KE

i.e. $\quad 3s = 4,$

giving $\quad s = \frac{4}{3}$

Hence the particle will travel $1\frac{1}{3}$ m up the incline.

(b)

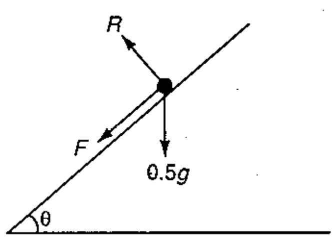


Fig. 25.6

There is now an additional resistance F due to friction (Fig. 25.6).

Resolving forces perpendicular to the plane,

$$\begin{aligned} R &= 0.5g \cos \theta \\ &= 0.5 \times 10 \times \tfrac{4}{5} \\ &= 4 \text{ N} \end{aligned}$$

Frictional force $F = \mu R$

$$\begin{aligned} &= \tfrac{1}{4} \times 4 \\ &= 1 \text{ N} \end{aligned}$$

Let the distance travelled by the particle be s_1, up the slope.

Work done against total resistance = loss in KE

$$\begin{aligned} (3 + 1)s_1 &= 4 \\ s_1 &= 1 \end{aligned}$$

Hence the particle will travel 1 m up the incline.

POTENTIAL ENERGY (PE)

Suppose I lift a mass of 4 kg vertically through a height of 2 m. The lifting force = the weight of the body = $4g$ N. Hence the work I do is $4g \times 2 = 78.4$ J (taking $g = 9.8$ m s^{-2}) and this work has been done against gravity. At this point the body is now at rest and has no KE, but if I let go, the body will acquire KE in falling and can do work on the downward path. Hence in its state of rest at a height of 2 m the body has a potential for doing work and we say it possesses **potential energy (PE).** PE is the ability to do work because of the position of the body, in the sense that if released, the body will move to a lower position and its PE will be converted into KE. The PE of a body has no absolute value, but is relative to some datum level, say the surface of the earth or some other level above which the body is raised and to which it can fall.

Suppose a body of mass m kg is raised through a height of h m from a floor (Fig. 25.7). The work done against gravity = mgh J and hence the body now possesses PE = mgh J. If the body now falls it will acquire a velocity v on reaching its original level, where $v^2 = 2gh$.

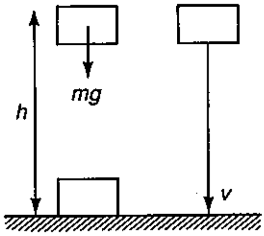


Fig. 25.7

Its KE is now $\frac{1}{2}mv^2 = \frac{1}{2}m \times 2gh = mgh$.

Hence all the PE has been converted into KE.

Note that this result is true if the particle descends by any route (provided it is smooth) through a vertical drop of h m (Fig. 25.8). The distance travelled by the point of application of the weight (the centre of gravity, CG) in the direction of the weight is always h. Hence the work done by gravity is mgh, which is converted into KE. Similarly, if a body of mass m kg is raised through a vertical height (h m) by whatever path (provided smooth), the work done against gravity = mgh and this is the value of the PE of the body.

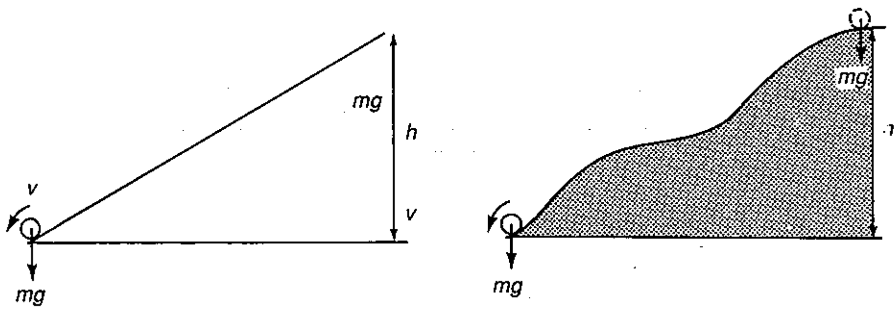


Fig. 25.8

If the body now strikes the floor, some KE will be lost, i.e. converted into another form of energy, for example heat, light or sound. This is a simple illustration of the principle of the **conservation of energy**, which states that the total energy in a closed system is constant. This principle is true provided *all* forms of energy, mechanical and non-mechanical are taken into account, such as heat, sound, light, chemical, electrical energy, etc. It also illustrates the fact that energy can be converted from one form to another. For example in a hydroelectrical scheme, the water in a reservoir possesses PE. This can be converted into KE by allowing the water to fall through a sluice gate. The water strikes the turbine wheels and its KE is converted into another form of KE, i.e. kinetic energy of rotation. This in turn is converted into electrical energy, which is used in factories and homes to be converted into light, heat and kinetic energy again.

From a mechanical point of view, energy dissipated through friction, heat, sound etc. is energy lost and wasted. If there were no such losses it would be possible to achieve perpetual motion mechanically. In applied mathematics, we deal only with KE and PE. The principle of conservation will then appear in the form:

$$\text{KE} + \text{PE} = \text{constant}$$

Hence, the (KE + PE) of a body at any time = original (KE + PE) + any work done by a force on the body.

Example 6

A car of mass 800 kg is travelling in a straight line on ground level with a speed of 30 m s⁻¹ when its engine is shut off. After moving a distance of 20 m, the ground slopes upwards at an angle of 30° to the horizontal. The frictional resistance of the ground is 5 N per kg. Find how far up the slope the car will travel before coming to rest.

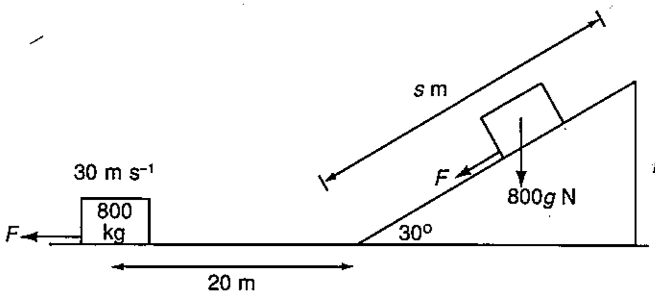


Fig. 25.9

The frictional resistance, $F = 800 \times 5 = 4000$ N

Initial KE of the car $= \frac{1}{2}(800)(30)^2 = 360\ 000$ J

Let the car travel s m up the slope equivalent to a vertical rise of h m where $h = s \sin 30°$.

In climbing the slope the car acquires

$$\begin{aligned} \text{PE} &= 800gh \\ &= 800gs \sin 30° \\ &= 800(10)(s)\left(\tfrac{1}{2}\right) \\ &= 4000s \text{ J} \end{aligned}$$

Work done against frictional resistance $= F(20 + s)$

$= 4000(20 + s)$ J

Now, the initial energy = final energy + work expended

$$360\ 000 = 4000s + 4000(20 + s)$$

i.e. $$280\ 000 = 8000s$$

giving $$s = 35$$

Hence the car will travel 35 m up the slope before coming to rest.

Example 7

One end of a light inextensible string 60 cm long is fixed and a particle of mass m kg is attached at the other end. The particle is released from rest when the string is taut and horizontal. Find the speed of the particle when the string makes an angle of 30° to the horizontal. Find also its maximum speed in the ensuing pendulum motion.

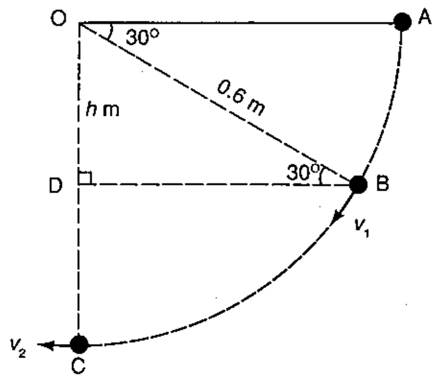


Fig. 25.10

Fig. 25.10 shows the positions of the string and the particle initially at A, then at B when the string makes an angle of 30° to the horizontal and at C when the particle is at the lowest point and the string is vertical. If we neglect air resistance, no work is done and thus the mechanical energy is conserved, that is, any loss in PE is converted into KE.

Now consider the particle at the position B.

The particle has descended a vertical distance h m given by

$$h = 0.6 \sin 30° = 0.3$$

$$\begin{aligned} \text{Loss in PE} &= mgh \\ &= m(10)(0.3) \\ &= 3m \text{ J} \end{aligned}$$

Let the speed of the particle at B be v_1 m s^{-1}.

Then gain in KE = $\frac{1}{2}mv_1^2$ J

Gain in KE = loss in PE

$$\frac{1}{2}mv_1^2 = 3m$$

$$v_1^2 = 6$$

$$v_1 = \sqrt{6}$$

Hence the speed of the particle at B is $\sqrt{6}$ m s^{-1}.

The particle will have the maximum speed when it has descended the maximum distance, that is 0.6 m, when the particle is at its lowest point in the path. Taking the initial position as reference, the loss in PE is now $mg(0.6) = 6m$ J.

The gain in KE is $\frac{1}{2}mv_2^2$ where the maximum speed (at C) is v_2 m s^{-1}.

So by conservation of energy,

$$\frac{1}{2}mv_2^2 = 6m$$
$$v_2^2 = 12$$
$$v_2 = \sqrt{12}$$

Hence the maximum speed is $\sqrt{12}$ m s^{-1}. Note that the results are independent of the mass of the particle.

Exercise 25.1 *(Answers on page 652.)*

[Take $g = 10$ m s^{-2}]

1 Find the work done when a load of 50 kg is lifted vertically through 10 m.

2 A block is pulled horizontally through 4 m at a steady speed by a force of 20 N, inclined at an angle of 60° to the line of motion. Find the work done.

3 A mass of 20 kg is pulled across a rough horizontal floor (coefficient of friction 0.4) through 2 m at a steady speed by a horizontal force. Find the work done.

4 If a mass of 10 kg at rest acquires a velocity of 2 m s^{-1} after being pulled through 1.5 m, what force is acting in the direction of motion?

5 A body of mass 1 kg travelling at 2.5 m s^{-1} on a horizontal surface meets a rough patch and comes to rest in 2 m. What is the resisting force? Also find the coefficient of friction of the rough surface.

6 The velocity of a body of mass 0.5 kg is reduced from 3 to 1.5 m s^{-1} in a distance of 1.5 m. What force is acting on the body?

7 A ball of mass 250 g is projected up a smooth plane inclined at angle θ to the horizontal where $\sin\theta = \frac{1}{70}$, with a velocity of 5 m s^{-1}.
How far will it travel before coming to rest?

8 What force is required to stop a mass of 5 kg travelling at 2 m s^{-1} in 1.5 m?

9 A car whose mass is 500 kg starts from rest at the foot of an incline of $\frac{1}{70}$ and after travelling for 0.5 km has reached a speed of 5 m s^{-1}.
If the resistances to motion amount to 250 N, what was the average tractive force of the engine?

10 A car of mass 400 kg travelling at 9 m s^{-1} comes to rest in 200 m. What was the resistance?

11 A train of mass 100 t travelling at 0.5 m s^{-1} hits the buffers at a station and comes to rest in a distance of 30 cm. What is the average resistance of the buffers?

12 A ship of mass 5000 t moving at 0.01 m s^{-1} hits a quayside and continues to move for 15 cm before coming to rest. What average force does the quay exert on the ship?

13 A particle of mass 1.5 kg is projected up an incline of $\frac{1}{7}$ with an initial speed of 1 m s^{-1}. How far will it travel up the incline if
(a) the surface is smooth,
(b) the coefficient of friction is 0.5?

14 Find the average force required to stop a 5 t lorry travelling at 10 m s^{-1} on a level road in a distance of 15 m.

15 A pendulum consists of a light string 60 cm long attached to a mass of 5 kg and can swing freely. It is held taut at an angle of 60° to the downward vertical and released. Find the velocity of the mass at its lowest point.

16 A mass of 4 kg suspended by a light string 2 m long and at rest is projected horizontally with a velocity of 1.5 m s^{-1}. Find the angle made by the string when the mass comes to momentary rest.

17 A mass of 10 kg slides down a slope of 30° from rest. The coefficient of friction is 0.5. If the length of the slope is 5 m, find the velocity of the mass at the foot of the slope.

18 Masses of 10 kg and 4 kg are connected by a light string passing over a smooth pulley. After the 10 kg mass descends from rest for a time of 2 s, find
(a) the velocity of each mass,
(b) the potential energy lost by the system.

19 A constant force acts on a body of mass 2 kg and does 45 J of work. The effect on the body is that its final velocity is 2 m s^{-1} more than its initial velocity. Find the initial velocity of the body.

20 A machine drives a conveyor belt which lifts 100 bottles per minute through a vertical height of 2 m and then pushes them forward with a speed of 3 m s^{-1}. The mass of each bottle is 1.2 kg. Calculate the amount of work done per second by the machine.

21 A bullet of mass 40 g strikes a fixed piece of wood 10 cm thick with a velocity of 300 m s^{-1} and emerges with a velocity of 120 m s^{-1}. Find the average resistance of the wood.

22 A particle is released from rest at the top of a rough inclined plane making an angle θ with the horizontal where $\sin\theta = \frac{5}{13}$. If the coefficient of friction is 0.4, what is the speed of the particle after travelling 3 m down the plane?

23 A particle is suspended by a light string of length 30 cm from a fixed point O. The particle is now held on the same level as O with the string taut and projected vertically downwards with speed 1 m s^{-1}. Find
(a) the speed of the particle at its lowest point,
(b) how high above O it will reach.

24 Fig. 25.11 shows a smooth track in the shape of a quarter circle AB, with centre O and radius 0.8 m. OB is vertical and B is 2 m above the ground level. A particle is released from rest at A. Calculate

(a) its speed at B,

(b) how far horizontally from B it will strike the ground.

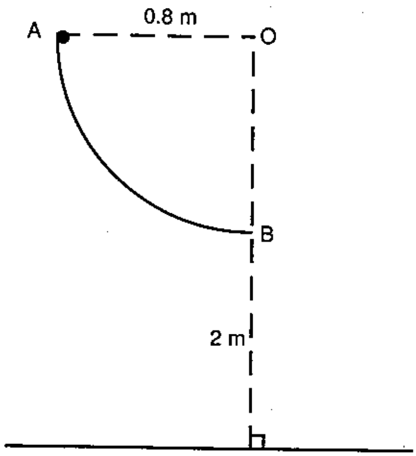


Fig.25.11

25 A particle is projected with speed u m s^{-1} directly up an incline of angle θ to the horizontal where $\sin\theta = \frac{4}{5}$. It comes to momentary rest after travelling a distance of 4 m. Given that the coefficient of friction is 0.4, calculate

(a) the value of u,

(b) the speed of the particle when it returns to its starting point.

POWER

Consider a machine that does 100 J of work in 1 s and a second machine that does 200 J of work in 1 s. We note that the second machine can accomplish more work (in fact twice as much) in the same time. The second machine is therefore more **powerful** than the first. Its **rate of doing work** is greater. We define **power** as the rate of doing work. Hence, power = amount of work (in J) done per second.

The unit of power is therefore 1 J s^{-1} which is given the name 1 **watt** (1 W). For very powerful machines, we can use a unit of 1 kW (kilowatt) = 10^3 W or 1 MW (megawatt) = 10^6 W.

A well-known unit in the British system of units was the horsepower (HP) which is approximately 746 W. This was the original unit of power, established by James Watt in the 18th century when he worked on the development of steam engines and the new unit has been named in his honour.

Example 8

A boy whose mass is 50 kg climbed 4 flights of stairs in 25 s. If the vertical height of each flight of stairs is 5 m, at what rate was he working?

Work done against gravity = gain in PE

$$= mgh$$
$$= 50(10)(4 \times 5) \text{ J}$$
$$= 10\,000 \text{ J}$$

This work was done in 25 s.

Therefore the power developed = $\frac{10\,000}{25}$ = 400 W.

Example 9

A cyclist is travelling at a steady speed of 5 m s^{-1} up a slope inclined at θ to the horizontal where sin $\theta = \frac{1}{100}$.
The total mass of the cyclist and the bicycle is 80 kg. The resistance due to friction amounts to 1.2 N per kg of mass. At what rate is he working?

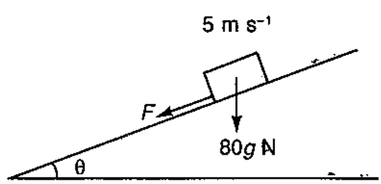


Fig. 25.12

At a steady speed, the driving force exerted by the cyclist

$$= \text{resistance} + \text{component of weight downslope}$$
$$= 80 \times 1.2 + 80g \sin\theta$$
$$= 96 + 80(10)\left(\frac{1}{100}\right)$$
$$= 104 \text{ N}$$

In 1 s the cyclist moves 5 m.

The work done by the cyclist in 1 s = 104 × 5 J = 520 J.

Hence his power is 520 W.

Example 10

A train of mass 200 t is travelling on a level track at a steady speed of 72 km h^{-1} and working at 70 kW.

(a) Find the resistance to motion.

(b) Assuming that power output and resistance to motion remain constant, calculate:

(i) the maximum speed up a slope inclined at an angle θ to the horizontal, where $\sin \theta = \frac{1}{400}$;

(ii) the acceleration up this slope at the instant when its speed is 20 km h^{-1}.

(a)

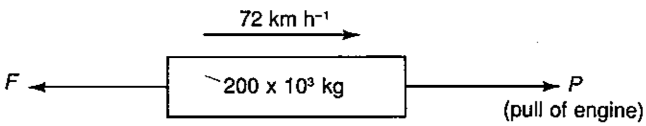


Fig. 25.13

Consider the train on the level (Fig. 25.13).

Let the resistance to motion be F N.

$$\text{The distance travelled per s} = \frac{72}{3600}\text{ km}$$

$$= \frac{72 \times 10^3}{3600}\text{ m} = 20\text{ m}$$

At steady speed, the pull of the engine equals the resistance which is F N.

Therefore the work done per s $= F \times 20$ J

i.e. power $= 20F$ W

Since the engine is working at 70 kW,

we have $20F = 70 \times 10^3$

giving $F = 3500$ N.

(b) Now consider the train going up the slope (Fig. 25.14).

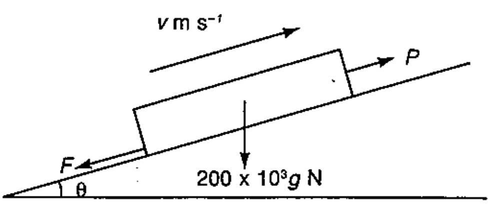


Fig. 25.14

(i) In addition to the resistance, F, there is the component of the weight down the slope which is

$$mg \sin \theta = 200 \times 10^3 \times 10 \times \frac{1}{400}$$

$$= 5000\text{ N}$$

Hence the total resistance to motion up-slope $= 3500 + 5000$
$= 8500$ N

When travelling at its maximum speed (v m s^{-1}) the train has no acceleration. The pull P equals the total resistance, i.e. 8500 N.

In 1 s the train moves v m.

Therefore work done by the engine in 1 s $= Pv$ J.

So the power of the engine is Pv W which is 70 kW

i.e. $\quad Pv = 70 \times 10^3$

or $\quad 8500v = 70 \times 10^3$

$$v = \frac{70 \times 10^3}{8500} \approx 8.24 \text{ m s}^{-1}$$

This is equivalent to $\frac{8.24}{1000} \times 60 \times 60 \approx 29.7$ km h^{-1}, which is the maximum speed up the incline.

(ii) 20 km h^{-1} is equivalent to $\frac{20 \times 10^3}{3600} = \frac{50}{9}$ m s^{-1}

The train is not travelling at its maximum speed (found in (i)) but the power developed remains the same. Hence there is spare power available to accelerate the train.

Let the pull of the engine at this instant $\left(\text{when the speed is } \frac{50}{9}\text{ m s}^{-1}\right)$ be P_1 N.

The work done per s $= P_1 \times \frac{50}{9}$ J

and this is equal to the power of the engine which is 70 kW.

Thus, $P_1 \times \frac{50}{9} = 70 \times 10^3$

giving $\quad P_1 = 12\,600$ N

But the total resistance (from (i)) on the incline is 8500 N.

So there is a resultant force P_2 of $12\,600 - 8500 = 4100$ N up the slope, which gives an acceleration a m s^{-2}.

Hence $\quad P_2 = ma$

i.e. $\quad 4100 = 200 \times 10^3 \times a$

giving $\quad a \approx 0.02$

Hence the acceleration is 0.02 m s^{-2}.

Example 11

An engine which is 80% efficient works at a steady rate to pump water, initially at rest, through a vertical height of 5 m and then discharges it at a speed of 8 m s^{-1} through a pipe of cross-section 10 cm^2. At what rate is the engine working? (1 m^3 of water has a mass of 10^3 kg.)

The work done by the engine consists of (a) giving it PE by lifting the water through a height of 5 m against gravity and (b) giving it KE by discharging it at a speed of 8 m s^{-1}.

In 1 s, a length of 8 m of water issues from the pipe with cross-section 10 cm^2 or $\frac{10}{10^4}$ m^2.

Therefore the volume of water carried per second $= 8 \times \frac{10}{10^4}$ m^3

which has a mass of $8 \times \frac{10}{10^4} \times 10^3 = 8$ kg.

When this mass of water is lifted through a height of 5 m,

the PE gained $= mgh$

$= 8 \times 10 \times 5 = 400$ J

The gain in KE $= \frac{1}{2}mv^2 = \frac{1}{2}(8)(8^2)$

$= 256$ J

Hence the total amount of work done in 1 s $= 400 + 256 = 656$ J.

As the engine is 80% efficient, this output is 80% of the actual work done by the engine. (The other 20% of work done is unproductive, mostly against friction.)

Therefore 80% of the work done by the engine in 1 s = 656 J

Hence the rate of work done by the engine $= 656 \times \frac{100}{8} = 820$ W.

Exercise 25.2 *(Answers on page 652.)*

[Take $g = 10$ m s^{-2}]

1 If a car travels at a steady speed of 15 m s^{-1} against resistances of 200 N, what power is being exerted by the engine?

2 A boy of mass 44 kg runs up a flight of stairs of vertical height 4 m in 5 s. What power is he sustaining?

3 A man runs 100 m in a time of 15 s. If the resistances to motion are estimated at 45 N, what power does he use?

4 The power of the engine of a car is 7 kW. What would be the maximum speed of the car on the level against resistances of 250 N?

5 A pump raises water through a height of 15 m at the rate of 0.05 m^3 per s. What is the power of the pump?

6 A train of total mass 300 t travels at a constant speed of 20 m s^{-1} on the level, the resistances being 100 N per t of mass. What is the power of the engine?

7 A fire hose delivers water horizontally at a speed of 20 m s^{-1} through a nozzle of cross-sectional area 10 cm^2. Find the power of the pump if it is only 70% efficient.

8 A car of mass 800 kg is travelling at a steady speed of 20 m s^{-1} on the level. The engine is developing a power of 8 kW. Find the resistance to motion.

9 A pump delivers water from a depth of 15 m and delivers it at a rate of 0.1 m^3 s^{-1} at a speed of 10 m s^{-1}. Find the power of the pump.

10 On the level a car develops a power of 15 kW. If the resistance to motion is 300 N what is the maximum speed of the car? Working at the same power and with the same resistance operating, what would be the maximum speed possible up an incline of $\frac{1}{70}$ if the mass of the car is 500 kg? What is the acceleration at the time when the car is moving up this incline at 30 m s^{-1}?

11 A car of mass 800 kg working at 12 kW can climb a slope of $\frac{1}{140}$ at a steady speed of 30 m s^{-1}. What is the resistance due to motion? If the resistance and the power are unchanged, what would be the maximum speed of the car on level ground?

12 A diesel electric engine has a power rating of 3000 kW. If it travels at a steady speed of 120 km h^{-1} on the level find the resistance to motion. If the total mass of the same engine and its train is 450 t and the same power is used, what is the acceleration on the level if the speed is 100 km h^{-1}?

13 If a car of mass 900 kg can travel at a maximum speed of 40 m s^{-1} on the level and at a maximum speed of 30 m s^{-1} up an incline of $\frac{1}{70}$, find the resistance to motion (assumed the same in both cases) and the power of the engine.

14 A car is rated at 30 HP. Taking 1 HP $\approx$ 750 W, find the maximum speed of the car up an incline of $\frac{1}{280}$ if the resistance to motion is 700 N and the mass of the car is 800 kg.

15 A car of mass 800 kg freewheels at a steady speed of 20 m s^{-1} down a slope of $\frac{1}{70}$. Find the resistance to motion. Assuming that the resistance to motion is proportional to the square of the speed, find the resistance at a speed of 30 m s^{-1}. Now find the power required to drive this car up the same incline at a steady speed of 30 m s^{-1}.

16 A train of total mass 300 t is moving up an incline of angle θ to the horizontal, where $\sin\theta = \frac{1}{280}$. The resistance to motion is 3000 N and the train is accelerating at 0.2 m s^{-2}. Find
(a) the driving force of the engine,
(b) the power exerted at the moment when the speed is 10 m s^{-1}.

17 A machine lifts 50 cranes per minute, each of mass 8 kg, through a vertical height of 1.5 m and sends each one to the loading bay at a speed of 4 m s^{-1}. Find the power used.

18 A water pump sends a mass of 0.3 kg of water each second into a vertical jet. If the power of the pump is 5.4 W, find the height of the jet and the speed with which the water leaves the pump.

19 A conveyor belt carries 1200 kg of grain up a slope inclined at an angle θ to the horizontal where $\sin\theta = 0.08$ against a frictional resistance of 120 N. If the belt is travelling at 10 km h^{-1}, find the power used to drive the belt.

20 A truck develops a constant power of 180 kW. If its maximum speed on the level is 25 m s^{-1}, find the resistance to motion.
If the truck of mass 3×10^3 kg, now climbs a slope of angle θ to the horizontal, where $\sin\theta = \frac{1}{20}$, against the resistance, calculate
(a) the maximum speed up the slope,
(b) the acceleration of the truck when its speed is 15 m s^{-1}.

SUMMARY

- Work (W) done by a force (P) in moving a body a distance (s) in the direction of the force is
  given by $W = Ps$
  Using vectors, $W = \mathbf{P}.\mathbf{s}$
  $= Ps\cos\theta$ where θ is the angle between the force and the direction of motion of the body.
- The unit of work is 1 joule (J) which is the work done when a force of 1 N moves a body through a distance of 1 m in the direction of the force.
- Kinetic energy, $\text{KE} = \frac{1}{2}mv^2$
- Potential energy, $\text{PE} = mgh$
- *Principle of conservation of energy*: The total energy in a closed system is constant. If no energy is dissipated through friction, heat, sound etc., the principle of conservation of energy will take the form:
  $$\text{KE} + \text{PE} = \text{constant}.$$
- If work is done by a force on a body then
  $$\text{work done} = \text{initial energy} - \text{final energy}$$
  (where energy = KE + PE)
- Power = rate of doing work
  Unit of power is 1 watt (W) = 1 J s^{-1}

REVISION EXERCISE 25 *(Answers on page 653.)*

1 A missile of mass 50 kg is projected vertically upwards with a speed of 200 m s^{-1} from a submarine lying on the bottom of the sea at a depth of 600 m. Assuming that the water offers a constant resistance of 160 N to the motion of the missile, calculate
(i) the kinetic energy of the missile as it leaves the water,
(ii) the maximum height above sea-level reached by the missile. (C)

2 Fig. 25.15 shows a mass of 5 kg placed on a rough horizontal table and attached to one end of a light inextensible string which passes horizontally over a smooth light pulley at the edge of the table. The other end of the string is attached to a mass of 3 kg which hangs freely. The coefficient of friction between the 5 kg mass and the table is 0.4, and the distance of the 5 kg mass from the pulley is 2.5 m. The system is released from rest. Calculate
(i) the time taken for the 5 kg mass to reach the pulley,
(ii) the loss of potential energy of the system during this time,
(iii) the kinetic energy of the system immediately before impact with the pulley. (C)

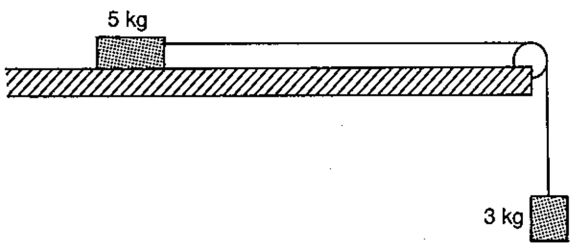


Fig. 25.15

3 The top of a chute whose length is 12 m is 3 m vertically above its lowest point. A parcel of mass 1.6 kg slides from rest from the top of the chute and reaches the lowest point with a speed of 5 m s^{-1}. Calculate, for the parcel,
(i) the gain in kinetic energy,
(ii) the loss in potential energy,
(iii) the work done in overcoming the frictional resistance,
(iv) the average value of this resistance.

After reaching the lowest point of the chute, the parcel slides along a horizontal floor, the resistance to motion being 4 N. Calculate how far the parcel travels before coming to rest. (C)

4 A pump forces oil from rest through a horizontal pipe so that it emerges with a speed of 6 m s^{-1} from a nozzle of cross–sectional area 2×10^{-2} m^2. The mass of 1 m^3 of the oil is 2500 kg. Calculate, in kW, the power developed by the pump.
Calculate by how much the power will have to be increased if the nozzle is raised through a height of 2 m and the oil emerges at the same speed. (C)

5 A car of mass 1200 kg and power output 90 kW experiences frictional resistances to motion of 3000 N.
Assuming that the power output remains constant, calculate
(i) the maximum speed on a horizontal road,
(ii) the maximum speed up a slope inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{12}$,
(iii) the acceleration up this slope when the speed is 15 m s^{-1}. (C)

6 A car of mass 960 kg has a maximum speed of 50 m s^{-1} on a horizontal road when the power output of the engine is 40 kW. Calculate the frictional resistance.
The car ascends a slope inclined at an angle α to the horizontal where $\sin \alpha = \frac{1}{6}$. The power output of the engine remains the same but the frictional resistance is now 900 N. Calculate
(i) the maximum speed of the car up the slope,
(ii) the acceleration of the car up the slope when its speed is 10 m s^{-1}. (C)

7

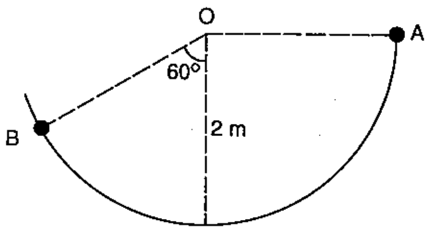


Fig. 25.16

Fig. 25.16 shows a body of mass 3 kg held at A, the top of a rough semicircular slide, centre O and radius 2 m.
The body is projected vertically downwards with velocity 8 m s^{-1} and comes to instantaneous rest at B where OB makes an angle of 60° with the downward vertical. Calculate
(i) the change in the kinetic energy of the body,
(ii) the change in the potential energy of the body,
(iii) the work done against the resistance of the slide,
(iv) the average resistance of the slide. (C)

8 A cyclist and his machine have a combined mass of 72 kg. Starting at X with a speed of 3 m s^{-1}, he freewheels down the hill XY, arriving at Y with a speed of 11 m s^{-1}. He continues to freewheel along the horizontal road YZ, coming to rest at Z. Given that the constant frictional resistance to his motion is 24 N, calculate the length of (i) XY, (ii) YZ.

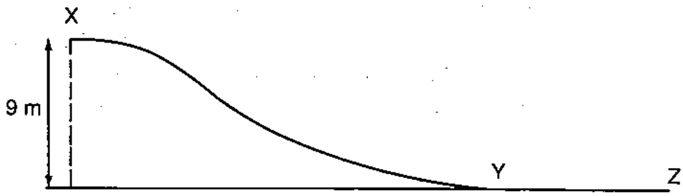


Fig.25.17

9 (a) An engine, working at an effective rate of 0.2 kW, pumps water, initially at rest, through a vertical height of 4.8 m, where it is ejected with a velocity of 2 m s^{-1}. Calculate the number of kilograms of water delivered per second.

(b) A car of mass m kg is being driven down a slope inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{75}$. Given that the frictional resistance to motion is 600 N, and that the acceleration is 0.8 m s^{-2}, when the velocity is 15 m s^{-1}, show that the effective power developed by the car at this instant is $\left(9 + \frac{m}{100}\right)$ kW. (C)

10 A car of mass 1800 kg ascends an incline at a constant speed of 14 m s^{-1}. Given that the frictional resistance is 400 N and the engine is working at a rate of 17.5 kW, calculate the angle that the incline makes with the horizontal.
Assuming that the frictional resistance and the rate of working remain constant, determine the acceleration of the car on a level road at an instant when the speed is 28 m s^{-1}. (C)

11 A four–engined aeroplane of mass 120 000 kg is flying horizontally at a constant speed of 240 m s^{-1}. Each engine produces a driving force of 52 000 N. Calculate
(i) the air resistance to motion,
(ii) the power produced by each engine.
If one engine has to be cut out, what deceleration will be produced?

12 A car of mass 600 kg is travelling along a level road at a constant speed of 90 km h^{-1}. The frictional resistance is 450 N. Calculate (in kW) the power being used.
The car now climbs a slope inclined to the horizontal at angle θ, where $\sin \theta = 0.02$, at the same speed. If the frictional resistance is the same as before, what power is now being used? If this power is used but the speed of the car up the slope is 72 km h^{-1}, what acceleration will it have?

13 A block of mass 6 kg is projected with a speed of 5 m s^{-1} directly up a rough plane inclined at θ to the horizontal, where $\tan \theta = \frac{3}{4}$. The block travels a distance of 1.25 m before coming to momentary rest and then slips back down the plane.
Calculate
(i) the initial kinetic energy of the block,
(ii) the gain in potential energy between projection and momentary rest,
(iii) the coefficient of friction between the block and the plane,
(iv) the velocity of the block as it passes through its starting point on the way down.

14 A cyclist and his machine together have a mass of 75 kg. If the cyclist works at the rate of 0.15 kW, his maximum speed on level ground is v m s^{-1}. If he climbs a slope of angle θ to the horizontal, where $\sin \theta = \frac{1}{50}$, using the same power and against the same resistance, his maximum is now $\frac{v}{3}$ m s^{-1}. Find the value of v.

15 Fig. 25.18 shows a track in the form of a quarter circle AB, centre O and radius 1.6 m and a straight horizontal section BC. OB is vertical. The section AB is smooth but BC is rough. A particle of mass 2 kg is projected vertically downwards at A with speed 2 m s^{-1} and comes to rest at S where BS = 3 m. Calculate

(i) the change in kinetic energy of the particle,

(ii) the change in potential energy,

(iii) the speed of the particle at B,

(iv) the coefficient of friction along BC.

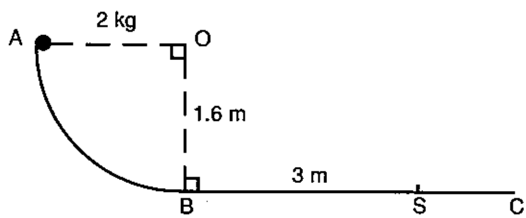


Fig. 25.18

16 A particle of mass 4 kg is falling vertically with speed of 10 m s^{-1} when it meets a dense layer of material 3 m thick which gives a constant resistance of 50 N. Find the speed with which the particle comes out of this layer.
If the particle had been moving upwards with speed u m s^{-1} when it entered the layer, what is the minimum value of u so that it will pass through the layer?

17 A particle P is attached by a light string 2 m long to a fixed point O. The particle is held so that OP makes an angle of 60° with the downward vertical through O and is then projected with speed v m s^{-1} downwards at right angles to OP. If it comes to momentary rest when it reaches the level of O, find the value of v.

18 A car of mass 800 kg moves up a slope inclined at angle θ to the horizontal, where $\sin\theta = \frac{1}{40}$, at a steady speed of 25 m s^{-1}. The road resistance is 100 N. Calculate the power being used.
If the car now travels down this slope, with the same resistance and using the same power, what would be its acceleration when the speed is 25 m s^{-1}?

19 A particle of mass 2 kg is dropped from a height h_1 m onto a concrete floor and rebounds to a height h_2 m. Given that it loses 64 J of energy, show that $h_1 - h_2 = 3.2$. Given also that the velocity of lift off from the floor is 4 m s^{-1} less than the velocity on hitting the floor, find the values of h_1 and h_2.

20 (a) As part of a manufacturing process a machine lifts a number of components, each of mass 0.6 kg, from rest through a height of 1.3 m and then projects each one with a speed of 2 m s^{-1}.
Given that the effective power output of the machine is 27 W, find how many components are dealt with per second.

(b) Fig. 25.19 shows the vertical plane through the portion ABC of a road along which a car of mass 750 kg travels against a resistance. The points A and C are 36 m and 28 m respectively higher than the lowest point B.

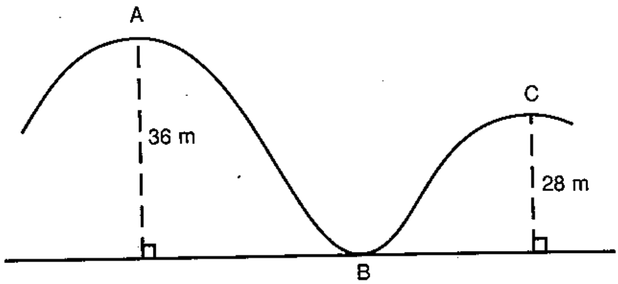


Fig.25.19

The car coasts from A to B, a distance of 150 m, with no power being transmitted to its wheels. Given that its speed at A is 12 m s^{-1}, its speed at B is v m s^{-1}, and that the average resistance is 720 N, find, as the car travels from A to B.

(i) the loss in potential energy,

(ii) the gain in kinetic energy, in terms of v,

(iii) the work done against the resistance.

Hence show that $v = 24$.

When the car arrives at B, power is supplied to the wheels and, 7.5 seconds later, the car arrives at C with a speed of 20 m s^{-1}. Given that the distance from B to C is 120 m, and that the average resistance is again 720 N, find, as the car travels from B to C,

(iv) the gain in potential energy,

(v) the loss in kinetic energy,

(vi) the work done against the resistance.

Hence calculate the average power supplied by the engine from B to C. (C)

Momentum and Impulse

26

Consider a body, mass m kg, moving in a straight line with velocity u m s^{-1}. Now suppose a force P N acts on the body in the direction of its motion for t s (Fig. 26.1).

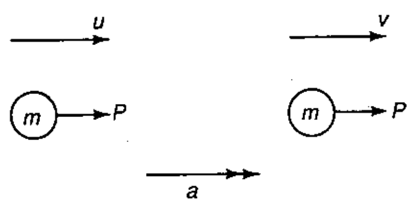


Fig. 26.1

The force will give the body an acceleration a

given by $\qquad P = ma \qquad$ (i)

The velocity v reached after t s is

given by $\qquad v = u + at \qquad$ (ii)

From (ii) $\qquad a = \frac{v - u}{t}$

Substituting this into (i), we have

$$P = m\left(\frac{v-u}{t}\right)$$

i.e. $\qquad P = \frac{mv - mu}{t} \qquad$ (iii)

or $\qquad Pt = mv - mu \qquad$ (iv)

This equation shows the relationship between two important quantities. Pt is called the **impulse** of the force. It is the product of the force (assumed constant) and the time for which it acts. Since force is measured in newtons and time in seconds, the unit of impulse is the newton–second (N s). Thus the impulse of a force of 50 N acting on a body for 0.1 s is $50 \times 0.1 = 5$ N s.

The right hand side of (iv) measures the change in a quantity called the **momentum** of the body. The momentum of a body is the product of its mass and its velocity

i.e. momentum = mass × velocity.

Hence the initial momentum is mu and the final momentum mv. $mv - mu$ is then the change in momentum. Equation (iv) can be restated in words as follows:

impulse of force on a body = change of momentum of the body

Since the unit of impulse is N s, it follows that the unit of momentum must also be N s. So a body of mass 3 kg travelling at 4 m s^{-1} has a momentum of $3 \times 4 = 12$ N s.

To find momentum in N s, mass must be expressed in kg and velocity in m s^{-1}.

Since momentum is the product of a scalar (mass) and a vector (velocity), it is itself a vector quantity. Care must therefore be taken with its direction.

If the initial and final velocities are in the same line as the impulsive force, we measure the momentum in the direction of the force and use the following equation:

$$\overrightarrow{\text{impulse}} = \overrightarrow{\text{final momentum}} - \overrightarrow{\text{initial momentum}}$$

The arrows show that all are measured in the same direction.

If the two momenta are in different lines, the change of momentum must be found by vector subtraction. Such cases will not be dealt with as they are beyond the scope of this book.

Returning to equation (iii), we note that $\frac{mv - mu}{t}$ represents the rate of change of momentum. Hence equation can be stated in words as follows:

force = rate of change of momentum

Example 1

A constant force of P N acts on a body of mass 2 kg travelling at 4 m s^{-1} for 0.3 s in the direction of its motion. If its final velocity is 7 m s^{-1}, what is the force?

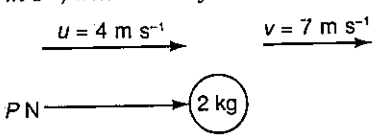


Fig. 26.2

Impulse of the force $= P \times 0.3$
$= 0.3P$ N s

Change of momentum in the direction of the force

$=$ final momentum $-$ initial momentum
$= 2 \times 7 - 2 \times 4$
$= 6$ N s

Impulse = change of momentum
$0.3P = 6$

giving $P = 20$

[Alternatively, we can use: Force = rate of change of momentum

i.e. $P = \frac{6}{0 \cdot 3} = 20$ as before.]

Hence the force is 20 N.

Example 2

A body of mass 2 kg travelling at 10 m s^{-1} encounters a constant frictional force of 5 N. How long does it take for the body to come to rest?

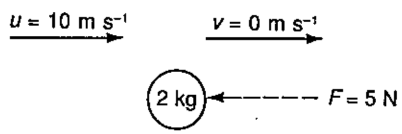


Fig. 26.3

Let the time taken be t s.

Impulse of the frictional force = $5t$ N s

Change of momentum in the direction of the force

$$= 2 \times 0 - 2 \times (-10)$$
$$= 20 \text{ N s}$$

therefore $5t = 20$

$t = 4$

Hence the time taken is 4 s.

Example 3

A ball of mass 0.2 kg is dropped from a height of 5 m onto a concrete floor and rebounds to a height of 1.8 m. Find the impulse of the floor on the ball. If the contact time is 0.05 s, find the average force on the ball.

Let the velocities on arrival and lift-off be v_1 and v_2 m s^{-1} respectively, and the impulsive force be P N (Fig. 26.4).

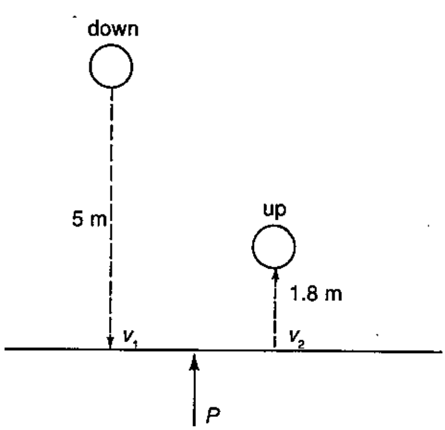


Fig. 26.4

We use the equation $v^2 = u^2 + 2as$ where $a = g \approx 10$ m s^{-2} to find v_1 and v_2.

For the downward motion,

$$v_1^2 = 0 + 2(10)(5)$$

giving $v_1 = 10$

For the upward motion,

$$0 = v_2^2 - 2(10)(1.8)$$

giving $v_2 = 6$

Impulse $=$ change of momentum in the direction of the force P (upwards)
$=$ momentum after impact – momentum before impact
$= 0.2 \times 6 - 0.2 \times (-10)$
$= 3.2$

Hence the impulse of the floor on the ball is 3.2 N s.

But impulse $= Pt$ where $t = 0.05$

therefore $P \times 0.05 = 3.2$

giving $P = 64$

Hence the average force on the ball is 64 N.

Example 4

A hose of cross–section 2 cm^2 delivers a jet of water horizontally with a speed of 20 m s^{-1}. With what average force does the water hit a vertical wall? (Assume that the water does not rebound and that the mass of 1 m^3 of water = 10^3 kg.)

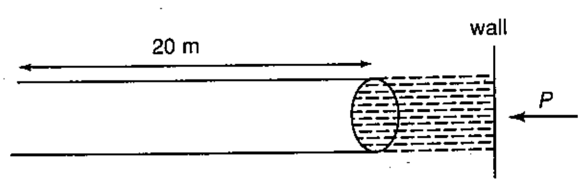


Fig. 26.5

The impulsive force P N exerted by the wall on the water (which equals the force exerted by the water on the wall) = momentum of water destroyed on impact since the water does not rebound.

Consider the momentum destroyed in 1 s.

Volume of water issued from the hose in 1 s

$= \text{length of water issued} \times \text{area of cross-section}$
$= 20 \text{ m} \times (2 \times 10^{-4}) \text{ m}^2$
$= 4 \times 10^{-3} \text{ m}^3$

Mass of water $= 4 \times 10^{-3} \times 10^3$
$= 4$ kg

thus the momentum destroyed in 1 s $= 4 \times 20$
$= 80$ N s.

The impulse lasts for 1 s,

therefore $P \times 1 = 80$

i.e. $P = 80$

Hence the average force on the wall is 80 N.

Exercise 26.1 *(Answers on page 653.)*

[Take g = 10 m s⁻²]

1 Find the momentum of the body in each of the following:
(a) mass 2 kg, velocity 5 m s^{-1}
(b) mass 0.25 kg, velocity 2 m s^{-1}
(c) mass 80 g, velocity 1.5 m s^{-1}
(d) mass 300 kg, velocity 20 km h^{-1}

2 A force of 10 N acts on a body of mass 2 kg for 0.5 s. What is the increase in momentum? If the body was originally travelling at 5 m s^{-1}, what is its final speed? How far will it travel in this time?

3 A body of mass 1.5 kg travelling at 3 m s^{-1} is acted on by a force P N for 0.75 s. If its velocity at the end of that time is 5 m s^{-1}, find the value of P.

4 A ball of putty, mass 0.75 kg, moving at 3 m s^{-1} hits a wall at right angles and stops dead. Find the impulse on the ball.

5 A ball of mass 60 g moving at 4 m s^{-1} hits a wall at right angles. If it rebounds with speed 2 m s^{-1}, what is the impulse on the ball?

6 A hammer of mass 5 kg, travelling at 4 m s^{-1}, hits a nail directly and does not rebound. What is the impulse on the hammer? If the contact effectively lasts 0.5 s, what is the average force between the two?

7 A mass of 1.2 kg travelling at 2 m s^{-1} strikes a wall at right angles and rebounds (also at right angles) with a velocity of 1.5 m s^{-1}. If the contact lasted 0.25 s, find the force on the mass.

8 A ball of mass 0.25 kg falls freely onto a concrete floor from a height of 20 m and rebounds to a vertical height of 5 m. If the ball was in contact with the floor for 0.8 s, find the average force exerted on the ball.

9 A truck of mass 50 kg has its speed reduced from 4 m s^{-1} to 1.5 m s^{-1} in 30 s. Find the braking force (assumed constant). After what further time will the truck come to rest under this force?

10 A tennis ball of mass 30 g travelling horizontally at 20 m s^{-1} is hit straight back at 30 m s^{-1}. If the impact lasted 0.04 s, find the average force on the ball.

11 A box of mass 10 kg is dragged across a rough floor (coefficient of friction 0.5) by a force P N. If the speed of the box is increased from 0.5 m s^{-1} to 1.9 m s^{-1} in 10 s, find P.

12 A hose (cross-section 4 cm^2) delivers water horizontally with a speed of 25 m s^{-1}. What is the impulse of the water on a vertical wall (assuming no rebound)? What average force acts on the wall?

13 A horizontal jet of water is emitted from a circular pipe of radius 1 cm at a speed of 12 m s^{-1}. Find the mass of water emitted per s and the average force exerted on a vertical wall.

14 A particle of mass 3 kg moves on a rough horizontal surface with coefficient of friction 0.4. When it is 3 m from a vertical wall its speed is 7 m s^{-1} and it is moving at right angles to the wall. If the impulse on the particle is 24 N s, calculate its speed before and after hitting the wall.

15 A ball of mass 100 g is thrown at a horizontal floor and hits it with speed 6 m s^{-1} at an angle of 30° to the floor. Given that the horizontal component of its velocity is unchanged by the impact and that the vertical impulse on the ball is 0.5 N s, find the components of its velocity after the impact and hence find the velocity in magnitude and direction.

CONSERVATION OF MOMENTUM

Let us now consider what happens when two bodies A and B collide. We will only consider direct collision, that is, the two bodies are moving in the same straight line. Suppose A with mass m_1 and B with mass m_2 are moving with velocities u_1 and u_2 before collision, and v_1 and v_2 after collision respectively (Fig. 26.6).

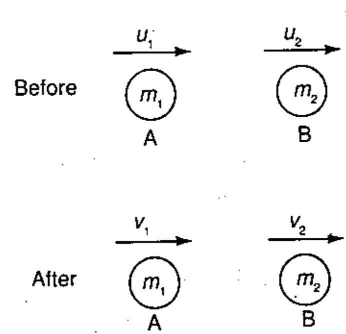


Fig. 26.6

Change in momentum of A = $m_1v_1 - m_1u_1$ (i)

Change in momentum of B = $m_2v_2 - m_2u_2$ (ii)

Now when A and B collide, each will exert an impulse on the other.

By Newton's Third Law, A will exert a force P on B and B will exert an equal but opposite force P on A. The time of contact, t s, is naturally the same for both. Hence the impulse of A on B equals the impulse of B on A (both = Pt) and hence the changes in momentum of A and B are equal but opposite in direction.

Thus, from (i) and (ii) we get

$$m_1v_1 - m_1u_1 = -(m_2v_2 - m_2u_2)$$

[negative sign indicates opposite direction]

or $$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$$

This shows that the total momentum after the collision equals the total momentum before the collision. This in essence is the principle of **conservation of momentum:**

In any collision between two bodies, the total momentum in any direction is unchanged, provided no external force acts in that direction.

In Examples 3 and 4, momentum was not conserved as the forces were external forces. Gravity is not however an external force in this context. We can also use the principle in the form: momentum before collision = momentum after collision (in the same direction). As an example of the principle, consider a gun being fired. The gun rests against the shoulder of a man. The explosion gives the bullet forward momentum and so the gun must acquire an equal amount of momentum backwards, thus producing the recoil of the gun against the man's shoulder. A more sophisticated example is how a spacecraft can change direction in space. As there is no resistance to motion in empty space, any object will move continually in a straight line. It is impossible to produce an external force to change direction or to slow down, as there is no atmosphere or friction to 'push against'. If however a small rocket is fired from the spacecraft, momentum in that direction is created and an equal but opposite amount of momentum affects the spacecraft, thus changing its direction of motion or its speed. So to increase the speed of a spacecraft, a retrorocket is fired backwards, giving the craft additional momentum (and hence increased speed) forwards. To slow it down, a forward-facing rocket is fired, so reducing the momentum of the spacecraft.

Example 5

A particle of mass 2 kg is moving at 8 m s^{-1}. It collides with a stationary particle of mass 3 kg and they move together. Find their common speed. Find also the loss of kinetic energy during the collision.

Let the common speed after collision be v m s^{-1}.

Momentum before collision = $2 \times 8 + 3 \times 0 = 16$ N s

Momentum after collision = $5v$ N s

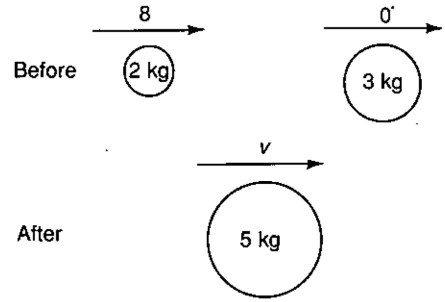


Fig. 26.7

By conservation of momentum,

$$5v = 16$$

giving $\quad v = 3.2$

Hence the common speed is 3.2 m s^{-1}.

$$\text{Loss of KE} = \tfrac{1}{2} \times 2 \times 8^2 - \tfrac{1}{2} \times 5 \times 3.2^2$$
$$= 38.4 \text{ J}$$

Thus the loss of KE during collision is 38.4 J. This loss of energy would mostly appear as sound and heat during the collision.

Example 6

Two trucks, of masses 5000 kg and 4000 kg are travelling on the same track with speeds of 2 m s^{-1} and 3 m s^{-1} respectively in opposite directions. They collide and the first truck is observed to have a speed of 1 m s^{-1} in the direction opposite to its original direction. What is the speed of the second truck?

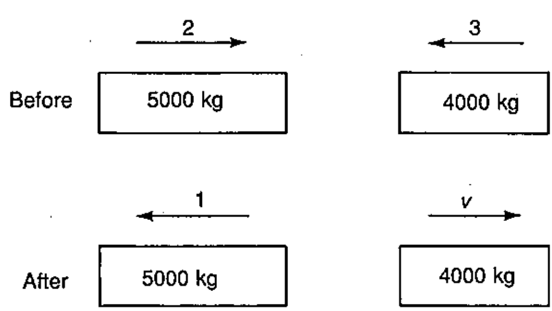


Fig. 26.8

We take momentum in the direction of the first truck as positive.

$$\text{Momentum before impact} = 5000 \times 2 - 4000 \times 3$$
$$= -2000 \text{ N s}$$

Suppose the second truck travels with speed v m s^{-1} opposite to its original direction after collision.

Momentum after impact $= -5000 \times 1 + 4000v$
$= 4000v - 5000$ N s

By conservation of momentum,

$$4000v - 5000 = -2000$$

giving $\quad v = 0.75$

Hence the second truck will travel at 0.75 m s^{-1} opposite to its original direction.

Example 7

A gun of mass 4 kg fires a bullet of mass 80 g horizontally at a speed of 400 m s^{-1}. With what initial speed does the gun recoil? If the bullet then hits a stationary block of wood of mass 0.8 kg resting on a smooth horizontal surface and remains embedded in it, find the final speed of the block of wood.

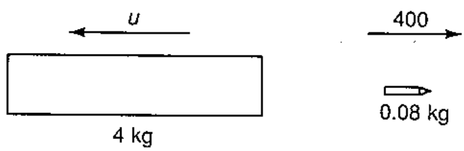


Fig. 26.9

First, consider the gun and the bullet.
The total momentum, which initially was zero, is unchanged by the explosion. Taking the direction of the bullet as positive for momentum (Fig. 26.9), we have by conservation of momentum,

$$0.08 \times 400 + 4(-u) = 0$$

where u m s^{-1} is the velocity of recoil of the gun.

This gives $u = 8$.

Hence the gun recoils with a speed of 8 m s^{-1}.

Now consider the bullet and the block of wood (Fig. 26.10).

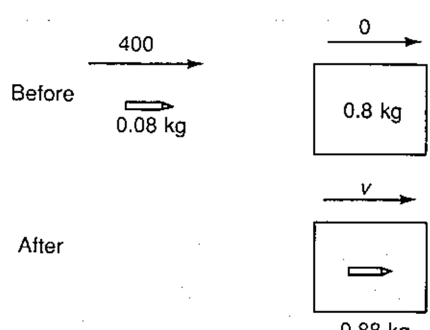


Fig. 26.10

Again, we take the direction of the bullet as positive for momentum.

Momentum before impact $= 0.08 \times 400 + 0.8 \times 0$
$= 32$ N s

Momentum after impact $= (0.8 + 0.08)v$
$= 0.88v$ N s

where v m s^{-1} is the speed of the block with bullet embedded.

By conservation of momentum,

$$0.88v = 32,$$

giving $$v \approx 36.4$$

Hence the speed of the block of wood is 36.4 m s^{-1}.

Example 8

A particle A of mass 0.5 kg is dropped from rest from a height of 8 m above ground level. At the same time a particle of mass 0.2 kg is projected vertically upwards with a velocity of 8 m s^{-1} from the ground so that it collides and coalesces with A. Find
(a) the height at which they collide,
(b) their velocities just before impact,
(c) their common velocity after the collision,
(d) the speed on reaching the ground.

(a) Let the particles collide h m above the ground (Fig. 26.11). We use the equation $s = ut + \frac{1}{2}at^2$.

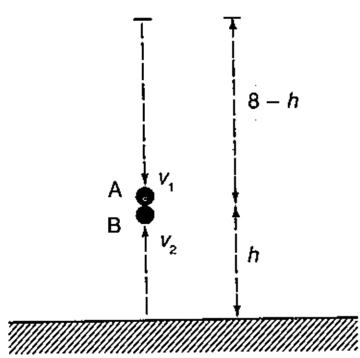


Fig. 26.11

For A, $8 - h = 0 + \frac{1}{2}gt^2$

i.e. $8 - h = 5t^2$ (i)

For B, $h = 8t - \frac{1}{2}gt^2$

i.e. $h = 8t - 5t^2$ (ii)

Adding (i) and (ii), we have

$$8 = 8t$$

giving $\quad t = 1$

Substituting $t = 1$ into (i) gives $h = 3$.

Hence collision occurs 3 m above the ground.

(b) Let the velocities of A and B just before collision be v_1 m s^{-1} and v_2 m s^{-1} respectively.

For A, $v_1 = gt = 10$

For B, $v_2 = 8 - gt = 8 - 10 = -2$

The negative sign indicates that B is moving downwards just before impact.

Note: If we use $v^2 = u^2 + 2as$, we would get $v_2^2 = 64 - 2(10)(3) = 4$ but we would not know if $v_2 = +2$ or -2 and it would be incorrect to assume $v_2 = +2$.

Hence the velocities of A and B are 10 m s^{-1} and 2 m s^{-1} respectively before impact and both are in the downward direction.

(c) Let the common velocity after collision be u m s^{-1}.

By conservation of momentum,

$$(0.5 + 0.2)u = 0.5 \times 10 + 0.2 \times 2$$
$$0.7u = 5.4$$

giving $\quad u \approx 7.71$

Hence the common velocity after collision is 7.71 m s^{-1}.

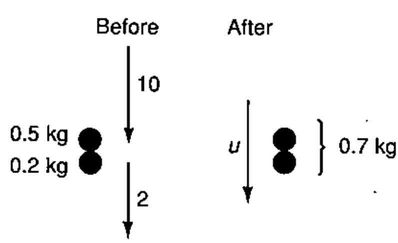


Fig. 26.12

(d) With u m s^{-1} as the initial speed and distance = 3 m, the final velocity v m s^{-1} on reaching the ground is given by

$$\begin{aligned} v^2 &= u^2 + 2as \\ &= 7.71^2 + 2(10)(3) \\ &= 119.4 \end{aligned}$$

giving $v \approx 10.9$

Hence the speed on reaching the ground is 10.9 m s^{-1}.

Example 9

Fig. 26.13(a) shows the particles A and B of masses 0.6 kg and 0.4 kg respectively, connected by a light string passing over a smooth pulley. The particle C of mass 0.5 kg is resting on a table and is attached to B by a string which is slack. A and B are held at rest with the connecting string between them taut and then released. Find

(a) the acceleration of A,

(b) the distance A descended and its velocity after 1 second.

At this time, the string connecting B and C becomes taut and C is lifted off the table. Calculate

(c) the velocity and acceleration of A immediately after C is lifted off the table,

(d) how much lower A will descend.

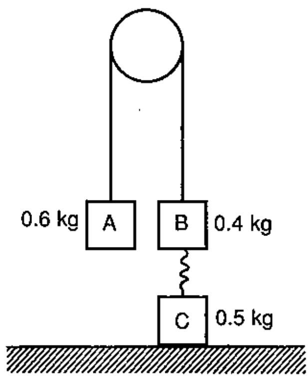


Fig. 26.13(a)

(a) The acceleration a m s^{-2} is found from Fig. 26.13(b).
C can be neglected at this stage as the string between B and C is slack and hence it will not affect the ensuing motion.

For A, $6 - T = 0.6a$

For B, $T - 4 = 0.4a$

Solving the two equations gives $a = 2$ m s^{-2}.

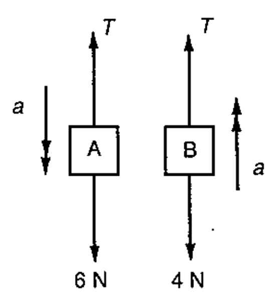


Fig. 26.13(b)

(b) A will descend a distance s m given by $s = ut + \frac{1}{2}at^2$

i.e. $s = 0 + \frac{1}{2} \times 2 \times 1^2 = 1$ m.

The velocity of A at that time will be $v = 0 + 2 \times 1 = 2$ m s^{-1}.

(c) When the string connecting B and C becomes taut, the velocities and acceleration immediately change. The total momentum is unchanged however, so we can find the new velocity. Since the particles are connected by tight strings, we can take them as one body (Fig. 26.13(c)).

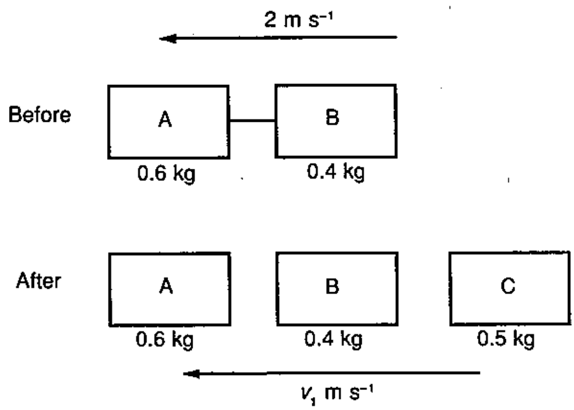


Fig. 26.13(c)

Momentum of the system before C is lifted = $(0.6 + 0.4) \times 2 + 0.5 \times 0$

$= 2$ N s

Momentum of the system after C is lifted $= (0.6 + 0.4 + 0.5)v_1 = 1.5v_1$ where v_1 is the common velocity of the particles immediately after C is lifted.

By conservation of momentum,

$1.5v_1 = 2$

giving $v_1 = \frac{4}{3}$

Hence the new velocity of A is $\frac{4}{3}$ m s^{-1}

The new acceleration is found as in (a) by taking B and C as one particle of mass 0.9 kg. Check that it is 2 m s^{-2} in the opposite direction to the previous acceleration.

(d) A now has a velocity of $\frac{4}{3}$ m s^{-1} downwards and an acceleration of 2 m s^{-2} upwards. It will descend a distance s given by $0 = \left(\frac{4}{3}\right)^2 + 2 \times (-2) \times s$ i.e. $s = \frac{4}{9}$.

Hence A will descend a further distance of $\frac{4}{9}$ m.

Exercise 26.2 *(Answers on page 653.)*

[Take $g = 10$ m s^{-1}]

1 Two masses of 30 kg and 20 kg, travelling at 4 m s^{-1} and 1 m s^{-1} respectively in the same direction, collide and continue together. Find their common speed after the collision.

2 A toy railway truck, mass 0.3 kg, travelling at 2 m s^{-1}, collides with another stationary truck, mass 0.25 kg, and they couple together. Find the common speed after impact and the impulse between them.

3 A gun of mass 450 kg fires a shell of mass 2 kg horizontally at a speed of 300 m s^{-1}. Find the initial recoil velocity of the gun. If the gun comes to rest (moving horizontally) in 10 s, find the average resisting force.

4 Two masses of 3 kg and 2 kg move towards each other at speeds of 1.5 m s^{-1} and 2 m s^{-1} respectively. After the collision they move together. Find their common velocity.

5 A mass of 0.1 kg travelling at 10 m s^{-1} overtakes and collides with a mass of 0.5 kg moving at 2 m s^{-1}. They move on together. Find their common velocity.

6 Two billiard balls of equal mass (0.8 kg) are moving in opposite directions (in the same line) with speeds of 12 m s^{-1} and 5 m s^{-1}. They collide and the slower ball is now seen moving at 8 m s^{-1} in the opposite direction. Find the new speed of the other ball and the impulse between them.

7 A body of mass 10 kg is moving horizontally with a speed of 20 m s^{-1}. It explodes and splits into two parts of masses 6 kg and 4 kg. The 4 kg part continues to move in the original direction but with a speed of 30 m s^{-1}. Find the speed of the 6 kg part, assuming it moves in the same direction.

8 A spacecraft of mass 450 kg is moving in space with a speed of 3×10^3 m s^{-1}. A rocket is fired straight ahead, emitting 1.5 kg of gas at a speed of 2×10^4 m s^{-1}. Ignoring the slight reduction in mass of the spacecraft, find its new speed.

9 Two masses of 4 kg and 3 kg are connected by a light string over a smooth pulley. After moving for 5 s, the 3 kg mass picks up a third mass of 1 kg instantaneously. Find the speed of the 3 masses after the pickup.

10 Two masses of 5 kg and 2 kg are connected by a light string over a smooth pulley. The 5 kg mass is at rest on a horizontal table (below the pulley) and the 2 kg mass is released from rest. After it falls freely for 2 s the string is tightened and the 5 kg mass is jerked off the table. Find the velocity with which the masses now continue. Also find their common acceleration.

11 Two masses of 5 kg and 2 kg are connected by a light string over a smooth pulley. The system is released from rest. Find the common acceleration. After falling for 2 s, the 5 kg mass hits a horizontal table and does not rebound. Find the velocity of the 2 kg mass at this time and find how much higher it will continue to rise before coming to rest. Find also the common velocity when the string tightens again.

12 Particle A of mass $\frac{1}{2}$ kg is let fall from a height of 5 m.

Simultaneously particle B of mass 1 kg is thrown vertically upwards from the ground with speed 10 m s^{-1} so that it collides and coalesces with A. Find

(a) the height at which they meet,

(b) their speeds and directions just before impact,

(c) the common speed after impact,

(d) how much higher the combined mass will rise.

13 A particle A, of mass 4 kg, is travelling in a straight line due north with a speed of 3 m s^{-1}; another particle B, of mass 3 kg, is travelling in the same straight line towards A with a speed of 5 m s^{-1}. After the collision A is moving south with a speed of 2 m s^{-1}. Calculate

(a) the velocity of B after the collision,

(b) the impulse between the particles.

14 A gun of mass 45 kg fires a shell of mass 0.9 kg at a speed of 100 m s^{-1} horizontally. Find the initial recoil velocity of the gun. If this recoil is opposed by a constant force of 250 N, how far does the gun recoil?

15 A horizontal force of 10 N is applied to a body A of mass 2 kg, initially at rest on a smooth surface, for 5 s. What velocity is gained by A? A now collides with another body B of mass 4 kg at rest and the two continue together. Find the common velocity.

16 A shell of mass 5 kg is travelling horizontally at 200 m s^{-1} when it explodes into two parts. One part (of mass 3 kg) continues in the same direction at a speed of 400 m s^{-1}. What will be the velocity of the other part? What are the KEs before and after the explosion?

17 A nail of mass 20 g is driven horizontally into wood by a hammer of mass 3 kg. Just before the impact the hammer is moving horizontally at 4 m s^{-1}. Find the common velocity of hammer and nail after the impact, if they move together, and if the nail penetrates the wood to a depth of 5 cm, find the average resistance of the wood.

18 Two masses of 5 kg and 3 kg move directly towards each other and collide. Their speeds before impact are 4 m s^{-1} and 3 m s^{-1} respectively. After the collision the 3 kg mass reverses at a speed of 2 m s^{-1}. Find the velocity of the 5 kg mass after the collision. Also find the percentage loss of KE.

19 Two pendulums with light strings each 1 m long carry masses of 1 kg and 2 kg and are suspended side by side from the same point. The larger mass is raised until its string is horizontal and taut and is then released. If the two masses stick together on impact, find the vertical height to which they rise after the impact.

20 A pendulum consists of a mass of 8 kg attached to a fixed point by a light string 50 cm long. The mass is at rest when it is struck a blow lasting 0.05 s by a force of 200 N, acting horizontally. Find the angle made by the string with the downward vertical when the mass first comes to rest.

21 A mass of 10 kg falls vertically through a height of 5 m onto a wooden stake of mass 2 kg and does not rebound. The stake is driven into the ground a vertical distance of 10 cm. Calculate

(a) the common velocity after the impact,

(b) the total energy just after impact to be converted into work against the resistance of the ground,

(c) the average resistance of the ground.

22 A truck of mass 6000 kg moving with a speed of 2.4 m s^{-1} hits a mass of 2000 kg moving in the opposite direction with speed 1.3 m s^{-1} and the two move together. Calculate

(a) the total kinetic energy after the impact,

(b) the distance in which a braking force of 1800 N would bring the trucks to rest.

23 Particles of A and B, of masses 1.4 kg and 0.6 kg are connected by a light string passing over a smooth pulley. They are released from rest. After moving freely for 1.5 seconds, B picks up a stationary particle C of mass 1 kg. Calculate

(a) the speeds of A before and after this pickup,

(b) how high the combined particle B and C will rise after the pickup, assuming that they do not reach the pulley.

24 A golf ball of mass 50 g is hit from a point O on level ground with speed 20 m s^{-1} at an angle θ where $\tan \theta = \frac{3}{4}$. At the top of its flight path, it just clears the branch of a tree but hits and adheres to a particle of mass 150 g resting on the branch. Calculate

(a) the height of the branch,

(b) the speed of the combined particle after the impact,

(c) how far from O the particles hit the ground.

SUMMARY

- Impulse of a force (P) on a body is the product of the force and the time t for which it acts.
  Impulse = Pt
- Unit of impulse: N s.
- Momentum of a moving body is the product of its mass (m) and its velocity (v).
  Momentum = mv
- Unit of momentum: N s
- Impulse = change of momentum
  = final momentum – initial momentum
- *Principle of conservation of momentum:* In any collision between two bodies, the total momentum in any direction is unchanged, provided no external force acts in that direction.
- Momentum before collision = momentum after collision in the same direction.

REVISION EXERCISE 26 *(Answers on page 653.)*

1 A gun of mass 1500 kg fires a shell of mass 2 kg horizontally at 300 m s^{-1}. Find the initial speed of the gun. If the gun recoils a distance of 10 cm, find the average resistance.

2 Two balls A and B of masses 0.5 kg and 0.3 kg respectively, are moving in the same direction in the same straight line with speeds 6 m s^{-1} and 2 m s^{-1}. After they collide the difference in their speeds is 2 m s^{-1}. Find the speed of each ball.

3 Two particles, A of mass 0.4 kg and speed 4 m s^{-1} and B of mass 0.2 kg and speed 8 m s^{-1}, are moving directly towards each other. After they collide, A has reversed with speed x m s^{-1} and B has reversed with speed y m s^{-1}. Show that $y = 2x$.
Given also that the loss of kinetic energy in the collision was 7 J, find the value of x and of y.

4 Fig. 26.14 shows two spheres A and B of masses M grams and 50 grams respectively hanging at rest and in contact with each other. Sphere B is pulled aside (its string remaining taut) through a vertical distance of 10 cm and is released so that it swings down and strikes sphere A. Calculate the velocity of sphere B immediately before it strikes sphere A.
The velocity of sphere B is reversed by the impact and its magnitude is halved. Sphere A subsequently swings through a vertical distance of 2.5 cm. Calculate the value of M.
(C)

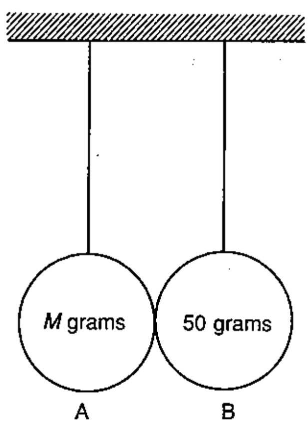


Fig. 26.14

5 A constant force of 8 N acts on a body of mass 5 kg for 10 s. If the initial speed of the body is 4 m s^{-1}, calculate its speed after 10 s. The body then carries on in a straight line at constant speed until it collides directly with a body of mass 8 kg travelling towards it with a speed of 2 m s^{-1}. Given that the direction of motion of the 8 kg mass after the collision is reversed and that its subsequent speed is 4 m s^{-1}, calculate the speed of the 5 kg mass after the collision.

6 Three particles A, B and C of masses 0.3 kg, 0.2 kg and 0.4 kg respectively lie at rest in this order inside a horizontal smooth tube. A and B are projected towards one another with speeds of v m s^{-1} and 5 m s^{-1} respectively. On impact, A is reduced to rest and B returns with speed 4 m s^{-1} to collide and coalesce with C. Calculate
(i) the value of v,
(ii) the final speed of B and C,
(iii) the loss of kinetic energy caused by the first collision. (C)

7 A particle A of mass 2 kg is dropped from rest from a height of 75 m above ground level. At the same time a particle B of mass 1 kg is projected vertically upwards from the ground so that it collides and coalesces with A after 3 seconds.
(i) Calculate the speed of projection of B.
(ii) Determine whether B is travelling upwards or downwards at the point of impact.
(iii) Find the common speed of A and B immediately after impact. (C)

8 A railway truck A of mass 4000 kg travelling at 2 m s^{-1} collides with another truck B of mass 6000 kg travelling at 1 m s^{-1} in the same direction. The speed of truck A after the collision is 1.25 m s^{-1} in the same direction. Calculate the speed of truck B after the collision.
A and B are now brought to rest by frictional forces which are in each case 50 N per 1000 kg mass. Calculate
(i) for how long A and B are each in motion after the collision,
(ii) the final distance between them. (C)

9 Fig. 26.15 shows three bodies A, B and C of masses 3 kg, 2 kg and 4 kg respectively moving in a smooth horizontal straight groove with velocities as shown. As a result of the collision between A and B, A is brought to rest and B moves towards C with velocity 7 m s^{-1}. Calculate the value of u.
When B and C collide they coalesce. Calculate
(a) the final velocity of B and C,
(b) the total loss of kinetic energy due to both collisions. (C)

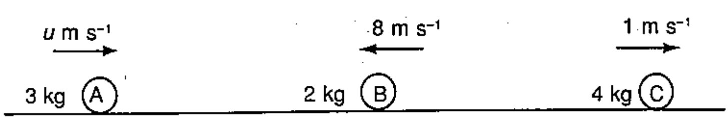


Fig. 26.15

10 Two spheres, A and B, of equal size but different masses, m_1 kg and m_2 kg respectively, travel towards each other along the line of centres with speeds of 8 m s^{-1} and 6 m s^{-1} respectively. After the collision A continues to travel in the same direction as before with speed 2 m s^{-1}, while the direction of motion of B is reversed and its speed reduced to 3 m s^{-1}. Given that the loss of energy on collision is 9.36 J, calculate the values of m_1 and m_2. (C)

11 A ball of mass 50 grams dropped from a height of 2 m on to a horizontal floor rebounds to a height of 1 m. Calculate the change of momentum due to the impact. If the impact between the ball and the floor lasts for 0.04 seconds, calculate the average force in newtons exerted by the floor on the ball. (C)

12 A particle of mass 1.5 kg is freely suspended from a fixed point by a light inextensible string of length 2 m. The particle receives a horizontal blow. When the particle comes instantaneously to rest, its horizontal displacement is 1.6 m. Calculate the corresponding vertical displacement of the particle and its initial velocity.
Given that the duration of the blow was 0.003 s, calculate the average value of the force acting on the particle during impact. (C)

13 Two spheres A and B of masses 0.5 kg and 2 kg respectively are moving in the same straight line on a smooth horizontal surface as shown in Fig. 26.16. Sphere A has a speed of 16 m s^{-1} and B has a speed of 4 m s^{-1}. On collision, A is brought to rest and B continues with speed u m s^{-1}. Calculate the value of u.
Sphere B then hits a vertical wall and rebounds with speed v m s^{-1} along the original line of motion. Given that the change in momentum of B on impact with the wall is 26 N s, and that B is in contact with the wall for 0.02 seconds, calculate
(a) the value of v,
(b) the loss in kinetic energy of B on impact with the wall,
(c) the average force exerted by the wall on B. (C)

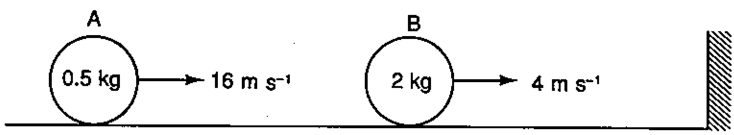


Fig.26.16

14 A ball P of mass m_1 moving with velocity u collides with another ball Q of mass m_2 which is at rest. After the collision Q moves with velocity $\frac{1}{4}u$ and P moves with velocity $\frac{1}{3}u$ in the opposite direction. Find the ratio $m_1{:}m_2$.

15 A particle of mass 100 g is projected with speed 20 m s^{-1} from a point O on level ground at an angle θ to the ground where $\tan\theta = \frac{3}{4}$. At the top of its flight it hits a vertical wall and the impulse on the particle is 2.8 N s. Calculate
(a) the height of the point of impact,
(b) the velocities of the particle before and after impact,
(c) how far from O it returns to the ground.

16 Fig.26.17 shows a smooth channel made up of a quarter circle AB, centre P and radius 0.8 m, a straight section BC and another quarter circle CD, centre Q and radius 1 m. PB and QC are vertical. A particle of mass 3 kg is released from rest at A. At C it hits and coalesces with another particle of mass 1 kg at rest. Calculate
(a) the speed of the 3 kg mass before the impact,
(b) the speed of the combined particles after the impact.
If the two come to momentary rest at E, find
(c) the height of E above the level of BC,
(d) the angle CQE.

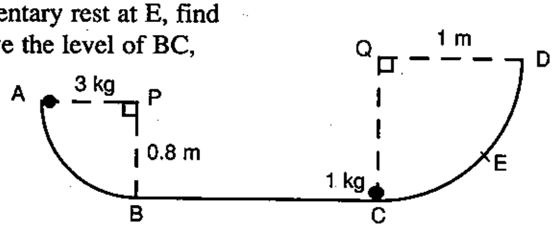


Fig.26.17

17 A light string passes over a smooth pulley. One end is attached to a particle A of mass 8 kg. The other end is attached to a particle B of mass 7 kg. A particle C, whose mass is greater than 1 kg, is attached to B by another light string. When the particles are released from rest, their acceleration is 2 m s^{-2}. Calculate

(a) the mass of C,

(b) the tension in each string.

After the particles have moved for 1 second, the 8 kg mass picks up a stationary particle D of mass 4 kg. Find

(c) the velocity of the particles immediately before this pickup,

(d) the velocity of the particles immediately after this pickup.

18 At a certain instant 3 particles A, B and C of masses 2 kg, 1 kg and 3 kg respectively are in a straight line where the distance AB = the distance BC. A is moving at u ($\neq 2$) m s^{-1} towards B,and C is moving at 2 m s^{-1} towards B. At each collision, the particles coalesce. If the final velocity of the three particles is 1 m s^{-1} in the direction of u, find the value of u.

19 Three balls A, B and C of masses 4 kg, 1 kg and 3 kg respectively, lie in that order in a straight line. C is initially at rest. A is projected towards B with speed u m s^{-1} whilst B is projected towards A with speed 4 m s^{-1}. A is reduced to rest in the first collision. If the speed of C is then $1\frac{1}{3}$ m s^{-1}, find the value of u and the loss of kinetic energy.

20 Particles A and B lie on a smooth horizontal table of a height 1.25 m above level ground. The mass of A is 2.5 kg and the mass of B is 1.5 kg. The particle A is propelled towards B at a speed of 9.6 m s^{-1}. Calculate

(a) the impulse applied to A.

On impact A and B coalesce and move towards the edge of the table. Calculate

(b) the common speed of A and B after the impact,

(c) the impulse exerted by A on B during the impact,

(d) the loss of kinetic energy at the impact.

When the combined particle reaches the edge of the table, it falls to the ground. Find

(e) the time that this fall takes.

By considering energy changes, calculate

(f) the speed of the combined particle when it strikes the ground. (C)

Revision Papers 11 – 18

PAPER 11 *(Answers on page 653.)*

1 Fig. R9 shows the v–t graph for a journey. Calculate
  (a) the rate of acceleration,
  (b) the total distance travelled.

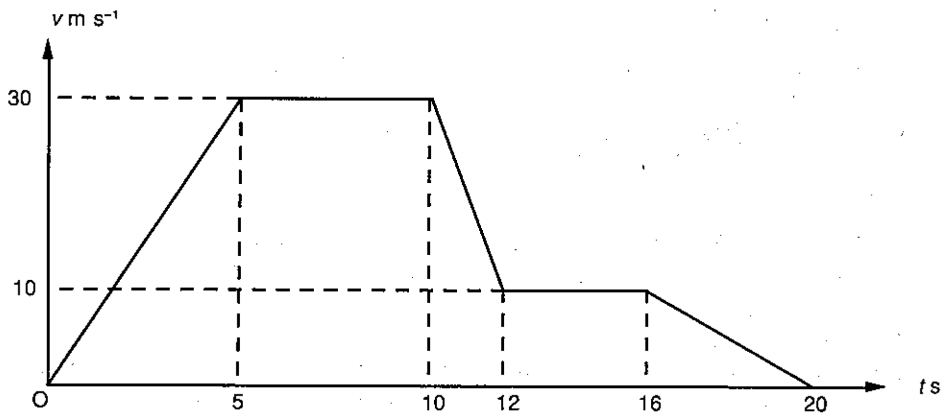


Fig.R9

2 Find the magnitude and direction of the resultant of the two forces 40 N in direction 000° and 30 N in direction 120°.

3 A particle covers a distance of 160 m accelerating uniformly from a speed of 10 m s^{-1} to a speed of 30 m s^{-1}. Find
  (a) the acceleration,
  (b) the time taken.

4 A horizontal force of 30 N is just sufficient to move a block of 8 kg on rough horizontal ground. Find the coefficient of friction.

5 A woman of mass 70 kg stands in a lift which is accelerating upwards at a m s^{-2}. If the reaction on her feet is 798 N, find the value of a.

6 A stone is thrown vertically upwards with a speed of 12 m s^{-1}. Find
  (a) the maximum height reached,
  (b) the time taken to reach this height.

7 A ball of mass 60 g travelling horizontally with speed u m s^{-1} is hit by a racket and rebounds with speed 30 m s^{-1}. If the impulse between the ball and the racket was 5 N s find the value of u.

8 An engine of mass 4 tonnes moving at 4 m s^{-1} hits a stationary wagon of mass 500 kg and is coupled to it. The two then hit a second stationary wagon of the same mass as the first and the three are coupled together. Find the final speed of all three.

9 A car of mass 750 kg is towing a trailer of mass 100 kg on level ground at a speed of 24 m s^{-1} and accelerating at 0.8 m s^{-2}. Taking the frictional resistances on each vehicle to be 2 N per kg of mass, calculate the tension in the tow-bar.

10 Particles A and B are hanging vertically from the ends of a light string which passes over a smooth pulley. When released they move with acceleration 2 m s^{-2}. If the mass of A is 5 kg, find the mass of B (two answers).

PAPER 12 *(Answers on page 653.)*

1 Ship A sailing due N at 15 km h^{-1} sees another ship, B, which is sailing at 12 km h^{-1} in the direction 300°. Calculate the velocity of B relative to A.

2 A ball is thrown with speed u m s^{-1} at an angle θ to the horizontal where $\tan\theta = \frac{3}{4}$. If the greatest height reached is 7.2 m, find the value of u.

3 A car of mass 800 kg is travelling at a steady speed of 40 m s^{-1} on a level road against a constant resistance of 500 N. What power is the engine using? Using this power, what would be the acceleration when the car is travelling at 20 m s^{-1}?

4 In Fig. R10, a body of mass 4 kg is in equilibrium on a rough horizontal plane. Forces of 20 N and F N act on the body in opposite directions. If the minimum possible value of F is 4, calculate
(a) the coefficient of friction,
(b) the maximum possible value of F.

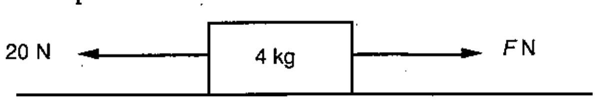


Fig.R10

5 Fig. R11 shows a sphere of mass 4 kg and diameter 12 cm resting against a smooth vertical wall and held in equilibrium by a string AB of length 4 cm. Draw a diagram showing the forces acting on the sphere. Hence, by drawing or calculation, find the tension in the string.

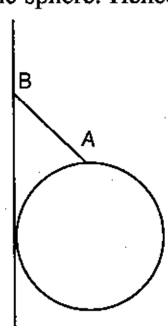


Fig.R11

6 A body of mass 5 kg is projected with speed 20 m s^{-1} directly up a rough inclined plane of length 10 m fixed on horizontal ground and making an angle θ with the horizontal where $\sin\theta = \frac{1}{5}$. The resistance to motion is 33.75 N. With what velocity does it reach the top of the plane?
It then leaves the plane and moves freely under gravity. How long does it take to reach the ground?

7 Particle A is travelling at a constant speed of 10 m s^{-1} in a straight line. Particle B, of equal mass, is 16 m distant from A on the line and is moving towards A with speed 5 m s^{-1} and uniform acceleration 2 m s^{-2}.
(a) How far has A travelled when the particles collide?
(b) Find their speeds at that point.
(c) At the collision the particles join together. Find their speed immediately after the collision.

8 ABCD is a square and E is the midpoint of DC. The forces 3 N, $\sqrt{5}$ N and 2 N act at A in the direction of the lines AB, AE and AD respectively. Find the magnitude of their resultant and the angle it makes with AB.

9 A pump raises 50 kg of water per second at rest from a well 8 m vertically down and discharges it with a speed of 4 m s^{-1}. Calculate
(a) the increase in potential energy of the water per second,
(b) the increase in kinetic energy of the water per second,
(c) the power of the pump in kW.

10 Fig. R12 shows two bodies, A of mass 5 kg and B of mass M kg, placed on the faces of a double wedge, whose faces make angles of 30° and 45° with the the horizontal. The face on which A stands is rough with coefficient of friction $\frac{1}{2}$ while the other face is smooth. Calculate the maximum value of M if A is to remain at rest.

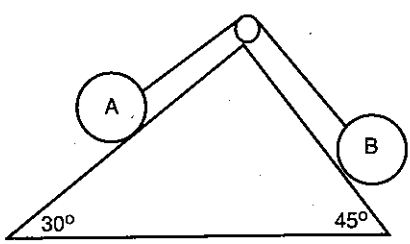


Fig.R12

PAPER 13 *(Answers on page 654.)*

1 A block of mass 3 kg rests on a rough plane inclined at an angle θ where $\tan\theta = \frac{3}{4}$. It is just prevented from slipping down by a horizontal force of 10 N. Find the coefficient of friction between the block and the plane.
What force, applied parallel to the plane, would be just sufficient to make the block move up the plane?

2 Two towns A and B are 500 km apart and the bearing of B from A is 030°. A wind is blowing from the direction 345°. A pilot wishes to fly from A to B and the airspeed of his aeroplane is 300 km h^{-1}. If he takes 2 hours over the flight, calculate
(a) the course he must take,
(b) the wind speed.

3 A particle is travelling in a straight line with constant acceleration a m s^{-2}. It passes a point A with speed u m s^{-1}. Between 3 and 4 seconds after passing A it travels 11.4 m and between 4 and 5 seconds after passing A it travels 11.8 m. Find the value of a and of u.

4 A particle is projected with speed u m s^{-1} at an angle θ to level ground where $\tan\theta = \frac{3}{4}$. If it reaches its maximum height 30 m horizontally from the point of projection, calculate
(a) the value of u,
(b) the maximum height reached.

5 (a) A mass of 10 kg is attached by two strings of lengths 15 m and 20 m to points A and B. If A and B are on the same level and AB = 25 m, find by drawing or calculation the tensions in the strings.
(b) The following coplanar forces acting at a point are in equilibrium: P N in direction 000°, $2P$ N in direction θ° ($0 < \theta < 90$), $3P$ N in direction 150° and 20 N in direction 270°. Find the value of θ and of P.

6 Fig. R13 shows two bodies A and B connected by a light string passing over a smooth pulley. A has a mass of 5 kg and lies on a rough plane inclined at angle θ to the horizontal where $\tan\theta = \frac{3}{4}$, while B hangs vertically. The coefficient of friction between A and the plane is 0.4. Calculate the mass of B if
(a) A moves up the plane with acceleration 1.4 m s^{-2},
(b) A is just about to slide down the plane.

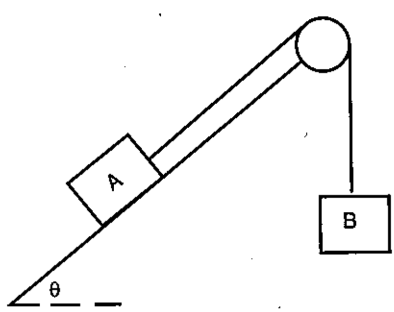


Fig.R13

7 A rocket consists of three parts, A of mass $6M$, B of mass $3M$ and C of mass M. The rocket is travelling in space with constant velocity V when an internal explosion causes part A to separate from the rocket with a backward speed of $\frac{V}{3}$. Find

(a) the new velocity of parts B and C in terms of V,

(b) the amount of KE generated by the explosion in terms of M and V.

(c) A second explosion later separates B and C giving B a backward speed of $2V$. Find the final velocity of C in terms of V.

8 In Fig. R14, the particles A and B of masses 1 kg and 2 kg respectively are connected by a light string passing over the pulley P. Because the bearing of the pulley is not smooth, the tension in string PB is $1.2 \times$ the tension in string PA. If the particles are released from rest, calculate

(a) their acceleration,

(b) the tension in PB,

(c) the force exerted on the pulley.

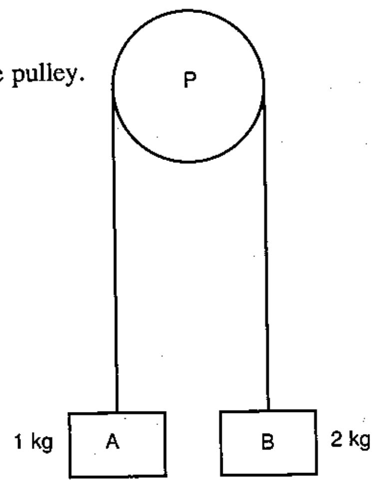


Fig.R14

9 A car of mass 750 kg has a maximum speed of 40 m s^{-1} on the level when the power output of the engine is 20 kW. The car climbs a slope at an angle θ to the horizontal where $\sin\theta = \frac{1}{15}$. If the power output and the frictional resistance are unaltered, calculate

(a) the maximum speed up the slope,

(b) the acceleration up the slope when the car has a speed of 5 m s^{-1}.

10 A straight stretch of a motorway has three parallel lanes. Car A is travelling in an outside lane at a steady speed of 15 m s^{-1}. At a certain instant, car B is travelling in the middle lane with speed 20 m s^{-1}, constant acceleration 0.4 m s^{-2} and is 30 m behind A. Car C is travelling in the third lane and at the same instant has speed 25 m s^{-1}, constant acceleration 1 m s^{-2} and is 20 m behind B. After what time will

(a) B overtake A,

(b) C overtake B?

(c) How far ahead is A when C overtakes B?

PAPER 14 *(Answers on page 654.)*

1 The three coplanar forces shown in Fig. R15 are in equilibrium. Calculate the value of θ and of P.

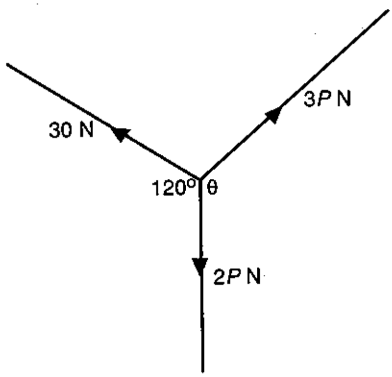


Fig.R15

2 A helicopter rises to a height of 240 m. First it accelerates uniformly to a speed of 8 m s^{-1}, maintains this speed for 6 seconds and then decelerates uniformly to rest. The magnitude of the acceleration is twice that of the deceleration. Draw a v–t graph. Hence, or otherwise, calculate

(a) the rate of acceleration,

(b) the time taken for the ascent.

3 To a motorcyclist, travelling due N at 25 km h^{-1}, the wind appears to be blowing at 15 km h^{-1} from the direction 030°. Calculate the true velocity of the wind.

4 A police patrol boat receives a report that a suspicious ship is 30 km away on a bearing of 135° and sailing at 30 km h^{-1} in the direction 045°. The patrol boat has a maximum speed of 50 km h^{-1}. Calculate

(a) the direction in which the patrol boat should be steered to intercept the ship as soon as possible, and

(b) the time it would then take to reach the ship, assuming that the ship does not alter its velocity.

5 Two particles P and Q are projected vertically upwards from ground level at the same instant. The initial speed of Q is three times that of P. When P has returned to the ground, the height of Q is 160 m. Calculate the initial speeds of P and Q.

6 Ball A of mass 0.5 kg is moving towards ball B with speed 4 m s^{-1} while B, of mass 0.2 kg, is moving towards A with speed 5 m s^{-1}. They collide and A's speed is reduced to 1 m s^{-1}. Calculate

(a) the speed with which B rebounds,

(b) the impulse between A and B.

7 Fig. R16 shows three bodies A, B and C of masses 2 kg, 1 kg and 0.5 kg respectively, moving in a smooth horizontal groove with the velocities shown. The bodies coalesce on each impact. Given that the final speed of all three is 14 m s^{-1} in the direction A → C, find the value of v.

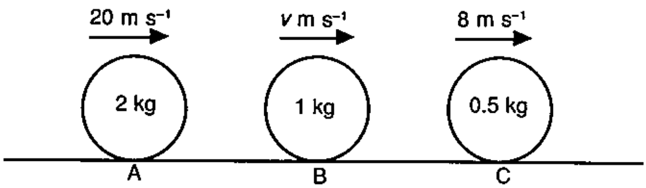


Fig.R16

8 (a) An electric motor drives a pump which raises a mass of 5 kg of water per second through a vertical height of 15 m and then sends it through a pipe with a speed of 8 m s^{-1}. If the motor is only 65% efficient, calculate the power of the motor.

(b) A hosepipe of internal diameter 2 cm is held pointing upwards at an angle of 25° to the horizontal. Water emerges with a speed of 20 m s^{-1} and hits a vertical wall at right angles and then drops vertically. Calculate the average force exerted on the wall.

9 Two particles P and Q of masses 3.4 kg and 2.6 kg respectively are connected by a light string passing over a smooth pulley. The particles are at rest hanging vertically and are released when P is 2 m above the ground. If P remains on the ground after impact, calculate

(a) the speed of Q when P reaches the ground,

(b) the total distance travelled by Q from the start until it comes to momentary rest.

10 A body P of mass $2m$ is projected from a point A on level ground with speed u at an angle θ where $\tan\theta = \frac{4}{3}$. At the same time another body Q of mass m is projected towards A from a point B with speed v at an angle ϕ where $\tan\phi = \frac{3}{4}$. Given that AB = 270 m and that P and Q collide when each is at its maximum height, find (a) the value of u and of v.

If P and Q coalesce on impact, find (b) the distance from A where they reach the ground.

PAPER 15 *(Answers on page 654.)*

1 A missile is projected vertically upwards from ground level. Between 1 and 2 seconds after leaving the ground it rises 15 m. Calculate

(a) the speed of projection,

(b) the maximum height reached,

(c) the length of time for which the missile is higher than 40 m.

2 (a) A body of mass 3 kg rests on a rough plane inclined at an angle θ to the horizontal where $\tan\theta = \frac{3}{4}$. The coefficient of friction between the body and the plane is $\frac{1}{2}$. What force, applied parallel to the plane, will cause the body to move up the plane with acceleration 2 m s^{-2}?

(b) Fig. R17 shows two bodies A and B with masses of 3 kg and 5 kg respectively placed on the rough sloping faces of a double inclined plane. The angles of the sloping sides are 30° and 60°. The coefficient of friction is $\frac{1}{3}$ between A and the left-hand face and $\frac{3}{5}$ between B and the right-hand face. Calculate the acceleration of the bodies when they are released from rest.

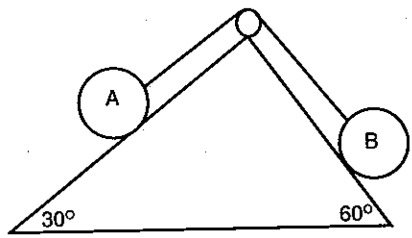


Fig.R17

3 Fig. R18 shows three bodies P, Q and R of masses m kg, 4 kg and M kg respectively in equilibrium connected by light strings passing over two smooth pulleys at A and B. A and B are on the same horizontal level with $\angle QAB = 45°$ and $\angle QBA = 30°$. Find the value of m and of M.

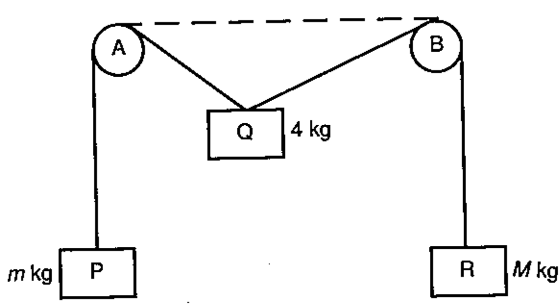


Fig.R18

4 At a certain time, two ships S and T are 20 km apart with T on a bearing of 060° from S. S is sailing due N at a speed of 15 km h^{-1} while T is sailing in the direction 330° with speed 20 km h^{-1}. Calculate (a) the velocity of T relative to S. If S maintains the same direction but wishes to intercept T, find

(a) by how much it must increase its speed,

(b) the time taken for it to intercept T.

5 Four horizontal forces of magnitude 5 N, 2 N, 10 N and P N act at a point in the direction 000°, 030°, 120° and 270°. If their resultant is in the direction θ° (000° < θ° < 090°) where tan θ = 5, find the value of P. Hence find the magnitude of the resultant of the forces.

6 Sphere A of mass 2 kg moving with speed 6 m s^{-1} collides with another sphere B of mass 4 kg which is at rest. B then goes on to hit a wall from which it rebounds with half its previous speed and strikes A again. After this collision, B is reduced to rest while A has a speed of 2 m s^{-1}. Calculate the speeds of A and B after the first collision.

7 A lorry of mass 2.5 tonnes starts to climb a straight hill inclined at angle θ to the horizontal where $\sin\theta = \frac{1}{10}$ with speed 6 m s^{-1}. The resistance due to friction amounts to 2500 N. If the power output of the engine is 45 kW, find
(a) the acceleration at the start of the climb and
(b) the maximum speed reached on the hill.

8 In Fig. R19 the two bodies A and B of masses 0.6 kg and 0.4 kg respectively are hanging vertically connected by a light string passing over a smooth pulley. The string passes through the ring R of mass 0.4 kg. A and B are held at rest with R at rest 1 m above B. When released B picks up R in the subsequent motion. Calculate
(a) the speed of B just before it picks up R,
(b) how far B will carry R upwards before they come to momentary rest.

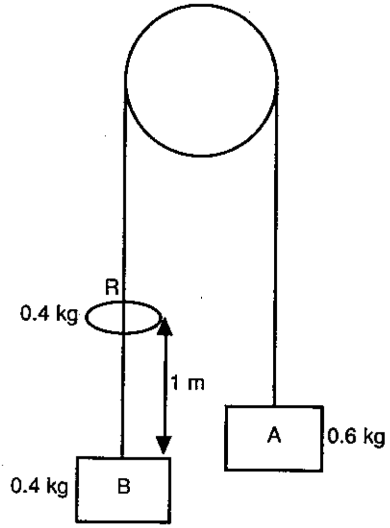


Fig.R19

9 A particle P of mass 6.5 kg is suspended by two light strings PA and PB of lengths 30 cm and 72 cm respectively attached to the points A and B on the same horizontal level. Given that AB = 78 cm, calculate the tensions in the strings.

10 Two cars A and B are moving along the two parallel lanes of a straight motorway. At a certain instant A is moving with speed 15 m s^{-1} and acceleration 0.5 m s^{-2}. At the same instant B is 39 m behind A and moving with speed 20 m s^{-1} and acceleration 1 m s^{-2}. Find
(a) the time taken for B to overtake A,
(b) the speeds of A and B at that time.
As soon as B is level with A, B stops accelerating while A increases his acceleration to 1 m s^{-2}. At what *further* distance will A overtake B?

PAPER 16 *(Answers on page 654.)*

1 A man in a boat wishes to cross a river 200 m wide, running between parallel straight banks with a speed of 3 m s^{-1}. If he sails in a direction making an angle of 60° upstream with the bank, with what speed relative to the water must he sail if he wants to land directly opposite?
If in fact he sails at 8 m s^{-1} relative to the water and at the same angle (60°), where will he land and how long will it take him to cross the river?

2 Two particles A and B of masses 0.8 kg and 0.4 kg respectively lie on a horizontal rough surface 2 m apart. A is pushed towards B with a speed of 8 m s^{-1}. In the collision A and B stick together and start to move with a speed of 4 m s^{-1}. Calculate
(a) the speed of A just before the collision,
(b) the coefficient of friction,
(c) the distance A and B will travel after the collision.

3 (a) Find the resultant, in magnitude and direction, of the forces 10 N in direction 350° and 15 N in direction 110°.
(b) The angle between two forces 10 N and P N is θ (< 90°) and their resultant is $\sqrt{175}$ N. If the P N force is reversed in direction, the new resultant is $\sqrt{75}$ N. Find the value of P and of θ.
(c) The following three coplanar forces act at a point O: 8 N in the direction 000°; Q N in the direction 150° and 2 N in the direction 240°. If the resultant of these forces acts in the direction 000°, find the value of Q and the magnitude of the resultant.

4 A car of mass 800 kg is towing a trailer of mass 150 kg on a level road. Frictional resistance to each vehicle amounts to 2 N per kg of mass. Calculate the tension in the tow-bar when the vehicles are
(a) travelling at constant speed,
(b) accelerating at 1.5 m s^{-2}.
The car and trailer now climb a straight slope of angle θ where $\sin\theta = \frac{1}{20}$.
If the frictional resistances are the same as before and the power of the engine is 50 kW, calculate
(c) the maximum speed up the slope,
(d) the acceleration when the speed is 15 m s^{-1}.

5 From a point P, 2 m above level ground, a ball is thrown with speed 15 m s^{-1} at an angle of θ to the horizontal where $\tan\theta = \frac{3}{4}$. After 1 second the ball hits a vertical wall and rebounds. Immediately after the impact the vertical component of the velocity is unchanged but the horizontal component is reduced to $\frac{3}{4}$ of its value before the impact and reversed in direction. Calculate
(a) the components of the velocity just before impact,
(b) the height of the point of impact above the level of P,
(c) how far horizontally from P the ball will strike the ground.

6 The pilot of a fighter aeroplane on patrol, flying horizontally at 400 km h^{-1} in the direction 045°, sees another aeroplane at the same height and due east of him flying horizontally with relative velocity 100 km h^{-1} due west. Calculate the true velocity of this aeroplane. The fighter pilot immediately alters course to due north with the same speed as before. Calculate the new relative velocity of the second aeroplane.

7 (a) A bullet of mass 50 g is fired horizontally from a gun of mass 15 kg, which is free to recoil. If the bullet has a speed of 300 m s^{-1} on leaving the gun, calculate the speed with which the gun recoils.
If the average force exerted by the gun on the the bullet is 3750 N, find how long the bullet was in the gun after firing.

(b) A body of mass 50 kg is moved from rest through a distance of 30 m in a straight line in 20 seconds by a constant force P N. Calculate (i) the final kinetic energy of the body, (ii) the value of P.

8 A car accelerates uniformly from rest to reach a speed of 24 m s^{-1}, continues at this speed for 20 seconds and then decelerates uniformly to stop at traffic lights. The total time from start to stop was 35 seconds. Given that the magnitude of the deceleration was twice that of the acceleration, calculate

(a) the total distance travelled,
(b) the rate of acceleration,
(c) the time taken to travel the first half of the journey.

9 (a) A tennis ball of mass 60 g moving in a straight line with speed 20 m s^{-1} is hit by a racket and returned along the same line with speed 35 m s^{-1}. Calculate the impulse on the ball.
If the ball is in contact with the racket for 0.005 seconds, with what average force does the racket hit the ball?

(b) A nail of mass 30 g is driven horizontally into a wooden fence by a hammer of mass 1 kg. Just before hitting the nail the hammer is moving with a speed of 10 m s^{-1}. If the hammer does not rebound, find the common speed of the hammer and nail after the blow. If the nail is driven 1.5 cm into the wood, calculate the average resistance of the wood.

10 Two balls A and B, of masses 1 kg and 0.5 kg respectively, lie in a straight line perpendicular to a wall with B 9 m from the wall. A is projected with speed 6 m s^{-1} towards B which is at rest. After the collision, A continues in the same direction with speed 3 m s^{-1} while B moves to hit the wall and rebound to meet A again. If the impulse of the wall on B is 3.5 N s, calculate

(a) the speed of B after the first collision,
(b) the distance of A from the wall at the moment B hits the wall,
(c) the speed of B after hitting the wall,
(d) the time between the first and second collisions.

PAPER 17 *(Answers on page 654.)*

1 Fig. R20 shows three particles A, B and C of masses 1 kg, 3 kg and 3 kg respectively. A and B are at rest on a rough horizontal surface with coefficient of friction 0.6 and are connected by a slack light string. C is held at rest and is connected to B by a light string passing over a smooth pulley. When C is released, the string AB will tighten after 0.7 seconds. Calculate

(a) the initial acceleration of B,

(b) the velocity of B just before and after 0.7 seconds,

(c) the acceleration of B and the tension in the string AB, when AB is taut.

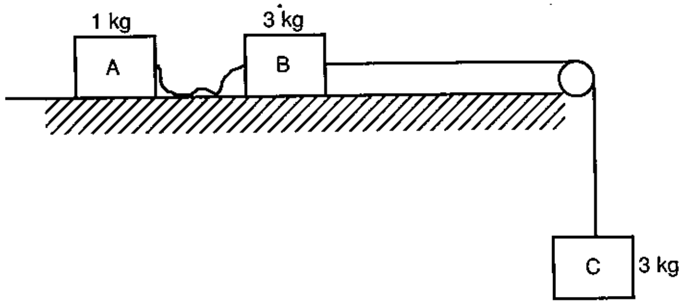


Fig.R20

2 The pilot of an aeroplane whose airspeed is 200 km h^{-1} sets a course of 020°.

(a) What is his position after a flight of 1 hour if there is a wind of 40 km h^{-1} blowing from the north?

He then decides to return to his starting point.

(b) What course should he take, the wind being unchanged in velocity?

(c) Calculate the time for the return flight.

3 Two small spheres of masses 4 kg and 2 kg are each attached by light strings of length 40 cm to the same point A and rest side by side. The 2 kg sphere is then held with its string taut and making an angle of 60° with the downward vertical through A. It is released from this position. Calculate

(a) its speed just before it hits the other sphere.

If the two spheres stick together on impact, calculate

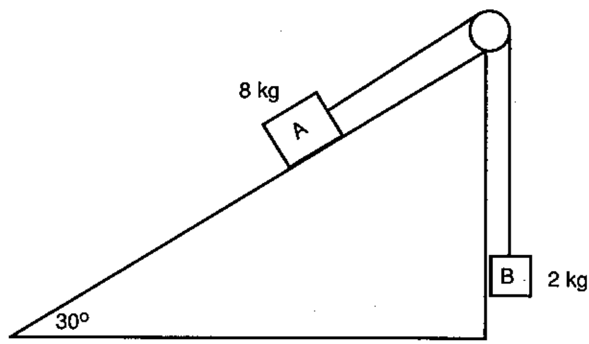
(b) their common speed and

(c) the vertical height they will reach, after the impact.

4 (a) Two bodies, A, of mass 3 kg, and B, of mass 4 kg, are moving towards each other on the same straight line with speeds 6 m s^{-1} and 2 m s^{-1} respectively. They collide and after the collision the relative velocity of B to A is 6 m s^{-1} in the direction A $\rightarrow$ B. Find their speeds after the impact and the loss of kinetic energy.

(b) Fig. R21 shows a body A of mass 8 kg on the smooth sloping face of a wedge of angle 30° connected by a light string passing over a smooth pulley to a body B of mass 2 kg hanging vertically. A is released from rest.

Fig.R21



Calculate (i) the acceleration of the bodies, (ii) the speed of B after the bodies have been travelling for 1 second.

At this instant A is stopped. Calculate (iii) how much higher B will rise, assuming that it cannot reach the pulley.

Just before the string tightens again, A is released. Calculate (iv) the common speed of the bodies just after the string tightens.

5 (a) A ball is thrown vertically upwards with speed u m s^{-1} from ground level. After 0.8 seconds it first reaches a height of 8 m. Calculate
   (i) the value of u,
   (ii) the maximum height reached,
   (iii) the time for which the ball is higher than 8 m.

(b) A particle is projected from a point O with speed V m s^{-1} at an angle θ to the horizontal. 5 seconds later it is 5 m vertically below O and 90 m horizontally from O. Calculate the value of V and $\tan\theta$.

6 (a) When the driver of a car has to stop he takes a certain time t seconds before he puts on the brakes. The car then immediately decelerates at the uniform rate of a m s^{-2}. When travelling at a uniform speed of 10 m s^{-1}, the car covers a distance of 15 m before coming to a stop and when travelling at 20 m s^{-1}, the car covers a distance of 50 m. Find the value of t and of a.

With these values, what is the shortest distance in which the driver can stop the car if it is travelling with a speed of 24 m s^{-1} and an acceleration of 2 m s^{-2}?

(b) Two bodies A and B are travelling along parallel straight lines. At a certain instant, A has a speed of 15 m s^{-1}, an acceleration of 2 m s^{-2} and is 8 m ahead of B. At the same instant, B has a speed of 20 m s^{-1} and an acceleration of 1 m s^{-2}. Find (i) the times after this instant when B overtakes A and when A overtakes B, (ii) the distance between these two events.

7 Particle A of mass 20 g is projected vertically upwards from the ground with speed 30 m s^{-1}. At the same instant, particle B of mass 5 g is let fall from rest 60 m vertically above A. On impact the particles coalesce. Find

(a) the time taken to the moment of impact,
(b) the height at which A and B meet,
(c) the speeds of A and B just before impact,
(d) the time taken for the combined particle to reach the ground.

8 A lorry of mass 1000 kg is pulling a trailer of mass 300 kg at a constant speed of 24 m s^{-1} up a straight slope making an angle θ with the horizontal where $\sin\theta = \frac{1}{50}$. The resistance (apart from gravity) to the lorry is 300 N and 120 N to the trailer. Calculate

(i) the tension in the tow-bar,
(ii) the force exerted by the engine,
(iii) the power developed by the engine in kW.

If the lorry and trailer now accelerate at a rate of 0.5 m s^{-2}, calculate

(iv) the tension in the tow-bar,
(v) the power developed by the engine in kW.

9 (a) In Fig. R22, the particle of mass 2 kg is held in equilibrium on a smooth plane of angle 30° to the horizontal by the force P N acting at an angle of 30° to the plane. Find the value of P and the normal reaction of the plane on the particle.

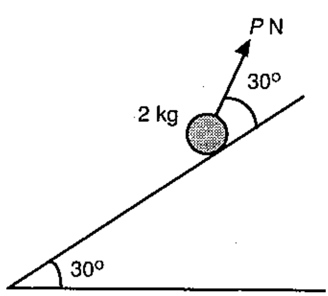


Fig.R22

(b) The following forces acting at a point are in equilibrium:
20 N in direction 000°, P N in direction 060°, $2P$ N in direction 120° and Q N in direction 270°. Calculate the value of P and of Q.

(c) The angle between two equal forces of T N acting at a point is θ where $\theta < 90°$ and their resultant is 5 N. If one of the forces is reversed in direction, the resultant becomes 3 N. By drawing or calculation, find the value of T and of θ.

10 If a man travels due north with speed 10 km h^{-1} the wind appears to blow from the east. If he travels at the same speed due south, the same wind appears to blow from the direction 150°. By drawing or calculation, find the true velocity of the wind.

PAPER 18 *(Answers on page 655.)*

1 The following coplanar forces act at a point: P N in direction 000°, P N in direction 060° and $2P$ N in direction $(180 + \theta)°$ where $\theta < 90$. If their resultant is 10 N in the direction 270°, show that $\cos\theta = \frac{3}{4}$ and find the value of P.

2 A sphere has a radius 5 cm and mass 5 kg. One end of a light string of length 8 cm is attached to a point on the sphere and the other end to a point on a smooth plane inclined at an angle of 30° to the horizontal. The sphere rests in equilibrium on the plane with the string taut. Calculate the tension in the string.

3 A wooden post of mass 4 kg is being driven vertically into the ground by a hammer of mass 16 kg. The hammer is let fall freely through a vertical distance of 1.8 m and does not rebound after striking the post. If the post is driven 15 cm into the ground by the blow, calculate
(a) the speed of the hammer just before hitting the post,
(b) the common speed after impact,
(c) the average resistance of the ground.

4 At a certain time, ship B is 20 km in a direction 135° of ship A. A is sailing at 15 km h^{-1} in the direction 045° while B is sailing at 10 km h^{-1} in the direction 330°.
(a) Calculate the velocity of B relative to A.
If B maintains the same course but wishes to intersect A, find
(b) the new speed at which it must sail,
(c) the time taken for it to intersect A.

5 Fig. R23 shows a body A of mass 2.6 kg on a rough inclined plane PQR connected by a light string passing over a smooth pulley Q to a body B of mass 5.4 kg hanging freely. PQ = 13 m, QR = 5 m, AQ = 4 m and QB =2 m. The coefficient of friction between A and the plane is $\frac{1}{2}$. When the bodies are released from rest, calculate
(a) the common acceleration.
After B has descended 2 m the string breaks. Calculate
(b) the speed of B at that instant,
(c) the time from the start for B to reach the level of PR,
(d) how much further up the plane A will move.

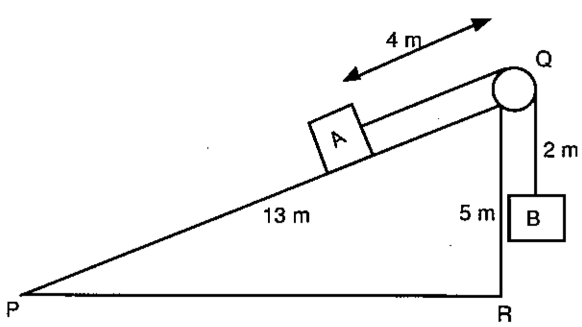


Fig.R23

6 In Fig. R24, particles A and B lie on a smooth horizontal table of height 0.8 m above the ground. The masses of A and B are 1.5 kg and 0.9 kg respectively. A is pushed towards B with speed 2 m s^{-1} and A and B stick together on impact. Find
(a) the impulse given to A at the start,
(b) the speed of the combined particle after impact,
(c) the impulse of A on B at the impact,
(d) the horizontal distance from the edge of the table of the point where the combined particle hits the ground.

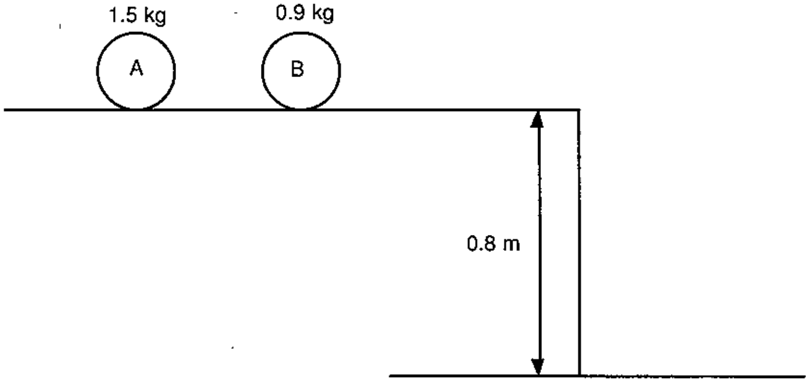


Fig.R24

7 A ferryman wishes to cross a river 300 m wide flowing at 4 m s^{-1} between parallel straight banks. He can sail his boat at 6 m s^{-1} relative to the water. At what angle should he set out so as to cross
(a) in the shortest time,
(b) by the shortest distance?
(c) Find the times taken to cross the river in each of these cases.
(d) If however he sails at an angle of 60° upstream to the bank, how long will he take to cross the river?

8 A ball is hit at 20 m s^{-1} from floor level in a sports hall whose ceiling is 8 m high.
(a) What is the maximum angle to the horizontal at which it can be hit if it is not to touch the ceiling?
(b) At what angle should it be projected with the same speed so as to reach a point on the floor 25 m distant and not to touch the ceiling?
(c) If the ball is hit at an angle $\theta°$ where $\sin \theta = 0.7$, again with the same speed, where would it hit the ceiling?

9 A train takes 5 minutes to travel from rest at one station to rest at the next station 5.25 km distant. For the first 500 m, it accelerates uniformly at the rate of a m s^{-2} to reach a speed of v m s^{-1}. It continues with this speed before decelerating at the rate of $2a$ m s^{-2}. Find the value of a and v.

10 An engine with a power of 1000 kW can pull a train of mass 300 tonnes (including the engine) up a slope making an angle of θ to the horizontal where $\sin \theta = \frac{1}{600}$ at a maximum speed of 25 m s^{-1}. Calculate the frictional resistance to motion.
What would be the acceleration up this slope if the speed of the train was 20 m s^{-1}, the power and resistance being the same?

Answers

EXERCISE 1.1

1 (a) $(1\frac{1}{2},1)$ (b) $(-3,2)$ (c) $(-1,3\frac{1}{2})$ (d) $(2,2)$ (e) $(-4\frac{1}{2},-1\frac{1}{2})$ (f) $(0,0)$ (g) $(2p,-p)$ (h) $(a,a+b)$ (i) $(a+1,a+1)$ (j) $(\frac{a}{2},\frac{b}{2})$ **2** $(2\frac{1}{2},1\frac{1}{2})$, $(4,-2)$, $(5\frac{1}{2},-5\frac{1}{2})$ **3** $(1,-2\frac{1}{2})$; $(2,-\frac{1}{2})$ **4** $(5,2)$ **5** 2, 7 **6** −1, 10 **7** $(1,1)$, $(4,7)$ **8** $(5,0)$, $(-1,-3)$ **9** $(3,5)$; $(15, 7)$ **10** $(2,1)$, $(-1,-2)$ **11** (a) $(4,5)$ (b) $(5,2)$, $(7,-4)$ **12** [Use coordinates of midpoint of diagonals] **13** No; $(4,-1)$

EXERCISE 1.2

1 (a) 5 (b) 5 (c) 13 (d) 5 (e) 2.2 (f) 6.1 (g) 4.5 (h) 5.1 (i) $\sqrt{2}a$ (j) $\sqrt{a^2+b^2}$ **2** 5 units **3** (a) Yes, at B (b) $(-1,2)$ [midpoint of AC] (c) 5 units **4** (a) $AB^2 = 25$, $BC^2 = 25$, $AC^2 = 50$ (b) Right-angled isosceles (c) 12.5 units^2 **5** (a) Isosceles [PQ = QR] (b) $(5,4)$ (c) 4 units, 8 units^2 **6** $\sqrt{41} \approx 6.4$, $\sqrt{65} \approx 8.1$, $\sqrt{122} \approx 11.0$ **7** (a) $(1,0)$ (b) 100, 68, 58, 26 **8** (a) $(3,4)$, $(6,1\frac{1}{2})$ (b) 15.25, 61 (c) $\frac{1}{2}$ **9** 5 units **10** [Fig.A1.1 circles touch internally] **12** $x^2+y^2=9$ **13** (a) Perpendicular bisector of AB (b) $[AP^2 = PB^2]$ (c) $7x+y=6$

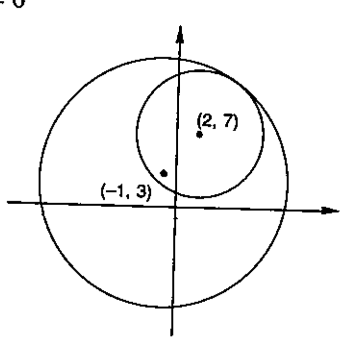


Fig. A1.1

OPTIONAL EXERCISE

(a) 16 (b) $11\frac{1}{2}$ (c) 63 (d) 33 (e) $37\frac{1}{2}$

EXERCISE 1.3

1 (a) −2 (b) −1 (c) 1 (d) 2 (e) 0 (f) undefined (g) 2 (h) $-\frac{2}{7}$ (i) 1 (j) −1 (k) $p+q$ **2** (a) 0 (b) −3 (c) $\frac{5}{4}$ (d) $-\frac{1}{2}$ **3** (a), (c), (f); (b), (e) **4** (a) 45° (b) 135° (c) 63° **5** Yes **7** $a+2b=4$ **8** 21 **10** (a) −5 (b) 7 **11** (b) 14 **12** 3 or −6

EXERCISE 1.4

1 (a), (e); (b), (d) **2** (i)(a) –2 (b) $\frac{1}{2}$ (ii)(a) 1 (b) –1 (iii)(a) $\frac{7}{5}$ (b) $-\frac{5}{7}$ (iv)(a) $-\frac{1}{7}$ (b) 7 **3** Yes **4** 4 **5** (b) $-\frac{1}{2}, 2, -3$ **6** 26.6°; 116.6° **7** $\frac{1}{4}$ **8** (a) (i) $(3\frac{1}{2},5)$ (ii) $-\frac{3}{4}$ (b) $6p + 8q = 61$ **9** (b) 4 **10** (a) (–5,0), (5,0), (0,5) **11** (a) $\frac{3}{b+1}, \frac{4}{b-6}$ (c) 3, 2

EXERCISE 1.5

1 (a) $y = x + 1$ (b) $3x - 4y = 1$ (c) $x + 5y = 16$ (d) $x + y = 1$ (e) $2x + y = 1$ (f) $y = -2$ (g) $x = -2$ (h) $y = 2x$ (i) $y = 3x + 8$ (j) $x = -1$ (k) $x = 0$ **2** (a) (–1,0), (0,1) (b) $(\frac{1}{3},0)$, $(0,-\frac{1}{4})$ (c) (16,0), $(0, \frac{16}{5})$ (d) (1,0), (0,1) (e) $(\frac{1}{2},0)$, (0,1) (f) does not cut x-axis, (0,–2) (g) (–2,0), does not cut y-axis (h) (0,0) (i) $(-\frac{8}{3},0)$, (0,8) (j) (–1,0), does not cut y-axis (k) (0,0) **3** $2x - 3y = 6$ **4** $3y - 2x = 0$ **5** (a) $3y + 2x = -14$ (b) $2y - 3x = -5$ **6** $y = 2x + 3$, $(-1\frac{1}{2},0)$ **7** $y - x = 5$, $y + 2x = 5$, $5y + 4x = 7$ **8** (a) $5x - 2y = 12$, $7y = x + 2$, $4x + 5y = 14$ (b) $4x - y = 12$ **9** $x + y = 7$ **10** (a) $(6\frac{1}{2},-1\frac{1}{2})$, (b) (7,–5) (c) $x + 9y = -7$, $7x + y = 44$ **11** (a) (1,3) (b) $\frac{1}{2}$ (c) $y + 2x = 5$

EXERCISE 1.6

1 (a) –1 (b) 1 (c) 2 (d) –2 (e) $-\frac{3}{2}$ (f) $\frac{5}{2}$ (g) 0 (h) $\frac{1}{2}$ (i) $-\frac{2}{3}$ (j) $\frac{2}{3}$ (k) $\frac{4}{3}$ (l) $\frac{5}{2}$ (m) t (n) $-\frac{1}{p}$ (o) $-\frac{a}{b}$ **2** (a) $x - y = -1$ (b) $2x + y = 1$ (c) $2y = x - 3$ (d) $x - 3y = 1$ (e) $y = 1$ (f) $3x + y = -9$ (g) $y = 3$ **3** (a) $x + y = 1$, $x - y = -3$ (b) $2x - y = -3$, $x + 2y = 6$ (c) $4x + 3y = -6$, $3x - 4y = 8$ (d) $x - 3y = 2$, $3x + y = -4$ **4** $y + 5x = -3$; $5y - x = 11$ **5** $3x + 2y = 12$

EXERCISE 1.7

1 5 units **2** $4y - x = 3$ **3** (a) $x + y = 5$, $y = 2x - 1$ (b) B(5,0), C(–1,–3) **4** $3x + y = -5$ **5** (2,0), $\sqrt{13}$ units **6** (a) $x - 2y = 1$ (b) (1,0) **7** (1,–1), (–4,–1), (–1,3) **8** (a) (6,1), (2,–1), (0,6) (b) $6x - 5y = 17$ **9** A(3,6), C(2,1) **10** (a) (1,1), (5,1), (2,4) (b) $x = 2$, $y = x$, $3y + x = 8$ (c) (2,2) **11** (a) $5y - 3x = 30$ (b) (–10,0) (c) 34 units² **12** (a) $2x - 5y = -27$, $5x + 2y = 5$ (b) (–11,1), (1,0) **13** (a) (–3,0), (2,0) (b) $3y + x = -3$, $x - 2y = 2$ (c) (0,–1) (d) $\sqrt{2} : 1$ **14** (a) $y - x = 1$, $y + x = 9$ (b) (4,5), (2,1) (c) 6 units² **15** (a) $2x + y = 15$ (b) (4,7) (c) (0,5) (d) 10 units² **16** (a) $3x + 2y = 11$ (b) $(\frac{a+5}{2}, \frac{b-2}{2})$ [Substitute in $2x - 3y = 3$] (c) 1, 4

REVISION EXERCISE 1

1 (a) $x + 2y = 4$ (b) $7x + 2y = -17$ (c) $2x + 3y = -5$ **2** (a) $3x + y = 13$ (b) $y = x + 1$; (3, 4) (c) 10 units² **3** 6 units **4** (a) $x + 2y = 11$ (b) (1,5) (c) (2,7) **5** (a) $(2,-1\frac{1}{2})$; 5 (b) $4x + 3y = 25$ **6** (a) $\frac{1}{2}$ (b) (5,7) **7** (a) $2y + x = 5$, $y = 3x - 22$ (b) (7,–1), (–6,–5) **8** (a) $x + 2y = 5$ (b) $y = 7$; (–9,7) **9** (a) $2y = x + 2$

(b) (0, 1); 20 units^2 **10** (a) (2, 4) (b) $\sqrt{8}$ units **11** (a) $3y - x = 22$ (b) (14, 12) (c) (16, 6) (d) 80 units^2 **12** $3x + 2y = 22$; (0,11), (8,–1); 6.45 units

13 (2, 3); $y + 5x = 13$; $2y = 3x - 13$ (i) (3, –2) (ii) (1, 8) (iii) 26 units^2

14 (a) (4,4), (–2,–3) (b) (0,1), (2,0) **15** (a) $y = 3x + 2$ (b) (0,2) (c) $y = 3x - 8$ (d) (2 ,–2); 40 units^2 **16** (a) $y = x + 3$; (6,9) (b) (–1,0) (c) 24 units^2 **17** (a) 4 (b) $x + 2y = 20$; (0,10) (c) [perp bisector is $y = 2x + 5$; OC is $y + 3x = 0$] (–1,3) (d) 25 units^2 **18** (a) $y = x + 1$, $2y + x = 5$, $3x - y = 1$ (b) (1,2) (c) 5 units

19 (a) $2y - x = 1$ (b) $(4, 2\frac{1}{2})$ (c) 2.5 units **20** $\frac{3}{2}$ **21** –3, –1 **22** (b) 3, $\frac{29}{5}$

23 (a) $(0, 2 - 3m)$, $(2m + 3, 0)$ (b) ±2 **24** $t + 2$; 6

25 [line is $y - 1 = m(x - 3)$ i.e. $y - mx = 1 - 3m$;

A is $(\frac{3m-1}{m}, 0)$, B is $(0, 1 - 3m)$

AC is $y = -\frac{1}{m}(x - \frac{3m-1}{m})$ so C is $(0, \frac{3m-1}{m^2})$

BD is $y - (1 - 3m) = -\frac{1}{m}(x)$ so D is $(m(1 - 3m), 0)$

Gradient of CD is $\dfrac{\frac{3m-1}{m^2}}{-m(1-3m)} = \frac{1}{m^3}$

26 (a) [consider point of intersection of diagonals] (b) [diagonals perpendicular] (c) [AB perp to Ad so $\frac{y_4 - y_1}{x_4 - x_1} \times \frac{y_2 - y_1}{x_2 - x_1} = -1$ multiply out to get $y_2y_4 + x_2x_4 = x_1(x_2 + x_4) - x_1^2 + y_1(y_2 + y_4) - y_1^2$ and substitute from (a)]

EXERCISE 2.1

1 (a) 3, 2 or 1, 4 (b) 3,1; –3, –5 (c) 3, –1 or 1, 3 (d) $-\frac{10}{3}, -\frac{8}{3}$ or 4, 1 (e) 4, –1, $-\frac{1}{2}$, 2 (f) $\frac{3}{13}, -\frac{41}{13}$ or 3, 1 (g) 5, 8 or 2, –1 (h) $-\frac{47}{13}, \frac{20}{13}$ or 4, –1 (i) 2, –1 or $\frac{14}{11}, -\frac{5}{11}$ (j) 3, 2 or $\frac{5}{7}, \frac{38}{7}$ (k) 1, $-\frac{4}{3}$ or 6, 2 (l) 8, 3 or $\frac{8}{3}$, –1 (m) 3, –1 or $-\frac{1}{3}$, –6 **2** (2,4) **3** (3,3) **4** (3,0) **5** Tangent at (2,1) **6** 7 cm, 4 cm **7** (4,9) **8** (0,–5) **9** 8, 6 or –6, –8 **10** 5, 20 or 10, 10 **11** $(3\frac{3}{4}, -3\frac{1}{4})$ **12** $(-4\frac{1}{2}, 5)$

REVISION EXERCISE 2

1 2, –1 or $-\frac{19}{4}$, –10 **2** (–5,–2) **3** $(2\frac{1}{2}, 4\frac{1}{2})$ **4** $(4, \frac{2}{3}), (\frac{1}{2}, 3)$ **5** –3, 4 or $\frac{12}{5}, \frac{2}{5}$ **6** 10 cm, 5 cm or 5 cm, 10 cm **7** (a) 4 (b) (3,1) **8** 3, 2 or –2, $4\frac{1}{2}$ **9** 5, 7 **10** $3\frac{1}{2}, \frac{1}{2}$ or $3\frac{1}{4}, \frac{5}{8}$ **11** 3, –1 or $\frac{31}{19}, -\frac{58}{19}$ **12** (b) 2, 9 or 8, 3 **13** (a) $4x$ (b) 1, 3 or $\frac{5}{6}, \frac{13}{3}$ **14** 2, 4 or 21, –15 **15** 9, 3 or –11, –7 **16** (a) [From $(x - 4)^2 + (y - 2)^2 = 5$] (b) (2,1), (6,3) (c) (5,0), (3,4)

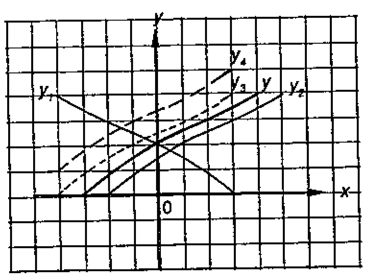
EXERCISE 3.1

1 (a) 7, –3, –5, –5, –3 –7 (b) 4, 0, 1, 4, 9, 25 (c) 2, undefined, –1, 0, $\frac{1}{3}, \frac{3}{5}$ (d) 10, 0, –2, –2, 0, 10 **2** –1 **3** (a) 2 (b) $1\frac{1}{2}$ (c) –1, 2 (d) –3 **4** (a) $f(x) = x^2 + 2$ (b) 3, 3, 2 (c) ±5 **5** (b) $F(x) = (x + 2)^2$ (c) 9, 1, 4 (d) 3, –7 (e) No **6** 2 **7** 5, 5, 5 **8** (a) 2, 4, 32 (b) 4 **9** (a) 1 (b) 3 **10** –2.19, 0.69 **11** –1, 6 **12** (a) 1, 4 (b) –1, 3 **13** –2, 3 **14** 7; 2, –3 **15** 3, –1, 4 **16** 5, –3 **17** 1, $2\frac{1}{2}$ **18** (a) $\frac{1}{3}, \frac{1}{6}$ (b) 1 (c) –1, $1\frac{1}{2}$

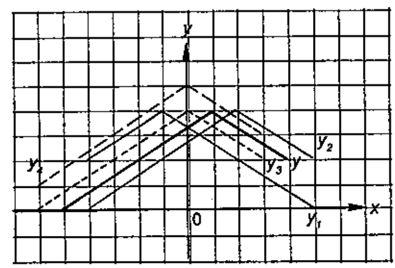
19 (a) $\{4, 5, 6, \ldots\}$ (b) 2, 5, 35 (c) 8 **20** $\{1, 3, 9\}$ **21** $\{-1, 3, 5\}$ **22** $\{-3, -1, 0, 1, 3\}$
23 $\{1, -2, -\frac{2}{3}\}$ **24** (a) 0.50, 0.77, 0.87 (b) 90 (c) $\{S(x): -1 \leq S(x) \leq 1\}$
25 (a) 0, 2, −3, −5 (b) 2, −3 (c) −1, 4 (d) 1, 2 **26** (a) −1, 6 (b) 0, 5 (c) $\frac{3}{4}$, 1
27 $\frac{1}{2}$ **28** $3a^2 - 3a$ **29** $p^2 - p + 3$; $4p^2 + 2p + 3$; $p^2 - 3p + 5$
30 $3a + 1, 3b + 1, 3a + 3b + 1$; No **31** $x^2 + 2hx + h^2 + x + h - 3$; $2x + h + 1$
32 Even (b); Odd (a), (c), (e), (f); Neither (d)

EXERCISE 3.2

1 (b), (c) if $x = 1$ is excluded **2** [Fig.A3.1]

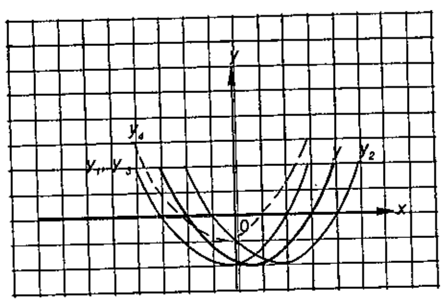


(a)



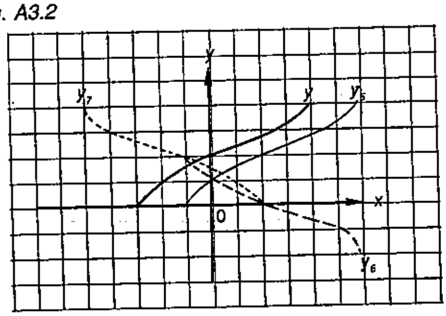
(b)

Fig. A3.1

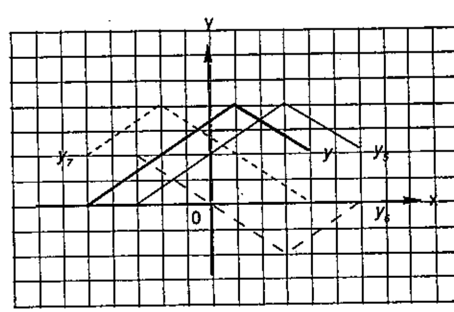


(c)

3 *Fig. A3.2*

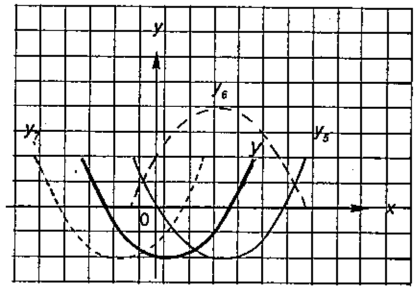


(a)



(b)

Fig. A3.2



(c)

4 $y_1 = -f(x)$, $y_2 = f(x) - 4$, $y_3 = f(x + 4)$ **5** (a) -1 to 4 (b) 1 to 6 (c) -2 to 3
6 (a) 0 to -5 (b) 1 to 6 (c) 0 to 5 (d) -3 to 2

EXERCISE 3.3

1 (a) 6 (b) $\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 27 **3** $\frac{1}{2}$ **4** $x \geq 3$ **5** (a) $x < -1$ or $x > 4$ (b) $-5 \leq x \leq 11$ (c) $x \leq -2\frac{1}{2}$ or $x \geq 3\frac{1}{2}$ (d) $-6 < x < 12$ **6** [Fig. A3.3]

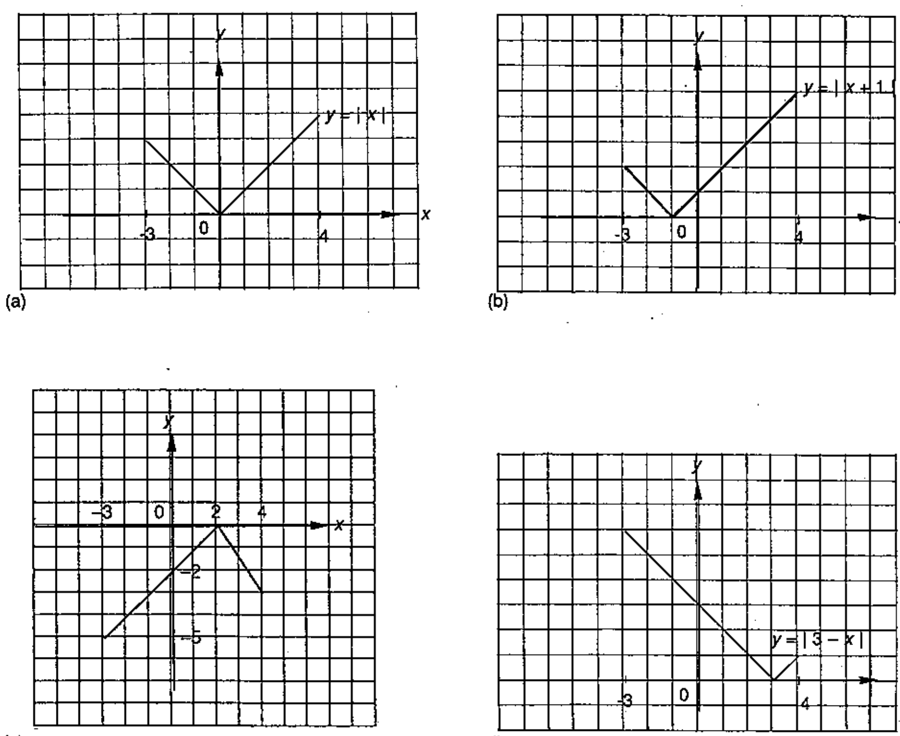


Fig. A3.3

7 (a) 0 to 4 (b) 0 to 5 (c) 0 to 7 (d) 0 to 6 **8** [Fig.A3.4] **9** [Fig.A3.5] $\frac{3}{4}$, $-1\frac{1}{2}$

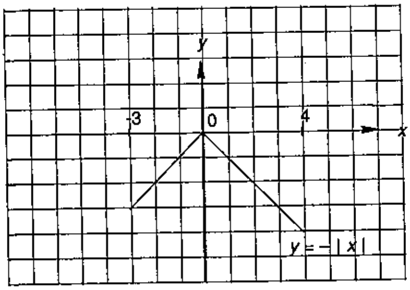


Fig. A3.4

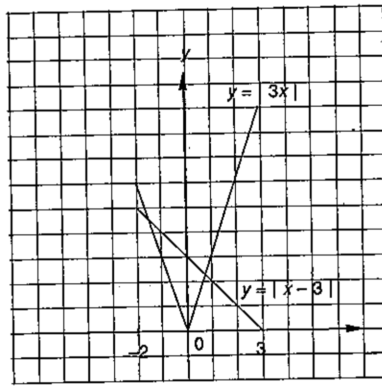


Fig. A3.5

10 [Fig.A3.6] 2, 4 **11** e.g. –1 to 4. –2 to 1 etc; –2 to 4 **12** –2 **13** [Fig.A3.7] 0 to 2

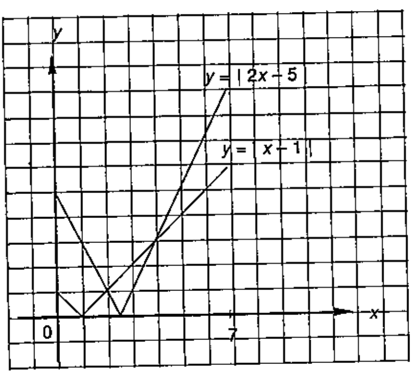


Fig. A3.6

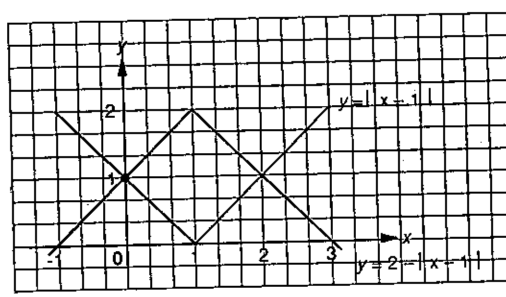


Fig. A3.7

EXERCISE 3.4

('$x \longrightarrow$' omitted)

1 (a) x (b) $x-2$ (c) $\frac{x+1}{2}$ (d) $\frac{x-4}{3}$ (e) $3x-1$ (f) $9-x$ (g) $\frac{x+5}{2}$ (h) $\frac{8-x}{2}$
(i) $3(1+x)$ (j) $\frac{8}{x}, x \neq 0$ (k) $\frac{5-x}{x}, x \neq 0$ (l) $\frac{3}{x-2}, x \neq 2$ (m) $\frac{2x+1}{x-1}, x \neq 1$
(n) $\frac{3x-1}{x-2}, x \neq 2$ **2** (a), (f), (j) **3** $\frac{x+3}{2}$ **4** $2x-3$ **5** 7 **6** $5\frac{1}{2}$ **7** $-2\frac{1}{2}$ **8** (a) 3
(b) $\frac{3+x}{x-1}, x \neq 1$ (c) self-inverse **9** (a) 2, –3 (b) 3 (c) $\frac{3x+2}{x-1}, x \neq 1$ (d) 1

10 [Fig.A3.8]

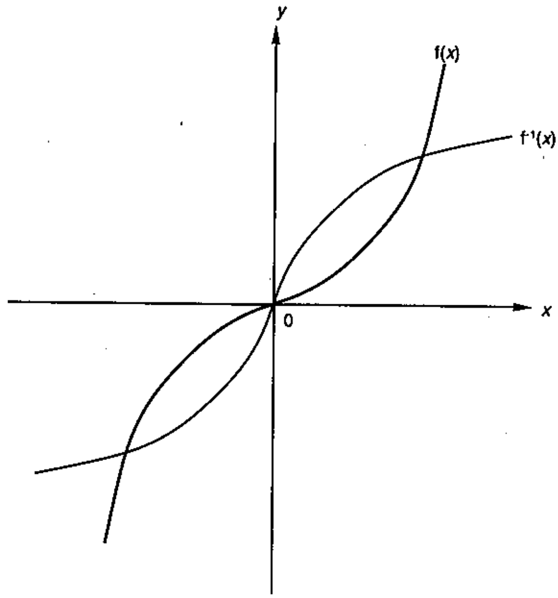


Fig. A3.8

11 [Fig.A3.9] self-inverse function
12 (a) $\frac{6-2x}{3}$ (b)(c) [Fig. A3.10]
13 [Fig.A3.11]
14 [Fig.A3.12]

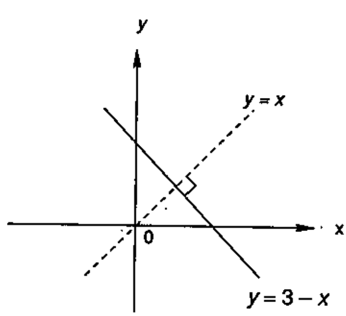


Fig. A3.9

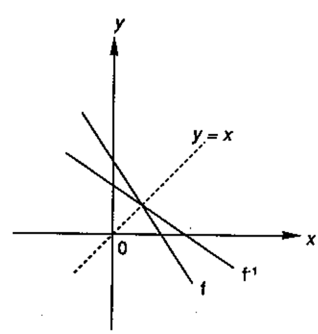


Fig. A3.10

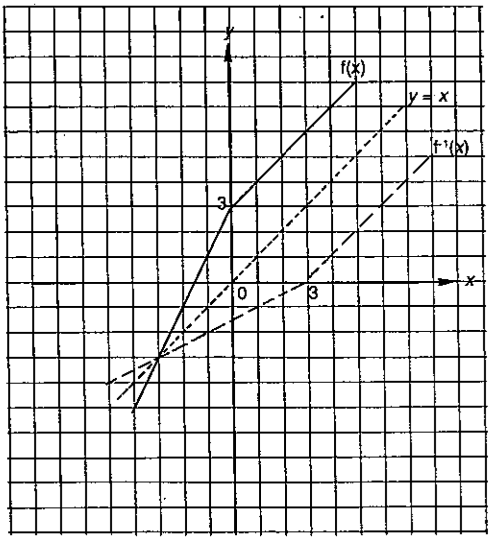


Fig. A3.11

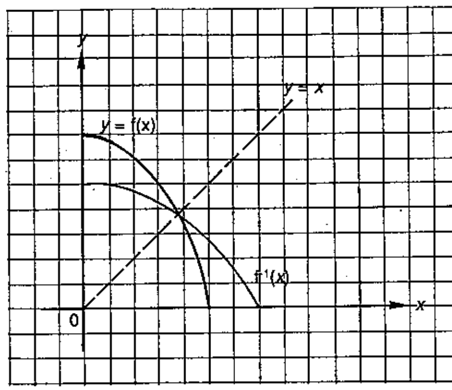


Fig. A3.12

15 (a) $x = 1$ or 2 (b)(c) [Fig.A3.13]

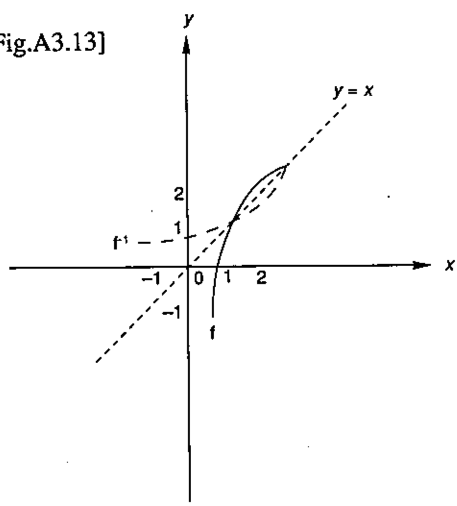


Fig. A3.13

EXERCISE 3.5

1 (a) $x - 1$ (b) $x - 1$ (c) $x + 4$ (d) $x - 6$ **2** (a) 0.64 (b) 0.68 **3** (a) $3x + 1$ (b) $3x + 5$ (c) $x - 2$ (d) $\frac{x+1}{3}$ (e) $3x - 3$ (f) $\frac{x+3}{3}$ **4** (a) $x + 2$ (b) $x + 3$ (c) $x + 4$ (d) $x + 5$ (e) $x + n$ **5** (a) $\frac{2x+3}{1-x}$ (b) $\frac{-2x-9}{3x+1}$ (c) $\frac{-x-9}{3x+2}$ **6** (a) $x^2 + 6x + 7$ (b) $x^2 + 1; -1$ **7** $-\frac{1}{2}$ **8** (a) $x + 4$ (b) $\frac{x+2}{3}$ (c) $\frac{x-10}{3}$ (d) $\frac{x+6}{3}$ **9** -2
10 (a) $\frac{1-x}{2x}, x \neq 0$ (b) $\frac{2}{x-1}, x \neq 1$ (c) $\frac{2}{x} + 1, x \neq 0$ (d) $2, -1$ **11** (a) $\frac{3}{x} + 2, x \neq 0$

(b) $\frac{1}{3x+2}, x \neq -\frac{2}{3}$ (c) $\frac{1-2x}{3x}, x \neq 0$ (d) $\frac{3}{x-2}, x \neq 2$ (e) $\frac{1}{3}, -1$ (f) -1 **12** x, x, g, f
13 (b) x **14** (a) $3a = b + 3$ (b) 5, 12 **15** [fg is $\frac{9}{2} - x$, gf is $11 - x$]
16 (a) $\frac{x+4}{1-2x}, x \neq -1, \frac{1}{2}$ (b) 5 **17** $\frac{-1}{x+1}, (x \neq -2, -1), \frac{-x-2}{2x+1}$ $(x \neq -2, -1, -\frac{1}{2})$; -1 **18** $x + 1$
19 $x - 2$ **20** (a) gh (b) hg (c) gg (d) hgg (e) hh (f) ghh (g) g^{-1} (h) hg^{-1}
21 (a) fg (b) gf (c) ff (d) gff (e) gf^{-1} (f) $f^{-1}g$ **22** (a) fg (b) gf (c) gg
(d) fgg (e) fg^{-1} (f) $f^{-1}g$ (g) $f^{-1}g^{-1}$ (h) $f^{-1}gg$ **23** $x^2 + 1$ **24** $\frac{9-3x}{x-2}$
25 (a) $\frac{x}{-2x+9}$ $(x \neq 3, \frac{9}{2})$ (b) $\frac{x}{7x-27}$ $(x \neq 3, \frac{9}{2}, \frac{27}{7})$ (c) $\frac{x}{-20x+81}$ $(x \neq 3, \frac{9}{2}, \frac{27}{7}, \frac{81}{20})$;
$\frac{x}{61x-243}$ $(x \neq 3, \frac{9}{2}, \frac{27}{7}, \frac{81}{20}, \frac{243}{61})$

REVISION EXERCISE 3

1 $-1 \leq x \leq 3$
2 (a) $\frac{1+3x}{x-2}$ (b) 3, 2
3 (a)(i) $x \leq \frac{1}{4}$ or $x \geq \frac{5}{4}$ (ii) $-4 < x < \frac{20}{3}$ (b) 2, 3
4 [Fig. A3.14]

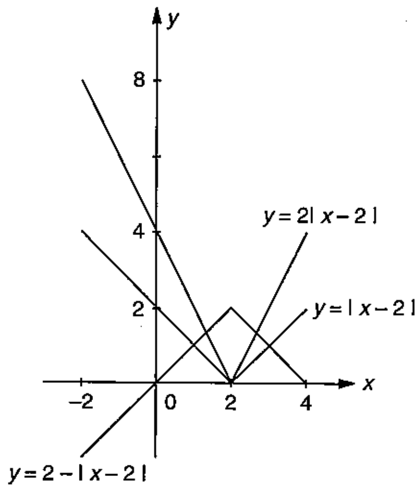


Fig. A3.14

5 [Fig.A3.15] 1
6 [Fig.A3.16]
7 1; 2, $-\frac{5}{3}$
9 (a) $4x^2 + 12x + 5$ (b) $2x^2 - 5$ (c) $-1, -5$
10 {0, 1, 2}; no
11 $\frac{x}{3x+1}$ $(x \neq -1, -\frac{1}{2}, -\frac{1}{3})$, $\frac{x}{nx+1}$ $(x \neq -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{n})$
12 $x^2 - 2$; gf^{-1}
13 (a) -11; 5 (b)(i) $\frac{8+3x}{x}$ $(x \neq 0)$, $\frac{4}{x-3}$, $\frac{25-3x}{x-3}$ $(x \neq 3)$
(ii) 7 (c) $f \geq -3$; $ff \geq -11$
14 3, -5, $2\frac{1}{2}$, $2\frac{1}{2}$ [$f(x) = x$]
15 (a)(i) $9x - 8$ (ii) $\frac{36}{x} - 14$ $(x \neq 0)$ (iii) $\frac{12}{x+4}$ $(x \neq -4)$
(b) -18, 30 **16** (a) $6 - \frac{4}{x}$ $(x \neq 0)$ (b) $\frac{1}{3-x}$ $(x \neq 3)$
(c) $\frac{6-x}{2}$ (d) $\frac{2}{x}$ $(x \neq 0)$ (e) $\frac{4}{6-x}$ $(x \neq 6)$
17 (a) fg (b) g^{-1} (c) gg (d) ggf (e) gf^{-1}
18 (a) 24, 36 (b) $\frac{4}{4-x}$ $(x \neq 4, 12)$ (c) 2

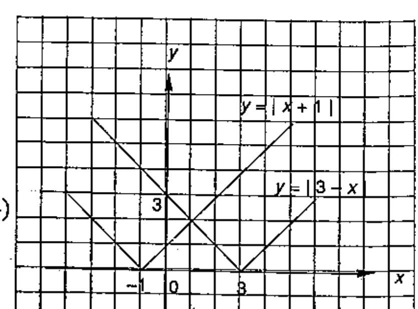


Fig. A3.15

19 (a) 5 (b) 1 (c) $\frac{3x-5}{7-x}$ $(x \neq 7)$ (d) $-1, 5$
20 (a) 2, 3 (b) 6 (c) $\frac{18-3x}{2x}$ $(x \neq 0)$
21 (a)(i) $\frac{6}{x} - 5$ $(x \neq 0)$ (ii) $\frac{3}{2x-5}$ $\left(x \neq \frac{5}{2}\right)$; $\frac{3}{5}$ (b)(i) $\frac{4-3x}{x}$ $(x \neq 0)$ (ii) $\frac{4}{x+12}$ $(x \neq -12)$
(c) -5
22 [Fig. A.3.17]
23 [Fig. A3.18]
24 (a) $\frac{1+2x}{x}$ $(x \neq 0)$; $\frac{x-2}{5-2x}$ $(x \neq 2\frac{1}{2})$; 2, $\frac{5}{3}$ (b) 3, -7
25 [Fig. A3.19] {0,1,2,3}
26 [Fig. A3.20]
27 [Fig. A3.21]

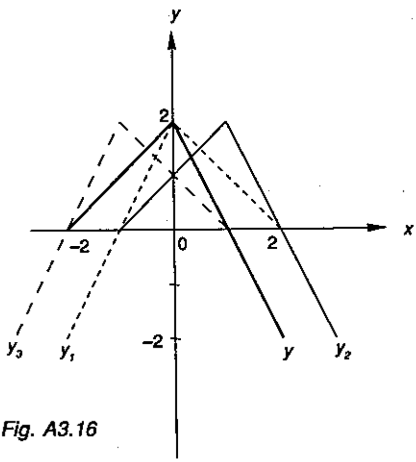


Fig. A3.16

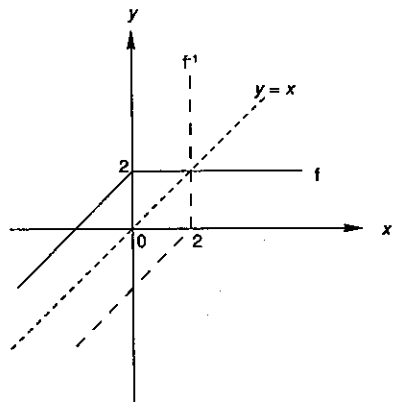


Fig. A3.17

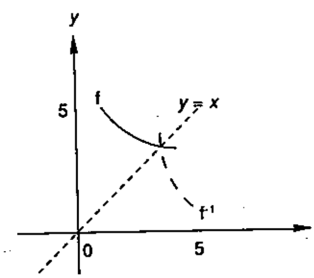
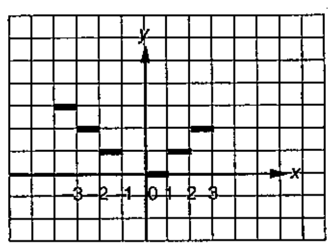


Fig. A3.18



A3.19

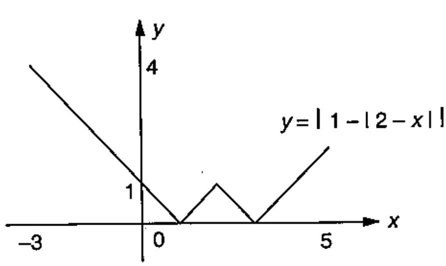


Fig. A3.20

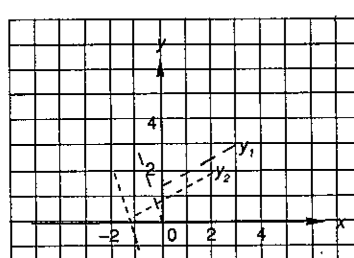


Fig. A3.21

28 (a) fgh (b) $gf^{-1}h$ (c) $g^{-1}f^{-1}h$ (d) $hg^{-1}f^{-1}$
29 (b) $[(7n + x)^2 \div 7 = 7n^2 + 2nx$, remainder $\frac{x^2}{7}]$; $g(5n + x) = g(x)$
30 (a) (3, 0), (3, 3) (b) (3, −2) (c) (1, 2) (d) $(x, y) \longrightarrow \left(\frac{2x+y}{3}, \frac{x-y}{3}\right)$
31 (a) −1 to 4 (b) 0 to 5

EXERCISE 4.1

1 (a) equal (b) not real (c) real (d) not real (e) real (f) equal (g) real (h) not real (i) real (j) not real **2** ±6 **3** $\frac{4}{3}$ **4** 2, −1 **5** 0,8 **7** ±4 **8** $k = -\frac{a}{3}$ **9** (a) $m > 4$ or $m < -4$ (b) $y = 4x + 5$, (2,13); $y = -4x + 5$, (−2,13) **10** $-1, \frac{1}{3}$ **11** ±12 **12** (a) $2mc = 1$ (b) $2y = 2x + 1$, $4y = x + 8$ **13** $-6 < c < 6$ **14** $p \le \frac{3}{2}$ **15** 0, 4 **16** 4, −1; $x = 4$, $x = -1$ **17** ±4

EXERCISE 4.2

1 (a) −10, $x = 3$ (b) −4, $x = -1$ (c) 3, $x = -1$ (d) $3\frac{1}{8}$, $x = -\frac{1}{4}$ (e) $-4\frac{1}{8}$, $x = \frac{1}{4}$ (f) 3, $x = 0$ (g) $-\frac{25}{16}$, $x = \frac{3}{8}$ (h) $5\frac{1}{4}$, $x = -\frac{1}{4}$ (i) $2\frac{1}{4}$, $x = -\frac{1}{2}$ (j) $c - b^2$, $x = -b$ **2** [Fig. A4.1]

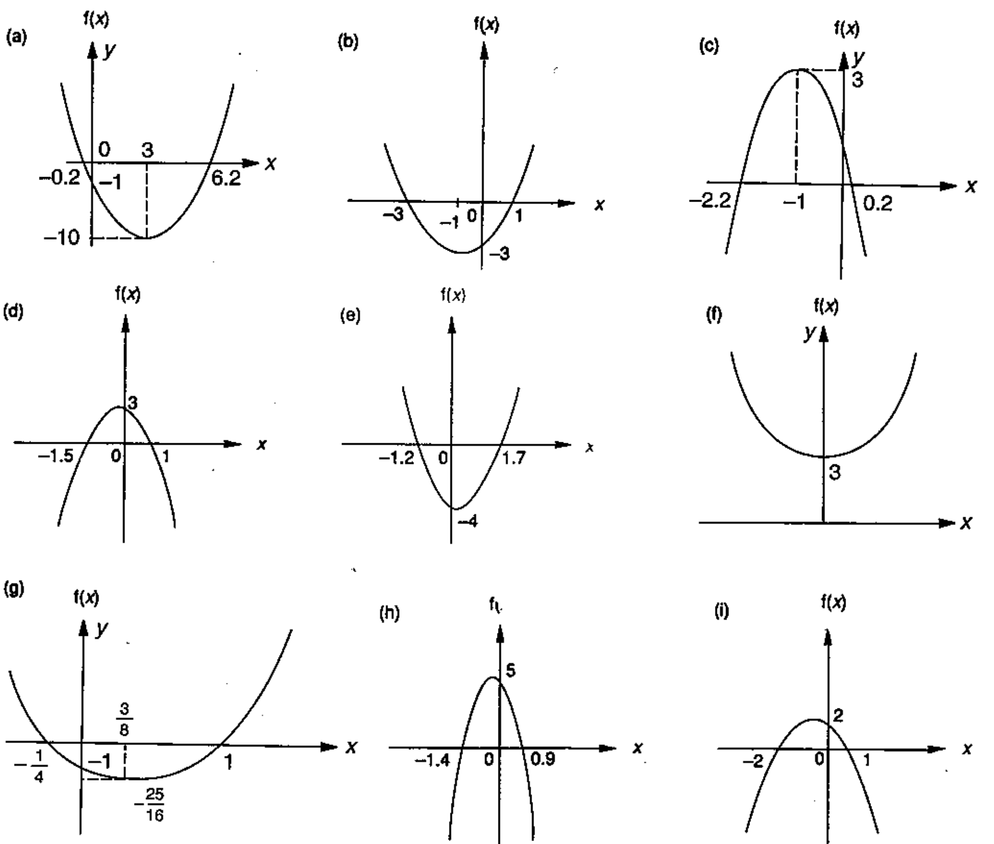


Fig. A4.1

3 -2
4 [Fig.A4.2] $0 \le f(x) \le 3$
5 [Fig. A4.3] $0 \le f(x) \le 8$
6 [Fig.A4.4] $0 \le f(x) \le 4$
7 $3 \le y \le 4$
8 (a) $22, t = 2$ (b) $22 \le V(t) \le 30$
9 $f(x) \ge -5\frac{1}{2}$
10 (a) -9 to 3 (b) $-3\frac{1}{8}$ to 3
11 (a) $-3\frac{1}{8}$ to 7 (b) -7 to 2
12 $(x+1)^2 + 9$; $9, -1$
13 (a) $7\frac{1}{12} - 3(x + \frac{1}{6})^2$, $a = 7\frac{1}{12}$, $b = 3$, $c = \frac{1}{6}$; $f(x) \le 7\frac{1}{12}$ (b) -3
14 0 to $11\frac{1}{4}$
15 14.25 cm, 3.5 sec
16 $-3, -6$
17 $4, -2, -5$
18 (b) 312.5 m^2, $x = 12.5$ m

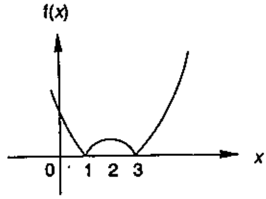


Fig. A4.2

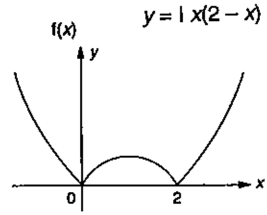


Fig. A4.3

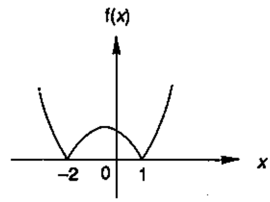


Fig. A4.4

EXERCISE 4.3

1 (a) $x \le -1$ or $x \ge 2$ (b) $-3 \le x \le 2$ (c) $x < -6$ or $x > 1$ (d) $x \le -\frac{1}{2}$ or $x \ge 1$ (e) $\frac{1}{3} \le x \le \frac{1}{2}$ (f) $x \le 0$ or $x \ge 4$ (g) $-\frac{2}{3} \le x \le 1$ (h) $x < -7$ or $x > 3$ (i) $-4 < x < \frac{1}{3}$ (j) $x \le -4$ or $x \ge \frac{1}{2}$

2 $\frac{1}{2}$ **3** $-2 \le t \le 6$ **4** $-\frac{1}{2} \le p \le 1$ **5** $-4 \le p \le 0$ **6** (a) $-4 < x < -2$, $1 < x < 3$ (b) $x \le -2$, $-1 \le x \le 6$, $x \ge 7$ (c) $-2 \le x \le -1$, $2 \le x \le 3$ **7** $-\frac{2}{3} \le p \le 2$
8 $[(2t+5)^2 - 4(t+3)(t+2) = 1 > 0]$ **10** $1 \le x \le 3$ **11** $-6 < t < 2$
12 [Fig.A4.5] $-1\frac{1}{2} \le x \le 0, \frac{1}{2} \le x \le 2$ **13** [Fig.A4.6] $-4 < x < -1, 0 < x < 3$ **14** (a) $x > \frac{2}{3}$ (b) $x < -3; -3 < x < \frac{2}{3}$ **15** $0 \le p \le \frac{12}{7}$ **16** $2; 1\frac{3}{4}$

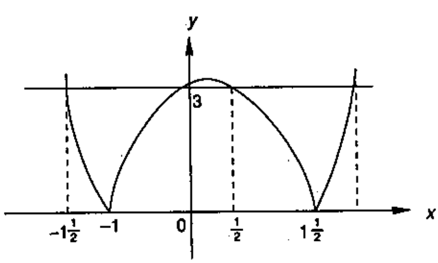


Fig. A4.5

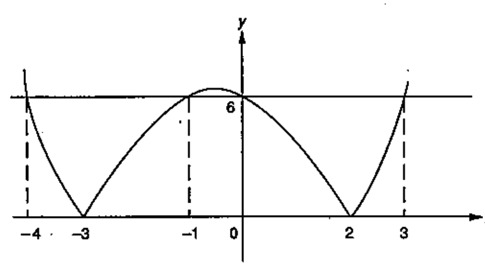


Fig. A4.6

REVISION EXERCISE 4

1 (a) real (b) real (c) not real (d) real (e) equal
2 $\frac{1}{3} < x < 3$
3 $[D = (2t-1)^2 - 4(t-3)(t+2) = 25$, perfect square$]$

5 (i) $-4, -12$ (ii) $r < -16$
6 $9 - (x-1)^2$; 0 to 9
7 (a) $-\frac{1}{2} < x < 2\frac{1}{3}$ (b) $(1\frac{1}{2}, 6)$ [Fig. A4.7]
8 $-3\frac{1}{8}$ to 7
9 (i) $k > \frac{1}{2}$ (ii) $k < -2$; $-2 < k < \frac{1}{2}$
10 (a) 5 (b) $q \geq 8$ or $q \leq -8$
11 $x < -1$, $0 < x < 2$, $x > 3$
12 (a) $k = \frac{a^2}{b - 2a}$ (b) $p \leq -\frac{1}{2}$
13 -1 or -2
14 [Fig. A4.8] 0 to $6\frac{1}{4}$
15 $-2, 8, 10$
16 $4 \leq x \leq 6$
17 (i) $x \leq -\frac{1}{2}$ or $x \geq 2\frac{1}{2}$ (ii) $(1,-9)$; $(1,9)$ [Fig. A4.9]
18 $\frac{1}{3} \leq x \leq 1$
19 (a) $-\frac{1}{2}$
20 (a)(i) $-7, 10$ (ii) $3, 4$ (b) $-2, 1, 6$
21 $\frac{1}{2} \leq x \leq 2$
22 $-1 \leq x \leq -\frac{1}{3}$, $1 \leq x \leq 1\frac{2}{3}$
23 $-2 < x < -1$, $\frac{1}{2} < x < 1\frac{1}{2}$
24 $k \leq -6$ or $k \geq 2$ (i) 1 (ii) -3
25 $2, -4, 7$
27 (a) $-6, 8$ (b) $-4 \leq x \leq -1$, $7 \leq x \leq 10$
28 (a) $\frac{-a \pm \sqrt{3a^2 + 5a - 2}}{a}$ (b) $a \leq -2$ or $a \geq \frac{1}{3}$
29 [$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta = 0$ is identical to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$: compare coefficients. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2a\beta$; $(a-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$]

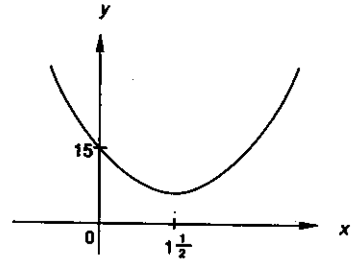


Fig. A4.7

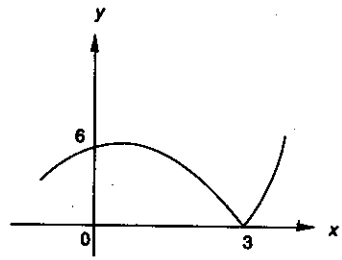


Fig. A4.8

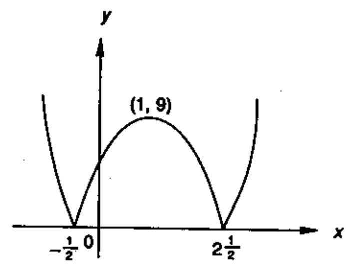


Fig. A4.9

EXERCISE 5.1

1 (a) $x^4 - 8x^3 + 24x^2 - 32x + 16$ (b) $8x^3 - 36x^2 + 54x - 27$
(c) $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$ (d) $x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{5}{4}x^2 + \frac{5}{16}x - \frac{1}{32}$
(e) $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$ (f) $\frac{x^4}{256} - \frac{x^3}{8} + \frac{3x^2}{2} - 8x + 16$
2 (a) $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$ (b) $16 - 96x + 216x^2 - 216x^3 + 81x^4$
(c) $64 - 96x + 60x^2 - 20x^3 + \frac{15x^4}{4} - \frac{3x^5}{8} + \frac{x^6}{64}$ (d) $1 - 3x^2 + 3x^4 - x^6$ **3** (a) $32 - 80x + 80x^2 - 40x^3$
(b) $1 - 14x + 84x^2 - 280x^3$ (c) $1 - 4x + 7x^2 - 7x^3$ (d) $1024 - 640x + 160x^2 - 20x^3$
4 (a) $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$ (b) $x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$
5 $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$; $\frac{45}{512}$ **6** 0.941 480
7 (a) $1 + 3x + 3x^2 + x^3$, $1 - 3x + 3x^2 - x^3$ (b) $2 + 6x^2$; 14 **8** 194
9 (a) $1 + 4x + 6x^2 + 4x^3 + x^4$, $1 - 4x + 6x^2 - 4x^3 + x^4$ (b) $8x + 8x^3$; 0.080 008
10 (a) $x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$, $x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$
(b) $10x^3 + \frac{20}{x} + \frac{2}{x^5}$ (c) $90\frac{1}{16}$

EXERCISE 5.2

1 $1 - 4x + 6x^2 - 4x^3 + x^4$; $1 - 4x + 8x^2$ **2** $1 + \frac{3x}{2} - \frac{9x^2}{4}$ **3** (a) $16 - 32x + 24x^2$
(b) $81 - 54x + \frac{27}{2}x^2$; -3456; 3888 (c) $8 - 4x - 6x^2$ (d) 3 **4** (a) $1 + 5x + 10x^2 + 10x^3$
(b) $1 + 5x + 5x^2 - 10x^3$ **5** (a) $1 + 8x + 24x^2 + 32x^3 + 16x^4$ (b) $1 - 3x + 3x^2 - x^3$; $1 + 5x + 3x^2$
6 $1 + 6x + 12x^2 + 8x^3$; $16 - 16x + 6x^2 - x^3 + \frac{x^4}{16}$; 102 **7** $-\frac{7}{2}$
8 $1 + 5x + 10x^2 + 10x^3$; $32 - 80x + 80x^2 - 40x^3$; -120 **9** $2, -3$ **10** 2
11 (a) $1 + 3ax + 3a^2x^2 + a^3x^3$; $b^4 + 4b^3x + 6b^2x^2 + 4bx^3 + x^4$ (b) $2, -2$ **12** $2, -\frac{1}{12}$; $\frac{x^2}{6}$
13 (a) $1 + 4px + 6p^2x^2$; $1 + 3qx + 3q^2x^2$ (b) $3, -2$ or $-1, 2$ **14** (a)(i) $1 + 3ax + 3a^2x^2 + a^3x^3$
(ii) $1 + 4bx + 6b^2x^2 + 4b^3x^3 + b^4x^4$ (b) $\frac{17}{7}, -\frac{4}{7}$ or $-1, 2$ **15** (a) $-70, 168$ (b) 350
16 $-\frac{160}{27}, \frac{20}{3}$; $-\frac{100}{9}$ **17** (a) $1 - 8x + 24x^2 - 32x^3$ (b) $1 - 8x + 28x^2$; $-\frac{1}{8}$
18 $1 + 4x + 6x^2$; $1 + 4ax + (4b + 6a^2)x^2$; $2, -3$ **19** 0.1 **20** $1 - 8x + 28x^2$; 0.99203

EXERCISE 5.3

1 (a) 6 (b) 12 (c) 504 (d) 495 **2** (a) 15 (b) 36 (c) 495 (d) 455 **3** 3
4 $1 + 10x + 45x^2$ (b) $x^{12} - 6x^{11} + \frac{33x^{10}}{2}$ (c) $x^9 - 9x^7 + 36x^5$ **5** (a) 7920 (b) -3240 (c) 126
6 20 **7** $9, \frac{2}{3}$ **8** 6, 3 **9** fourth, $-\frac{21}{2}$ **10** 7

REVISION EXERCISE 5

1 (i) $1 - 15x + 90x^2 - 270x^3$ (ii) $1 + 35x + 525x^2 + 4375x^3$; 90
2 $a^6 + 6a^5\frac{x}{b} + 15a^4\frac{x^2}{b^2}$; $2, \pm 3, \pm 64x$ **3** $1 - 10x + 40x^2$; $2, 7, -13$
4 (a) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$, $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
(b) $20x + 160x^3 + 64x^5$ (c) 0.020 000 160 000 064 **5** (i) $1 + 5\alpha t + 10\alpha^2t^2$
(ii) $1 - 8\beta t + 28\beta^2t^2$; $10\alpha^2 - 40\alpha\beta + 28\beta^2$ **6** 10, 10
7 $1 - 5p + 10p^2 - 10p^3 + 5p^4 - p^5$; $1 - 5x + 5x^2 + 10x^3$; 0.01 **8** sixth
9 (i) $64 + 192x^2 + 240x^4 + 160x^6$ (ii) 48 **10** $\frac{1}{5}$ or $\frac{1}{45}$ **11** $\frac{7}{9}, -\frac{35}{3}$; -7 **12** $-\frac{1}{2}$, 12
13 $1 + 6ax + 15a^2x^2$; $\frac{1}{2}, -3$ or $-\frac{1}{2}, 3$ **14** 28:5 **15** $3, -2$ **16** 8 **17** fourth; 280 **18** $7, -1$
19 (a) $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, $1 + 5x^2 + 10x^4 + 10x^6 + 5x^8 + x^{10}$ (c) $1 + 5x + 15x^2 + 35x^3$
20 $7\frac{1}{2}$

EXERCISE 6.1

1 (a) 60° (b) 18° (c) 120° (d) 720° (e) 30° (f) 20° (g) 135° (h) 330° (i) 225°
(j) $22\frac{1}{2}$° (k) 114.6° (l) 85.9° **2** (a) $\frac{\pi}{6}$ (b) $\frac{3\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) 3π (e) $\frac{7\pi}{12}$ (f) $\frac{2\pi}{9}$ (g) $\frac{10\pi}{9}$
(h) $\frac{\pi}{8}$ (i) $\frac{20\pi}{9}$ (j) $\frac{5\pi}{12}$ **3** (a) 0.5 (b) 0.5 (c) 1 (d) -0.7071 (e) 1 (f) 0.9093 (g) 0.8776
4 (a) $\frac{\pi}{6}$ (b) 0.93 (c) 0.98 (d) 1.32 **5** 0.068 **6** Values tend to 1

EXERCISE 6.2

1 1.2 rad **2** 1.67 rad **3** 0.5 rad, 3 cm **4** (a) 10 cm (b) 0.3 rad **5** (a) 5.2, 10.4
(b) 10, 40 (c) 0.5, 4 (d) 12, 0.25 (e) 6, 2.4 **6** (a) 9.6 cm² (b) 12.8 cm **7** (a) 4.47

(b) 0.841 (c) 11.5 (d) 11.0% **8** 4 cm **9** (a) 1.85 rad (b) 9.27 cm **10** 40 cm, 1.2 rad; 24 cm, $3\frac{1}{3}$ rad **11** 2.7 m s^{-1} **12** (a) $\frac{10\pi}{9}$ (b) 0.52 m s^{-1} **13** (a) 1.85 rad (b) 22.1 m **14** (a) 3730 cm^2 (b) 256 cm **15** (a) 2.46 rad (b) 0.292 **16** (a) 5 cm (b) 0.4 rad (c) 4.8 cm^2 **17** (a) 1.85 (b) 11.2 cm^2 (c) 0.28 **18** 2.5 **19** (c) $3 \le r \le 4$ **20** (c) 3 cm (d) 2 rad

REVISION EXERCISE 6

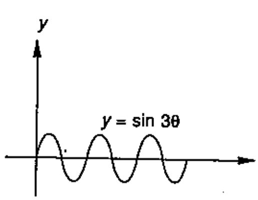
1 (a) $\frac{\pi}{3}$ (1.05) rad (b) 209 cm^2 **2** 44 cm^2 **3** 12.93 cm **4** (i) $r^2\theta$ (ii) $r^2(\theta - \frac{1}{2}\sin 2\theta)$ (iii) $\frac{r^2}{2}(\pi \sin^2\theta - 2\theta + \sin 2\theta)$ **5** 18.82 cm^2 **6** (i) 2.02 rad (ii) 10.1 cm (iii) 14.8 cm^2 **7** (i) 42.15 cm^2 (ii) 0.57 **8** 48.33 cm^2 **9** (i) 51.6 cm (ii) 4763 cm^2 (iii) 94.7% **10** (i) $\frac{121\theta}{2}$ cm^2 (ii) $\frac{75\theta}{2}$ cm^2 (iii) 0.8 rad (iv) 26 cm **11** (a) 4.44 m ($\sqrt{2}\,\pi$) (b) 4.57 cm^2 ($4\pi - 8$) **12** 6.17 cm^2 **13** 43.2 cm^2 **14** 3; 2 rad **15** (a) $\frac{2\pi}{3}$ rad (b) 1.63 cm^2 **16** [total area of sectors = $2 \times \frac{1}{2} \times \frac{x^2}{9}\theta + 2 \times \frac{1}{2} \times \frac{x^2}{9}(\pi - \theta)$; area of rhombus = $x^2 \sin\theta$] 0.77, 2.37 **17** 10:1

EXERCISE 7.1

1 (a) 19.5°, 160.5° (b) 40.4°, 319.6° (c) 49.0°, 229.0° (d) 110.5°, 249.5° (e) 194.5°, 345.5° (f) 141.0°, 321.0° (g) 186.8°, 353.2° (h) 220.5°, 319.5° (i) 76.7°, 283.3° (j) 123.7°, 303.7° (k) 34.9°, 145.2° (l) 98.0°, 262.0° (m) 150.4°, 209.6° (n) 61.6°, 241.6° (o) 61.1°, 298.9°
2 (a) 0°, 23.6°, 156.4°, 180°, 360° (b) 18.4°, 198.4°, 161.6°, 341.6° (c) 18.3°, 116.6°, 198.3°, 296.6° (d) 90°, 138.6°, 221.4°, 270° (e) 39.2°, 140.8°, 219.2°, 320.8° (f) 210°, 221.8°, 318.2°, 330° (g) 117.7°, 202.3° (h) 170.5°, 350.5° (i) 0°, 19.5°, 160.5°, 180°, 360° (j) 36.9°, 143.1°, 216.9°, 323.1° (k) 39.2°, 140.8°, 219.2°, 320.8° (l) 210°, 41.8°, 138.2°, 330° (m) 109.5°, 250.5° (n) 112.3°, 347.7° (o) 168.6°, 251.4° **3** 35.3°, 144.7°, 215.3°, 324.7° **4** 158.2°, 338.2°
5 39.2°, 140.8°, 219.2°, 320.8° **6** (a) 26.7°, 153.3° (b) 129.1° (c) 65.1° **7** (a) –0.5 (b) $\frac{1}{\sqrt{2}}$ (c) –0.36 (d) $\frac{2}{\sqrt{3}}$ (e) 0.58 (f) $\frac{\sqrt{3}}{2}$ (g) –1.05 **9** (a) 200.5°, 339.5° (b) 15.7°, 164.3° (c) 129.8°, 230.2° (d) 126.7°, 306.7°

EXERCISE 7.2

1 120° [Fig.A7.1] **2** [Fig.A7.2] **3** 4 [Fig.A7.3]



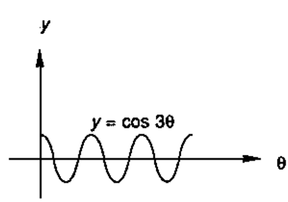


Fig. A7.1

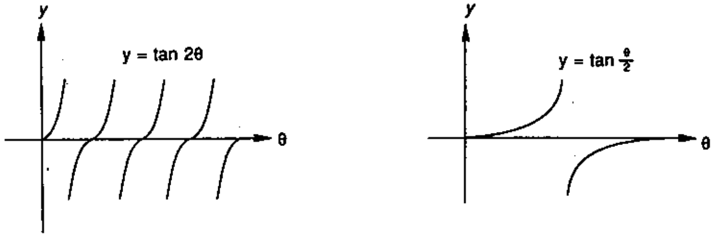


Fig. A7.2

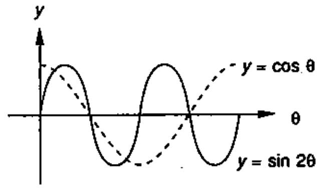


Fig. A7.3

4 2 [Fig.A7.4] **5** 6 [Fig.A7.5]

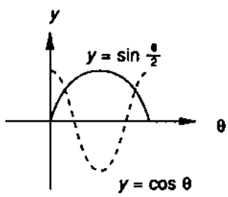


Fig. A7.4

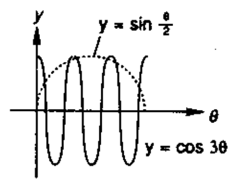


Fig. A7.5

6 (a) 21.0°, 69.0°, 201.0°, 249.0° (b) 18.2°, 101.8°, 138.2°, 221.8°, 258.2°, 341.8° (c) 112.6° (d) 29.4° (e) 24.1°, 155.9°, 204.1°, 335.9° (f) 252.1° (g) none (h) 80.8°, 170.8°, 260.8°, 350.8° (i) 62.7°, 207.3° (j) 92.4°, 147.6°, 332.4° (k) 110.9°, 159.1°, 290.9°, 339.1° (l) 73.1° (m) 55.2° (n) 67.5°, 157.5°, 247.5°, 337.5° (o) 9.7°, 80.3°, 135°, 189.7°, 260.3°, 315° (p) 120°, 180° (q) 114.7°, 155.3°, 294.7°, 335.3° (r) 115.7° (s) 65.3°, 114.7°, 245.3° 294.7° (t) 174.8° (u) 271.1° (v) 83.6°, 276.4°
7 (a) 24.1° (b) 58.6°, 148.6°, 238.6°, 328.6° (c) 82.8° (d) 66.3°, 123.7°, 246.3, 303.7° (e) 222.3°, 77.7° **8** (a) 0.5 (b) 0.64 (c) 1 **9** (a) 0 (b) 0.5 (c) 1.73
10 90°, 180°, 360°

EXERCISE 7.3

1 (a) 0.34 (b) 0.5 (c) −0.34 (d) 0.58 **2** 4; 8 **3** [Fig.A7.6] **4** [Fig.A7.7] 4
5 [Fig.A7.8]

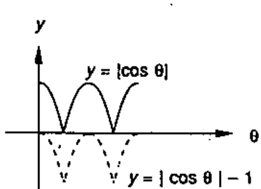


Fig. A7.6

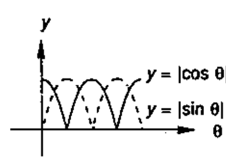


Fig. A7.7

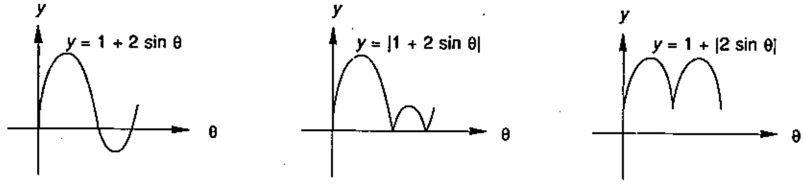


Fig. A7.8

6 [Fig. A7.9] **7** [Fig. A7.10] **8** [Fig. A7.11] **9** [Fig. A7.12] 3 **10** [Fig. A7.13] 4
11 [Fig. A7.14] 6; 12 **12** [Fig. A7.15] 2

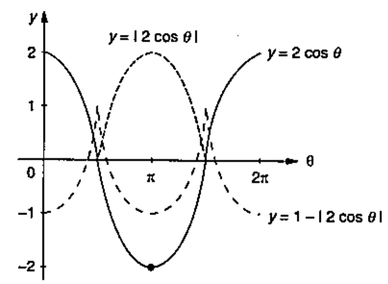


Fig. A7.9

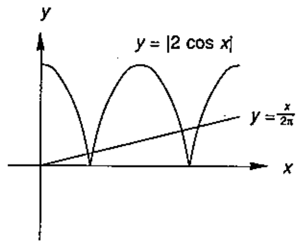


Fig. A7.10

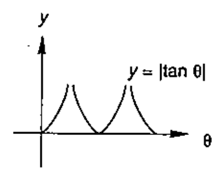
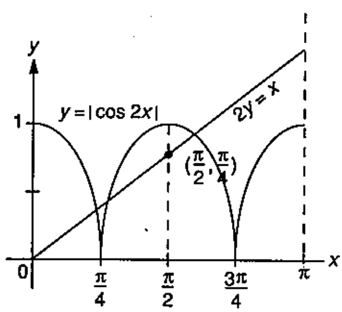


Fig. A7.11

Fig. A7.12



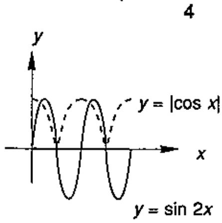


Fig. A7.13

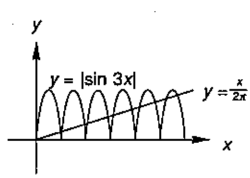


Fig. A7.14

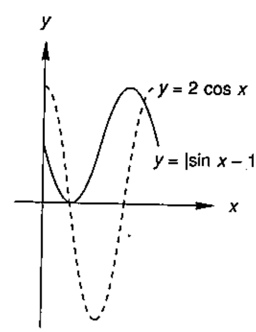


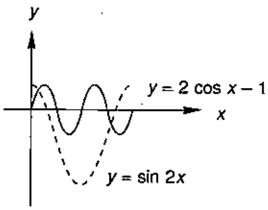
Fig. A7.15

EXERCISE 7.5

1 70.5°, 289.5° **2** 90°, 228.6°, 311.4° **3** 19.5°, 160.5° **4** 30°, 150°, 210°, 330°
5 30°, 150° **6** 45°, 71.6°, 225°, 251.6° **7** 54.7°, 125.3°, 234.7°, 305.3° **8** 30°, 90°, 150°
9 19.5°, 41.8°, 138.2°, 160.5° **10** 48.2°, 270°, 311.8° **11** 0°, 101.5°, 258.5°
12 60°, 120°, 240°, 300°

REVISION EXERCISE 7

1 (a) 58.3°, 148.3°, 238.3°, 328.3° (b) 56.3°, 236.3° **2** [Fig. A7.16] 2 **3** [Fig. A7.17]



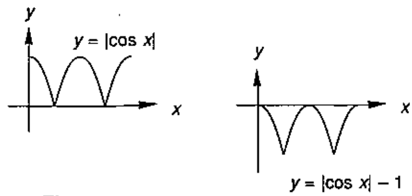


Fig. A7.16

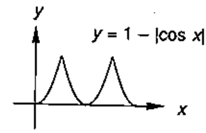


Fig. A7.17

5 (a) 72.3° (b) 48.6°, 90°, 131.4° (c) 63.4°, 146.3° **6** (a) 9.7°, 80.3°, 189.7°, 260.3° (b) 53.1°, 90°, 126.9°, 270° (c) 84.3°, 275.7° **8** [Fig. A7.18] 3 **9** (a) 155° (b) 159.4° (c) 18.4°, 116.6° **10** [Fig. A7.19] (i) 3 (ii) 1 **11** $1 \leq y \leq 2$ **12** [Fig. A7.20] 5; 8 **13** (a) 30°, 150° (b) 32.3°, 147.7°, 212.3°, 327.7° **14** 0°, 30°, 150°, 180°, 210°, 330°, 360° **15** (a) 30°, 150° (b) 60°, 120°, 240°, 300° **16** [Fig. A7.21] 3 **17** [Fig. A7.22] 2.3 **18** (a) [AC = $2r\sin\theta$, BC = $2r\cos\theta$] (b) max $\frac{\pi r^2}{2}$ when $\theta = 0$, min $r^2(\frac{\pi}{2} - 1) \approx 0.57r^2$ when $\theta = 45°$ (c) 25.9°, 64.1° **19** (a) 1.46 (b) $2.76r^2$ (c) 69% **20** 0.97 rad

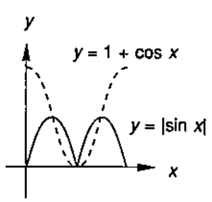


Fig. A7.18

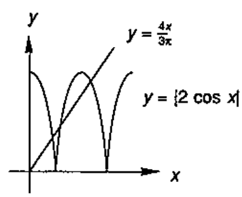


Fig. A7.19

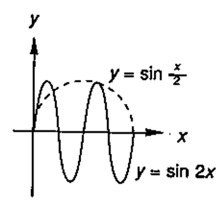


Fig. A7.20

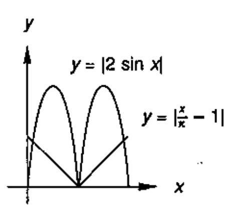


Fig. A7.21

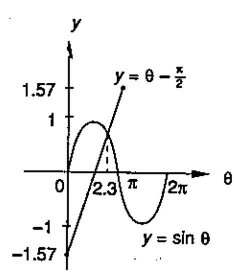


Fig. A7.22

EXERCISE 8.1

1 [Fig.A8.1] **2** $-2\mathbf{a}$, $\frac{1}{2}\mathbf{a}$, $3\mathbf{a}$ **3** (a) $\frac{3}{2}\mathbf{a}$ (b) $\mathbf{a}$ (c) $-\frac{1}{2}\mathbf{a}$ **4** (a) $-\mathbf{a}$, (b) $-\mathbf{b}$, (c) $-\mathbf{c}$, (d) 2c, (e) 2b **5** Collinear **6** 5 **7** 13 **8** $(\sqrt{5},2)$ or $(-\sqrt{5},2)$

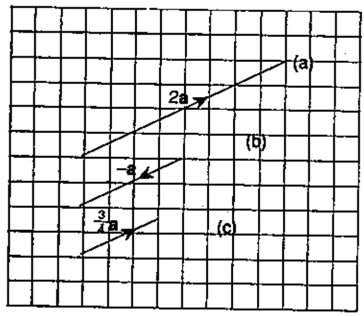


Fig. A8.1

EXERCISE 8.2

1 [Fig.A8.2] **2** (a) $\mathbf{a}+\mathbf{b}$ (b) $-\mathbf{a}-\mathbf{b}$ **3** [Fig.A8.3] **4** The parallelogram is a rhombus

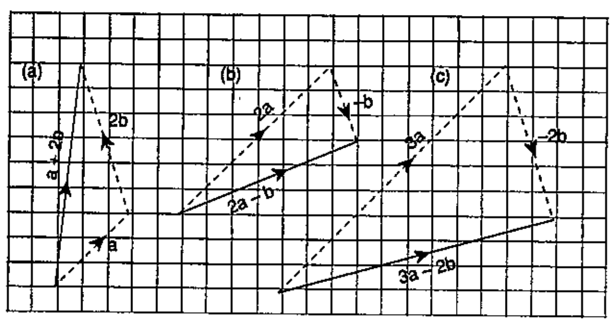


Fig. A8.2

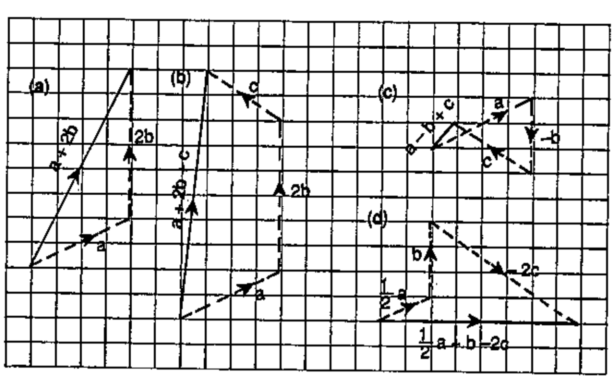


Fig. A8.3

5 (a) $\mathbf{b}-\mathbf{a}$ (b) $\frac{1}{2}(\mathbf{b}-\mathbf{a})$ (c) $\frac{1}{2}(\mathbf{a}+\mathbf{b})$ **6** (a) $\frac{1}{2}\mathbf{p}$ (b) $\mathbf{q}-\mathbf{p}$ (c) $\frac{3}{4}\mathbf{q}$ (d) $\frac{3}{4}\mathbf{q}-\frac{1}{2}\mathbf{p}$
7 (a) $\mathbf{b}-\mathbf{a}$ (b) $2\mathbf{a}$ (c) $\mathbf{b}+2\mathbf{a}$ (d) $\mathbf{b}+\mathbf{a}$ **8** (a) $\frac{3}{2}\mathbf{b}$ (b) $\mathbf{a}+\frac{3}{2}\mathbf{b}$ (c) $\mathbf{a}+\frac{1}{2}\mathbf{b}$
9 (a) $\frac{1}{2}\mathbf{a}$ (b) $\mathbf{b}-\frac{1}{2}\mathbf{a}$ (c) $\frac{1}{3}\mathbf{b}-\frac{1}{6}\mathbf{a}$ (d) $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$ **10** (a) $\mathbf{p}-2\mathbf{q}$ (b) $\frac{1}{2}(\mathbf{p}-2\mathbf{q})$ (c) $\frac{3}{2}\mathbf{p}$

EXERCISE 8.3

1 3, –4 **2** 2, –3 **3** –3, 2 **4** 2, –3 **5** 1:2 **6** 18; 1:4 **7** 3 **8** 3; 1:–2 **9** (a) 1; 1:2 (b) –5 **10** (a) $\frac{1}{3}(5\mathbf{a}+6\mathbf{b})$ (b) $\frac{\mathbf{a}}{3}$ (c) $\frac{1}{2}(4\mathbf{a}+5\mathbf{b})$ (d) $5\mathbf{a}+7\mathbf{b}$ (e) $-9\mathbf{a}-14\mathbf{b}$
11 (a) $\mathbf{a}-2\mathbf{b}, 3\mathbf{a}+\mathbf{b}$ (b) $\frac{1}{3}(6\mathbf{a}+2\mathbf{b})$ (c) $\overrightarrow{OG}=\frac{1}{3}(6\mathbf{a}+2\mathbf{b})$, $\overrightarrow{GQ}=\frac{1}{3}(3\mathbf{a}+\mathbf{b})$; 2:1 (d) 2 **12** (a) $\overrightarrow{QP}=\overrightarrow{AP}-\overrightarrow{AQ}=\mathbf{a}+\frac{2}{3}(\mathbf{b}-\mathbf{a})-\frac{2}{3}\mathbf{b}=\frac{\mathbf{a}}{3}$ **13** $\frac{1}{4}$, $\frac{4}{5}$, 1:3, 4:1
14 (a)(i) $(1-p)\mathbf{a}+\frac{1}{2}(1+p)\mathbf{b}$ (ii) $q\mathbf{a}+q\mathbf{b}$ (b) $\frac{1}{3}$, $\frac{2}{3}$; 2:1, 2:1
15 (a) $\frac{2}{3}(1-q)\mathbf{a}+q\mathbf{b}$; $(1-p)\mathbf{a}+\frac{p}{3}\mathbf{b}$ (b) $\frac{3}{7}$, $\frac{1}{7}$ (c) 1:6, 3:4
16 [$\overrightarrow{DE}=\frac{4\mathbf{a}-\mathbf{b}}{20}$, $\overrightarrow{DA}=\frac{4\mathbf{a}-\mathbf{b}}{4}$]; 1:4 **17** (a) $\frac{p}{3}(2\mathbf{a}+\mathbf{b})$; $\frac{2}{5}(1-q)\mathbf{a}+q\mathbf{b}$ (b) $\frac{1}{2}$, $\frac{1}{6}$ (c) 1:1, 1:5 **18** (a) $\frac{p}{2}\mathbf{a}+\frac{p}{2}\mathbf{b}$; $(1-q)\mathbf{a}+\frac{2q}{3}\mathbf{b}$ (b) $\frac{4}{5}$, $\frac{3}{5}$ (c) 4:1 (d) $(1+k)\mathbf{a}-k\mathbf{b}$; $\frac{1}{2}$
19 (a) $(1+p)\mathbf{a}+2p\mathbf{c}$; $\frac{3q}{2}\mathbf{a}+(1-q)\mathbf{c}$ (b) $\frac{1}{8}$; $\frac{3}{4}$; 1:7, 3:1 (c) $\frac{3}{8}$[$\overrightarrow{BF}=r\mathbf{a}+(2r-1)\mathbf{c}$, $\overrightarrow{CE}=\frac{(3\mathbf{a}-2\mathbf{c})}{2}$ hence $r: 2r-1=3:-2$] (d) 1:4
20 $\frac{1}{2}(\mathbf{a}+\mathbf{b})$; $(p-pq)\mathbf{a}+\frac{3p}{2}\mathbf{b}$; $\frac{5}{6}$, $\frac{2}{5}$; 5:1, 2:3 **21** $\frac{2p+1}{3}\mathbf{c}$; $(1-\frac{q}{2})\mathbf{a}+q\mathbf{c}$; $\frac{5}{8}$, $\frac{3}{4}$
22 4 **23** $5\mathbf{a}-\mathbf{b}$, $\mathbf{a}-\frac{\mathbf{b}}{2}$, $13\mathbf{a}-2\mathbf{b}$; 1:2 **24** $\frac{9\mathbf{a}+\mathbf{b}}{5}$, $\frac{15\mathbf{a}-7\mathbf{b}}{4}$, $7\mathbf{a}-5\mathbf{b}$; 3:5

EXERCISE 8.4

1 [Fig. A8.4] **2** $-\mathbf{i}+5\mathbf{j}$, $3\mathbf{i}-3\mathbf{j}$, $-2\mathbf{i}-2\mathbf{j}$ **3** (a) $-3\mathbf{i}-\mathbf{j}$, $-4\mathbf{i}+\mathbf{j}$ [Fig. A8.5] (b) $6\mathbf{i}+2\mathbf{j}$ (c) $\sqrt{40}$, 18.4° **4** (a) $\sqrt{5}$, 116.6° (b) $\sqrt{18}$, 315° (c) $\sqrt{13}$, 56.3° (d) $\sqrt{20}$, 206.6° **5** (a) $-5\mathbf{i}-2\mathbf{j}$ (b) $\binom{-15}{20}$, $\binom{-17\frac{1}{2}}{14}$ **6** $7\mathbf{i}$ **7** $3\mathbf{i}+6\mathbf{j}$ **8** (a) $\frac{3\mathbf{i}-4\mathbf{j}}{5}$, $\frac{\mathbf{i}+3\mathbf{j}}{\sqrt{10}}$ (b) $\frac{-3\mathbf{i}+4\mathbf{j}}{5}$, $\frac{4\mathbf{i}+3\mathbf{j}}{5}$ **9** (i) $\frac{3\mathbf{i}+\mathbf{j}}{\sqrt{10}}$ (ii) $\frac{-\mathbf{i}+2\mathbf{j}}{\sqrt{5}}$ (iii) $\mathbf{i}+5\mathbf{j}$ (iv) $9\mathbf{i}-6\mathbf{j}$ (v) $-4\mathbf{i}+3\mathbf{j}$ (vi) $\sqrt{26}$ (vii) $\sqrt{117}$ (viii) 5 **10** (a) $\mathbf{i}-4\mathbf{j}$ (b) $\mathbf{i}+3\mathbf{j}$ (c) $2\mathbf{i}+7\mathbf{j}$ (d) $2\mathbf{i}-\mathbf{j}$
11 (1, –1) **12** $\mathbf{i}+7\mathbf{j}$ **13** $\mathbf{i}-\mathbf{j}$ **14** $-11\mathbf{i}+3\mathbf{j}$ **15** (a) $-\mathbf{i}+4\mathbf{j}$, $4\mathbf{i}-2\mathbf{j}$ (b) [Fig. A8.6] (c) $\sqrt{61}$, 309.8°

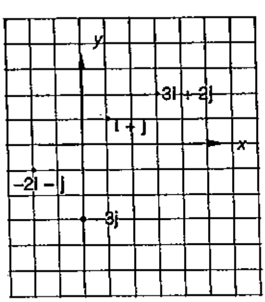


Fig. A8.4

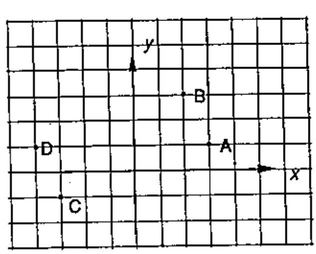


Fig. A8.5

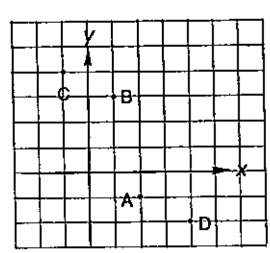


Fig. A8.6

16 $10\mathbf{i} + 19\mathbf{j}$ **17** $6\mathbf{i} + \mathbf{j}$ **19** -2 **21** (a) $\sqrt{10}$ m s^{-1}, $71.6°$ (b)(i) $2\mathbf{i} + 4\mathbf{j}$ (ii) $4\mathbf{i} + 10\mathbf{j}$ (iii) $\mathbf{i} + \mathbf{j} + t(\mathbf{i} + 3\mathbf{j})$ (c) 6 secs **22** $7\mathbf{i} - 5\mathbf{j}$; 5 sec **23** (a) $5\mathbf{i} - \mathbf{j}$ (c) $-\frac{1}{2}$; $x + 2y = 3$ **24** -1; $x + y = 0$ **25** (a) $\mathbf{i} - 3\mathbf{j}$ (b) $5\mathbf{i} + 5\mathbf{j}$ (c) 5 **26** $\frac{4}{3}$ **27** $-1\frac{1}{2}$ **28** (a) $2\mathbf{i} + 2\mathbf{j}$; $4\mathbf{i} + 5\mathbf{j}$; $5\mathbf{i} + 7\mathbf{j}$ (b) $\frac{4}{5}, \frac{2}{5}$; 4:1 **29** (a) $m_1 m_2 = -1$ (b) $-\frac{3}{2}, \frac{2}{3}$ **31** (a) $-\frac{3}{2}, \frac{2}{k+1}$ (b) 2 **32** 7, 3

EXERCISE 8.5

1 (a) 0 (b) 4 (c) -7 **2** $\mathbf{a}$ and $\mathbf{c}$ **3** 5 **4** (a) $150.3°$ (b) $123.7°$ **5** $-\frac{1}{3}$ **6** $164.7°$ **7** $m + 3n = 0$ **9** $70.6°$ **10** $45°$ **11** [Fig.A8.7] (b) $90°$; 2.5 **12** $171.9°$ **13** ± 4 **14** ± 3 **15** -24; $26.6°$ **16** 2; $82.9°$ **17** (a) $-4\mathbf{i} + 5\mathbf{j}$, $\mathbf{i} + 4\mathbf{j}$ (b) 16; $52.7°$ **18** (a) $\mathbf{i} + 10\mathbf{j}$ (b) $-\mathbf{i} + 2\mathbf{j}$; $2\mathbf{i} + 4\mathbf{j}$ (c) $42.3°$ **19** (a)(i) $\mathbf{b} - \mathbf{a}$ (ii) $\mathbf{d} - \mathbf{c}$ (iii) $\mathbf{c} - \mathbf{a}$ (iv) $\mathbf{d} - \mathbf{b}$ (b) trapezium (c) $(\mathbf{a} - \mathbf{c}).(\mathbf{b} - \mathbf{d}) =$ (d) $60°$ **20** (a) $3\mathbf{i} + \mathbf{j} + k(-4\mathbf{i} - 4\mathbf{j})$ (b) $-\frac{5}{4}$ **21** $\frac{4}{3}, -1$ **22** (a) $\mathbf{i} + 2\mathbf{j}$, $4\mathbf{i} + 2\mathbf{j}$ (b) $3\mathbf{i}$, $3\mathbf{i} - 3\mathbf{j}$ (c) $45°$ **23** (a) $67.8°$ (b) $2\mathbf{i} + 5\mathbf{j}$ (c) $54.2°$ **24** (a) $\frac{\mu + 16}{3}\mathbf{i} + 5\mathbf{j}$ (b) $\frac{\mu + 7}{3}\mathbf{i} + 2\mathbf{j}$ (c) 2 **25** (a) $3m + 4n$ (b) 5; $\sqrt{m^2 + n^2}$ (c) $\frac{3m + 4n}{5\sqrt{m^2 + n^2}}, \frac{8m - 6n}{10\sqrt{m^2 + n^2}}$ (d) $m = 7n$ (e) $7y = x$

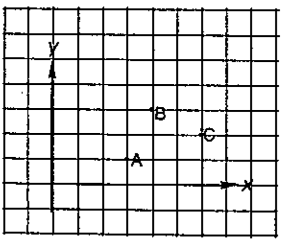


Fig. A8.7

REVISION EXERCISE 8

1 (a) 50 (b) $44.4°$ (c) $m = -n$ **2** 68; $26.6°$ **3** $-6\frac{1}{2}$ **4** $34 - 4p$ (i) $8\frac{1}{2}$ (ii) $\frac{5}{13} = 0.3846$ **5** $(\mu - 4)\mathbf{p} + (\mu + 1)\mathbf{q}$, $-3\mathbf{p} + 3\mathbf{q}$; $1\frac{1}{2}$ **6** (a) $\binom{-12}{9}$ (b) $\pm 3, \pm 4$ **7** $\frac{2}{5}(1 - q)\mathbf{a} + \frac{2}{5}q\mathbf{b}$; $-\frac{4p}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$; $\frac{1}{4}, \frac{1}{2}$ **8** (a) 4; $26.6°$ (b)(i) $(1 - \lambda)\mathbf{p} + 3\lambda\mathbf{q}$ (ii) $2(1 - \mu)\mathbf{p} + 2\mu\mathbf{q}$; $\frac{1}{2}, \frac{3}{4}$; $\frac{1}{2}(\mathbf{p} + 3\mathbf{q})$ **9** [Fig. A8.8] B is midpoint of AC **10** (i) 19, -3 (ii) 0; right-angle (iii) 90; $33.7°$ **11** (a) 1, $81.9°$ (b)(i) $\frac{9}{13}$ (ii) $\frac{5}{9}$ **12** $\frac{1}{2}(\mathbf{a} + \mathbf{b})$, $\frac{4}{5}(\mathbf{a} + \mathbf{b})$, $-\frac{1}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$, $-\frac{4}{5}\mathbf{a} + \frac{16}{5}\mathbf{b}$, [$\overrightarrow{RP} = 4\overrightarrow{AR}$] 1:4; 4 **13** 2, $-\frac{2}{3}$; 0, $90°$; $-\frac{8}{9}$, $118.1°$ **14** rhombus; square **15** (a) $\frac{p}{2}\mathbf{a} + \frac{p}{2}\mathbf{b}$; $(1 - q)\mathbf{a} + \frac{2q}{3}\mathbf{b}$; 4:1 (b) 1:1 (c) 5:3:2 **16** $\frac{1}{7}(17\mathbf{a} + 27\mathbf{b})$, $\frac{1}{5}(\mathbf{a} + 12\mathbf{b})$, $-5\mathbf{a} - \mathbf{b}$; $\frac{3}{7}$ **17** $\frac{2}{11}$, 2 **18** $x^2 + y^2 = 1$ [$\cos^2\theta + \sin\theta = 1$] **19** [$\mathbf{a}.(\mathbf{b} - \mathbf{c}) = 0$]

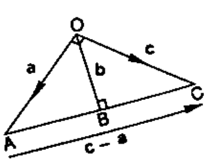


Fig. A8.8

20 (a) $\mathbf{h} - \mathbf{b}$ (b) [$\mathbf{h}.\mathbf{a} = \mathbf{b}.\mathbf{a}$ and $\mathbf{h}.\mathbf{b} = \mathbf{h}.\mathbf{a}$ hence $\mathbf{h}.\mathbf{b} = \mathbf{b}.\mathbf{a}$ i.e. $\mathbf{b}.(\mathbf{h} - \mathbf{a}) = 0$] (c) the 3 altitudes intersect at H.
21 [$\cos A = \frac{\mathbf{a}.\mathbf{b}}{ab}$, $\sin A = \sqrt{1 - \cos^2 A}$: area $= \frac{1}{2}ab \sin A = \frac{ab}{2}\sqrt{1 - \frac{(\mathbf{a}.\mathbf{b})^2}{(ab)^2}}$]; 13
22 $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$; 2:1

EXERCISE 9.1

1 4, -2 **2** 8 **3** 0.25

EXERCISE 9.2

1 (a) $6x$ (b) $6x$ (c) $2x+1$ (d) $-2x$ (e) $-\frac{1}{x^2}$ (f) $3x^2-2x$ (g) $\frac{-1}{(x-1)^2}$

2

	(i)	(ii)	(iii)
(a)	8	-4	0
(b)	12	-6	0
(c)	3	-3	-1
(d)	-4	2	0
(e)	$-\frac{1}{4}$	-1	undefined
(f)	8	5	0
(g)	$-\frac{1}{16}$	-1	$-\frac{1}{4}$

3 (a) $6x-6$ (b) 1 (c) -1 **4** (a) $3x^2-2x$ (b) $0, \frac{2}{3}$ (c) $0, -\frac{4}{27}$
5 [gradient function $= -4-2x$] max **6** $-\frac{1}{x^2}+4$; $\pm\frac{1}{2}$ **7** $2ax+b$

EXERCISE 9.3

1 (a) $6x$ (b) $-8x$ (c) $12x^2$ (d) $1+\frac{1}{x^2}$ (e) $x-1$ (f) x^2 **2** 0 **3** (a) $4x-4$
(b)(i) $(1,-1)$ (ii) $(-1,7)$ **4** (a) $2x+a$ (b) -4

EXERCISE 9.4

1 (a) 5 (b) $8x$ (c) 0 (d) $6x$ (e) $6x-1$ (f) $3x^2-2x-1$ (g) $-6x$ (h) $2x-2$
(i) $-\frac{4}{x^2}$ (j) $6x-\frac{1}{x^2}$ (k) $3x^2-12x+12$ (l) $2x-\frac{2}{x^3}$ (m) $x^4+x^3+x^2+x$
(n) $2x-2+\frac{3}{x^2}$ (o) $\frac{4}{x^2}-\frac{2}{x^3}$ (p) $-\frac{2}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}$ **2** $6t-4$ **3** $9r^2-4r+1$ **4** $18t-12$
5 6 **6** 2 **7** -4 **8** $(\frac{2}{3},\frac{2}{9}), (-\frac{2}{3},\frac{34}{9})$ **9** $(0,-5), (2,-7)$ **10** $1, -\frac{2}{3}$ **11** $(2,-38)$ **12** $1, -\frac{3}{2}$
13 $\frac{1}{3}, -\frac{1}{2}$ **14** $8s-12$; 2 **15** $2, -1$; 6 **16** $3, -2$ **17** $10, -13$ **18** $x \le -1$ or $x \ge 2$
19 $\frac{4}{5}, -\frac{9}{5}$ **20** $4mp^3+2np$; $4, -2$

EXERCISE 9.5

1 (a) $5(x-3)^4$ (b) $21(3x-1)^6$ (c) $-6(5-2x)^2$ (d) $40(4x-5)^9$ (e) $16(4x-3)^3$
(f) $3(2x-1)(x^2-x+1)^2$ (g) $5(-1-4x)(3-x-2x^2)^4$ (h) $-\frac{1}{(x-2)^2}$ (i) $\frac{12}{(1-3x)^2}$ (j) $\frac{-8}{(3+2x)^2}$
(k) $4(1+\frac{1}{x^2})(x-\frac{1}{x})^3$ (l) $\frac{-2x}{(x^2+3)^2}$ (m) $\frac{-8(2x-1)}{(x^2-x-1)^2}$ (n) $na(ax+b)^{n-1}$ (o) $\frac{-16}{(2x-3)^5}$
(p) $\frac{3(3+4x)}{(1-3x-2x^2)^4}$ (q) $3(2+\frac{1}{2x^2})(2x-\frac{1}{2x})^2$ **2** (a) $6(2t-1)^2$ (b) $1\frac{1}{2}, -\frac{1}{2}$ **3** -96
4 $t+\frac{3}{5}(1-t)^2 = \frac{3t^2-t+3}{5}$; $1, -\frac{2}{3}$ **5** $\frac{6}{(4-2r)^2}$; $\frac{1}{2}$ $(r=-1)$, $-\frac{1}{2}$ $(r=5)$ **6** (a) $2+\frac{4}{(x+1)^2}$
(b) 3 **7** -6 **8** $(1, \frac{1}{2}), (-3, -\frac{1}{2})$ **9** $(2,1)$ **10** (a) $\frac{12}{(1-4t)^2}$ (b) $-\frac{1}{4}, \frac{3}{4}$ **11** (a) $3-\frac{2}{(1+2t)^2}$
(b) $\frac{1}{2}, -\frac{3}{2}$ **12** $-2, -4\frac{1}{2}$ **13** $4, -3$ **14** $1, -2$

EXERCISE 9.6

1 (a) $12x^2-10x$; $24x-10$ (b) $6(2x-7)^2$; $24(2x-7)$ (c) $-16(1-4x)^3$; $192(1-4x)^2$
(d) $-\frac{1}{x^2}$; $\frac{2}{x^3}$ (e) $2x+\frac{1}{x^2}$; $2-\frac{2}{x^3}$ (f) $\frac{3}{(2-x)^2}$; $\frac{6}{(2-x)^3}$ (g) $4x^3-2x+\frac{2}{x^3}$; $12x^2-2-\frac{6}{x^4}$
2 $6t+\frac{4}{t^3}$; $6-\frac{12}{t^4}$ **3** ± 3 **4** $1, -2$ **5** $3, -2$ or $-3, 2$ **6** -4; 4 **7** $\frac{1}{(2-x)^4}$; $\frac{2}{(2-x)^3}$ **8** $\frac{2}{3}$, 6; $3\frac{1}{3}$
9 $x \ge \frac{2}{3}$

REVISION EXERCISE 9

1 (a) $3(x-5)^2$ (b) $-10(1-2x)^4$ (c) $\frac{8}{(1-4x)^2}$ (d) $12x(2x^2-1)^2$
(e) $3(-3-4x)(1-3x-2x^2)^2$ (f) $\frac{2x-1}{(x-x^2)^2}$ (g) $3(2+\frac{1}{x^2})(2x-\frac{1}{x})^2$ (h) $\frac{1}{2}-\frac{1}{x^3}$ (i) $1+\frac{4}{x^2}$
(j) $-\frac{3}{2}-\frac{6}{(1-3x)^2}$ **2** $0 \leq x \leq 2$ **3** ± 8 **4** (a) $-\frac{2}{3}$ (b) $2x+3y=12$ (c) (0,4),(6,0)
(d) $\sqrt{52}$ **5** -8 or 8 **6** $4-\frac{2}{(1-2x)^2}$; $\frac{-8}{(1-2x)^3}$ **7** $6x^2-4$, $12x$; 4 **8** 5 or $1\frac{1}{4}$ **9** -2, -5
10 (a) 2, -3 (b) [gradient $= -3\frac{1}{2}$] **11** (b) $y=3x-6$, $y=-3x-3$ **12** -3, 5
13 $\frac{7}{3}$, -2; 26, -26 **14** 3, -1 **15** $-3 < x < 1$ **16** $\frac{2}{3} \leq x \leq 4$ **18** (a) $2\mathbf{i}-\mathbf{j}$ (b) 1

EXERCISE 10.1

1 $t < -\frac{1}{2}$ or $t > \frac{1}{2}$ **2** $x < \frac{3}{4}$ **3** $x < -2$ or $x > 2$; $(-2,-1)$, $(2,1)$ **4** $\frac{1}{3} < x < 1$
5 $-2 < x < 1$; $(1,-3)$, $(-2,24)$ **6** $t < \frac{3}{4}$ **7** (a) $y+6x=-11$, $6y-x=45$
(b) $y-6x=-4$, $6y+x=13$ (c) $y-5x=4$, $5y+x=-6$ (d) $y=5x+3$, $5y+x=-11$
(e) $x+y=-4$, $x-y=0$ (f) $3y+x=5$, $y-3x=-5$ (g) $y-2x=-3$, $2y+x=1$
(h) $y-x=-4$, $y+x=-8$ (i) $y=8x-11$, $8y+x=42$ (j) $y=2x+2$, $2y+x=-1$
(k) $y=8x+11$, $8y+x=55\frac{1}{2}$ (l) $2y=2x+3$, $2y+2x=1$ **8** $4y-4x=3$; $(-\frac{3}{4},0)$
9 (a) $(-1,2)$ (b) $4y=x+9$ **10** (a) $y=x$ (b) (2,2) **11** (a) $y+x=7$ (b) $(\frac{4}{3},\frac{17}{3})$
(c) $y+7x=15$ **12** [Normal is $18y+8x=35$] $-\frac{53}{44}$ **13** $y=8x-12$, $3y=8x-12$ $(\frac{4}{3},0)$
15 $-3 < a < 3$ **16** (a)(i) $2y+x=3$ (ii) $(\frac{3}{2},\frac{3}{4})$ (b) $(\frac{1}{4},\frac{47}{16})$

EXERCISE 10.2

1

	(i)	(ii)
(a)	max at $x=-3$	16
(b)	min at $x=1\frac{1}{2}$	$-\frac{9}{4}$
(c)	max at $x=-1$	0
	min at $x=1$	-4
(d)	pt of inf at $x=0$	-2
(e)	max at $x=-3$	85
	min at $x=2$	-40
(f)	max at $x=0$	1
	min at $x=\frac{1}{3}$	$\frac{26}{27}$
(g)	pt of inf at $x=0$	1
	min at $x=1$	0
(h)	max at $x=-5$	-10
	min at $x=5$	10
(i)	none	
(j)	pt of inf at $x=2$	10
(k)	none	
(l)	min at $x=2$	8
	min at $x=-2$	8
(m)	max at $x=1$	0
	min at $x=3$	4

(n)	min at $x = \frac{5}{6}$	$-\frac{49}{12}$
(o)	min at $x = 0$	0
(p)	min at $x = 0$	1
	max at $x = -\frac{2}{3}$	$\frac{31}{27}$
(q)	max at $x = -1$	2
	min at $x = \frac{5}{3}$	$-\frac{202}{27}$
(r)	pt of inflexion at $x = -1$	-5
(s)	none	
(t)	min at $x = 2$	3
	max at $x = 0$	-1
(u)	min at $x = 2$	12
(v)	max at $x = \frac{2}{3}$	$-\frac{2}{3}$
	min at $x = \frac{8}{3}$	$7\frac{1}{3}$

2 (a) 1 (b) min **3** (a) 3 (b) (–2,16) **4** (a) 2 (b) min at $x = 1$, max at $x = -\frac{7}{3}$
5 (a) Max (0,1), min $\left(\frac{4}{3}, -\frac{5}{27}\right)$ (b) $\left(\frac{2}{3}, \frac{11}{27}\right)$ **6** (a) 1 (b) 5 **7** $\frac{2}{3}$ **8** (a) 2, –3
(b) min at $x = 2$, max at $x = -1$ **9** (a) $-\frac{1}{3}$ (b) $\frac{1}{2}$; min at 60°, 300°; max at 109.5°, 250.5°
10 $r = 0$, max; $r = \frac{4}{\pi}$, min **11** $-8 < a < 8$; $x = -1$, min; $x = -4$, max **12** min at $\left(\frac{3}{2}, 4\right)$

EXERCISE 10.3

1 $7\frac{3}{4}$ **2** 25 **3** 14 **4** $\frac{145}{27}$ **5** $-\frac{2}{3}$ **6** 2 **7** 10 **8** (b) 3 **9** 5 **10** $66\frac{2}{3}$ **11** (b) 5, 2
12 (b) 12; 576π **13** (a) $y = mx + 3 - 2m$ (b) $3 - 2m$ (d) $-\frac{3}{2}$ **14** (b) 2
15 (a) $\$(30x - \frac{3x^2}{4} - 50)$ (b) 20 **16** (a) $10t$, $100 - 20t$ (c) 4; 44.7 km
17 (b) $\pi^2(8r^2 - 2r^3)$; $\frac{8}{3}$ **18** (b) 5 (c) 86.6 cm² **19** (a) $x^2 + 25$; $x^2 - 6x + 25$ (b) $1\frac{1}{2}$
20 4; $\sqrt{3}$:4 **21** (a) –1, 2 (b) $(1 - t)\mathbf{i} - (2t + 3)\mathbf{j}$ (c) –1 (d) $\sqrt{5}$ **22** (c) 3; 45π cm²
23 (a) $10 - 2t$, $4t$ (b) $28t^2 - 80t + 100$ (c) $\frac{10}{7}$ (d) 6.5 cm **24** (b) $20(3x^2 - x^3)$
(c) 2 (d) 80 cm² **25** (b) $\frac{576 - 96x + (\pi + 4)x^2}{\pi}$; 6.72 cm **26** $\frac{1}{4}(7x^2 - 4kx + k^2)$; 4:5
27 (a) $50 - 5x + \frac{3x^2}{4}$ (b) $3\frac{1}{3}$ **28** $\frac{2}{27}m^3[x = \frac{2}{3}]$ **29** $\frac{4}{\sqrt{3}}$; $\frac{256}{3\sqrt{3}}$

EXERCISE 10.4

1 (a) 4 (b) –4 m s^{-1} **2** (a) –2 m s^{-1} (b) 1 s (c) 2 s (d) 2 m s^{-2} **3** (a) 6 m
(b) –5 m s^{-1} (c) $2\frac{1}{2}$ s (d) 2 s and 3 s (e) $6\frac{1}{4}$ m **4** (a) 24 (b) 1 s and 4 s (c) 4
(d) 1 s and 4 s (e) –18, 18 **5** (a) –5 m, –4 m s^{-1} (b) 2 s, –9 m (c) 5 s (d) 2 m s^{-2}
6 (a) –1, 12 (b) 1 s and 2 s (c) $\frac{3}{2}$ (d) $1\frac{1}{2}$ s **7** (a) 3 (b) 2 s (c) 3 cm s^{-1}

EXERCISE 10.5

1 0.175 **2** 2.8 **3** 0.16 **4** 6.4 **5** 0.12 **6** 0.495 **7** 9% **8** –0.5 **9** (a) 5% (b) 10%
10 (b) $1\frac{1}{3}$% **11** $\frac{4\pi}{3}$ **12** (a) $24k$ (b) $150k$% **13** $3p$ **14** 0.00575 **15** 0.5%
16 6; 0.00286 **17** 3; 0.004 **18** –6, 2 **19** –0.006 **20** 1.0225, 0.9775 **21** 0.03
22 0.01 increase

EXERCISE 10.6

1 20.1 $\text{cm}^2\,\text{s}^{-1}$ **2** 0.106 cm s^{-1} **3** (a) 314 $\text{cm}^3\,\text{s}^{-1}$ (b) 62.8 $\text{cm}^2\,\text{s}^{-1}$ **4** 1.59 cm s^{-1}
5 2 $\text{cm}^2\,\text{s}^{-1}$ **6** 54π $\text{cm}^2\,\text{s}^{-1}$ **7** 4.77 cm s^{-1} **8** 2400 **9** 0.4 **10** 0.75 **11** 41.89 $\text{cm}^3\,\text{s}^{-1}$
12 96 $\text{cm}^3\,\text{s}^{-1}$ **13** 0.637 cm s^{-1} **14** 1.59 cm s^{-1} **15** −0.125 **16** 0.018 units s^{-1}
17 1.02 cm s^{-1} (increasing) **18** (a) $56x - 4x^2$ (b) 2 (c) −10 $\text{cm}^2\,\text{s}^{-1}$ **19** $\frac{25}{7\pi}$ cm s^{-1}
20 (c) 2.39 cm s^{-1} **21** (c) 2.5 $\text{cm}^2\,\text{s}^{-1}$ **22** (a) $\left[\frac{dL^2}{dt} = \frac{dL^2}{dL} \times \frac{dL}{dt}\right]$ (c) $x^2 - 5x + 25$ (d) 11
(e) $\frac{11}{14}$ cm s^{-1}

REVISION EXERCISE 10

1 $x < -1$ or $x > 5$ **2** $1\frac{1}{2}$ **3** 0.05 **4** (i) $\frac{6}{x^2}$; $\frac{3p}{8}$ (ii) 2.4 **5** 0.012 $\text{cm}^2\,\text{s}^{-1}$; 5 km
6 $\frac{60 - 2r - \pi r}{2}$; 8.40; maximum **7** $\frac{13p}{7}$% **8** $V = x^3$, $A = 6x^2$; $\frac{dV}{dx} = 3x^2$, $\frac{dA}{dx} = 12x$
(i) 300 $\text{cm}^3\,\text{s}^{-1}$ (ii) 0.12 cm^2 **9** 17.3 **10** $\frac{100 - 3x}{4}$; 18.83 minimum **11** $\frac{3}{4}$
12 $y = \frac{27 - 4x^3}{\pi x^2}$ $[A = 4 \times 2x^2 + 4x^2 + (4x^2 - \pi x^2) + \pi x^2 + 2\pi xy]$ (i) $\frac{3}{2}$ (ii) 54, $\frac{6}{\pi}$; minimum
13 (a) $\frac{6 + 5b - 2b^2}{2}$ (b) $1\frac{1}{4}$ **14** $\sqrt{100 - h^2}$; $\frac{4000\pi}{3\sqrt{3}}$ **15** 8 **16** (a) $L = 4r$, $\theta = 2$, max
17 $(0, 2 - 3m)$, $(\frac{3m-2}{m}, 0)$; 12 $[m = -\frac{2}{3}]$ **18** (a) 30 cm s^{-1}, 42 cm s^{-2} (b) $6\frac{3}{4}$ cm
19 10 cm, $6\frac{2}{3}$ cm, 4 cm **20** −4% **21** (i) 9 m s^{-1} (ii) 12 m s^{-1} (iii) 8 m s^{-1} **22** 8 cm
23 1, −2, −7, −3 **24** $x = -1$, min; $x = 0$, pt of inf. **25** (b) $\pi r^2(22 - 3r)$ (c) $\frac{10}{3}\left[r = \frac{44}{9}\right]$
26 (b) $-\frac{2}{3}$% **27** (b) 2.11 cm s^{-1} **28** (a) $\left[\frac{PQ}{x-5} = \frac{20}{x}\right]$ (b)(i) 2.78 cm s^{-1} (ii) 1 cm s^{-1}
(c) $-\frac{3200}{x^3}$ cm s^{-1} [If PQ = h, $\frac{d(\frac{dh}{dt})}{dt} = \frac{d(\frac{dh}{dt})}{dx} \times \frac{dx}{dt}$] **29** (a) $\frac{80}{x}$ (b) $x^2 + \frac{6400}{x^2} - 80$;
$4\sqrt{5}$ (≈ 8.9); $4\sqrt{5}$ **30** (4,4); $\frac{1}{2}(\sqrt{8}\,x - \frac{x^2}{2})$ [A is $(2, \sqrt{8})$, B is $(x, \frac{x^2}{4})$]; 2 $[x = 2\sqrt{2}]$

EXERCISE 11.1

1 (a) $2x^2 + c$ (b) $x^4 + c$ (c) $-7x + c$ (d) $x^3 + c$ (e) $3x - \frac{x^2}{2} + c$ (f) $\frac{4x^5}{5} + c$
(g) $\frac{x^6}{3} + c$ (h) $\frac{x^3}{3} - 3x + c$ (i) $x - \frac{x^2}{2} - \frac{x^3}{3} + c$ (j) $\frac{x^3}{3} - \frac{x^2}{8} + c$ (k) $x - \frac{3x^2}{2} - \frac{4x^3}{3} + c$
(l) $\frac{x^6}{6} - x^3 + c$ (m) $-\frac{1}{x^2} + c$ (n) $\frac{x^3}{3} + 2x^2 + 4x + c$ (o) $\frac{x^4}{4} - x^3 + \frac{3x^2}{2} - x + c$ (p) $x + \frac{1}{9x^3} + c$
(q) $4x - 2x^2 + \frac{x^3}{3} + c$ (r) $\frac{x^3}{3} + \frac{1}{x} + c$ (s) $\frac{x^3}{3} - \frac{x^2}{2} - 6x + c$ (t) $\frac{x^2}{2} - \frac{1}{x} + c$
(u) $-\frac{1}{3x} - \frac{1}{6x^2} + c$ (v) $\frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x + c$ **2** (a) $\frac{1}{x^2} - 4x + c$ (b) $-\frac{3}{y} + c$ (c) $\frac{x}{3} + c$
(d) $2x + \frac{1}{x} + c$ (e) $\frac{3x^2}{2} - 2x + c$ (f) $\frac{t^2}{3} + c$ (g) $-\frac{4}{y} + c$ (h) $-\frac{1}{u} - \frac{1}{u^2} + c$
3 (a) $-\frac{1}{u} - \frac{1}{2u^2} + c$ (b) $3r^3 - 6r^2 + 4r + c$ (c) $p^3 - \frac{11p^2}{2} + 6p + c$ (d) $x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4} + c$
(e) $\frac{2t^2}{3} - t + \frac{1}{3t} + c$ (f) $\frac{s^3}{3} - s - \frac{1}{4s} + c$ (g) $2s - \frac{5s^2}{2} - 4s^3 + c$ (h) $\frac{x^3}{12} + \frac{x^2}{4} + \frac{x}{4} + c$

(i) $\frac{x^4}{3} - \frac{x^2}{4} + c$ (j) $y - 2y^2 + \frac{4y^3}{3} + c$ (k) $\frac{t^4}{2} - 2t^2 + \frac{t}{3} + c$ (l) $\frac{4x^3}{3} + 6x - \frac{9}{4x} + c$
(m) $2t - \frac{1}{t} + \frac{2}{3t^3} + c$ (n) $\frac{3p^4}{2} + \frac{5p^3}{3} - 3p^2 + c$

EXERCISE 11.2

1 $y = x^2 - x + 4$ **2** (a) $y = \frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{6}$ (b) max at $(0, \frac{1}{6})$, min at (1,0) **3** $1\frac{1}{2}$ **4** $9\frac{1}{2}$
5 $s = \frac{2t^3}{3} - \frac{3t^2}{2} + \frac{29}{6}$; 1 m s^{-2} **6** $y = \frac{x^3}{3} + \frac{x^2}{2} + 3x + \frac{35}{6}$ **7** $y = \frac{3x^2}{2} - 2x + 4$
8 (a) $y = \frac{x^3}{3} - 2x^2 + 3x - 1$ (b) min at (3, –1), max at $(-1, -6\frac{1}{3})$ **9** –3; $-\frac{1}{6}$ **10** $6\frac{1}{4}$
11 53 **12** 17 **13** (a) $y = x^3 - x^2 - x + 1$ (b) max at $(-\frac{1}{3}, \frac{32}{27})$, min at (1,0) (c) $x < \frac{1}{3}$
14 (a) 1 s 3 s (b) 10 m (c) 2 s **15** (a) 4 s (b) 0 m s^{-1}, –4 m s^{-2} (c) –9 m
16 (a) $s = t - \frac{9}{t} + 6$ (b) 8 m **17** –4; $y = \frac{2t^3}{3} + \frac{3t^2}{2} - 5t + 2$ **18** (a) 18 m s^{-1}
(b) $26\frac{2}{3}$ m **19** $u = 3t - \frac{3t^2}{2} + 4$

EXERCISE 11.3

1 (a) 3 (b) $-\frac{1}{2}$ (c) 3 (d) 6 (e) $2\frac{1}{2}$ (f) 0 (g) $\frac{2}{3}$ (h) $-12\frac{2}{3}$ (i) $13\frac{1}{2}$ (j) $b - a$
(k) $\frac{7t^3}{3} - 3t$ (l) $-\frac{2}{3}$ (m) –2 (n) $\frac{17}{72}$ (o) 7 (p) 20 (q) $4\frac{1}{2}$ (r) 28 (s) $8\frac{5}{6}$ (t) 1
(u) $-\frac{16}{3}$ **2** –2, 10 **3** –3, 4 **4** $1\frac{1}{2}$ **5** –2 **6** 2 **7** (a) $5\frac{1}{3}$ units^2 (b) $4\frac{1}{2}$ units^2
(c) $\frac{7}{6}$ units^2 (d) $10\frac{2}{3}$ units^2 (e) $3\frac{1}{3}$ units^2 (f) 3 units^2 (g) 8 units^2 (h) $8\frac{2}{3}$ units^2
8 $\frac{1}{3}$ units^2 **9** (i) $8\frac{2}{3}$ units^2 (ii) $17\frac{1}{3}$ units^2 [curve is symmetrical about y-axis] **10** 4
11 4 **12** $23\frac{1}{3}$ **13** $1\frac{5}{6}$ **14** $9\frac{1}{3}$ **15** $20\frac{2}{3}$ **16** (a) 2, –3, –2 (b) $3\frac{1}{6}$units^2 **17** (a) 6
(b)(i) 21 (ii) 3 (iii) 3 **18** $v = t^2 - 4t + 3$; 1 s, 3 s; $-\frac{4}{3}$ m (i.e. $\frac{4}{3}$ m in negative direction)
19 (a) (1,3) (b) 13 units^2 **20** (a) (1,3), (2,0), (4,0) (b) $7\frac{1}{3}$ units^2 **21** (a)(i) (0,4)
(ii) 2 (c) 8 units^2

EXERCISE 11.4

1 18 **2** $5\frac{1}{6}$ units^2 **3** (–2,5), (3,5); $20\frac{5}{6}$ units^2 **4** 1, 2; $\frac{1}{2}$ units^2 **5** (a) $\frac{4}{3}$ (b) $10\frac{2}{3}$ units^2
(c) $5\frac{1}{3}$ units^2 (d) $\frac{1}{12}$ units^2 (e) $\frac{9}{8}$ units^2 (f) $\frac{1}{6}$ units^2 (g) $10\frac{2}{3}$ units^2 (h) $4\frac{1}{2}$ units^2
(i) $4\frac{1}{2}$ units^2 **6** (a) (0,–1), (0,3) (b)(i) $10\frac{2}{3}$ units^2 (ii) $2\frac{1}{3}$ units^2 **7** (a) $y = x^2 - 4x + 3$
(b) (0,3), (5,8) (c) $20\frac{5}{6}$ units^2 **8** (b) $3y + x = 0$, $\frac{10}{3}$ (c) $\frac{500}{81}$ units^2 **9** (a) (–2,0)
(b) $1\frac{1}{3}$ units^2 **10** (a) (1,0) (b) $y = 2x - 2$ (c) 1:1 [both $\frac{1}{2}$] **11** (a) (1,1) (b) 7:25
12 (b) $y = 4x$, $y = -2x + 9$ (c) $1\frac{1}{2}$; $2\frac{1}{4}$ units^2 **13** (a) (–1,4), (3,8); $y = x + 5$
(b) $10\frac{2}{3}$ units^2 **14** (a) –4, 1 (b) $20\frac{5}{6}$ units^2 **15** $11\frac{1}{3}$ units^2

EXERCISE 11.5

1 (a) $\frac{\pi}{5}$ units^3 (b) $\frac{16\pi}{15}$ units^3 (c) $\frac{\pi}{2}$ units^3 (d) $\frac{5\pi}{6}$ units^3 (e) $\frac{15\pi}{2}$ units^3 (f) $\frac{5\pi}{3}$ units^3
2 (a) $\frac{206\pi}{15}$ units^3 (b) $\frac{243\pi}{5}$ units^3 (c) $\frac{\pi}{30}$ units^3 (d) $\frac{2\pi}{3}$ units^3 (e) 8π units^3 **3** $\frac{15\pi}{2}$ units^3
4 $\frac{16\pi}{15}$ units^3 **5** (a) (0,0), (2,4) (b) $\frac{64\pi}{15}$ units^3 **6** $\frac{28\pi}{15}$ units^3 **7** 107 units^3
8 (a) $x^2 + y^2 = 4$, circle (b) $\frac{32\pi}{3}$ units^3 **9** (a) (–3,0), (3,0) (b) 16π units^3 **10** (a) $y = 2x$
(b) $\frac{32\pi}{5}$ units^3 (c) $\frac{416\pi}{15}$ units^3 **11** $\frac{88\pi}{5}$ units^3 **12** $\frac{8\pi}{3}$ units^3 **13** (a) (1,2) (b) 2:1
14 (a) (1,1) (b) $\frac{7\pi}{6}$ units^3 **15** $\frac{16\sqrt{2} - 14}{5}\pi$ units^3 **16** (a) (1,3), (2,0) (b) $\frac{98\pi}{15}$ units^3
17 (a) (1,0), $(2, 1\frac{1}{2})$ (b) $\frac{11\pi}{3}$ units^3 **19** (a) $y = \frac{rx}{h}$ (b) cone (c) $\frac{1}{3}\pi r^2 h$ units^3
20 (a) $y = -x + 2$; (2,0) (b) $\frac{\pi}{3}$ units^3 **21** (a) $y = 4x - 2$ (b) $(\frac{1}{2}, 0)$ (c) $\frac{106\pi}{15}$ units^3

REVISION EXERCISE 11

1 -4, $y = \frac{x^2}{2} - 4x + 11$ **2** (a) -2 (b) $\frac{7}{8}$ **3** 1:7 **4** (i) 20 m s^{-1} (ii) 32 m
5 (i) $(-1,5)$, $(2,2)$ (ii) 4:3 **6** (a) (2,9) (b) $y = \frac{9x}{2}$ (c) 27 units^2 **7** 1:7
8 8:5 [volumes are $\frac{64\pi}{15}$ about y-axis, $\frac{8\pi}{3}$ about x-axis] **9** (a) (1,4), (4,1) (b) 9π units^3
10 4, -3; 4 m **11** (a) $-\frac{1}{x} + \frac{1}{2x^2} + c$ (b) $\frac{\pi}{6}$ units^3 **12** 5; $9\frac{1}{3}$ **13** $\frac{5\pi}{3}$ units^3 **14** 3
15 (a) (1,0) (b) $\frac{16\pi}{15}$ units^3 **16** (a) (0,4), (2,0), (1,3) (b) 15:28
17 (a) Area above x-axis = area below (b)(i) 18 (ii) 2 (iii) 4 **18** (a) (12,6)
(b) 36 units^2 [Use $\int x\,dy$] **19** (i) $4y = 5x - 3$ (ii) $(\frac{3}{5}, 0)$ (iii) (1,0) (iv) $\frac{9}{40}$ units^2
(v) 9:49 **20** (i) [area of P = $\frac{4a^3}{3}$, area of Q = $\frac{2a^3}{3}$] (ii) [Both = πa^4] **21** 0; 3, -1
22 (a) 12 (b) 2 **23** $\frac{9}{7}$ **24** $\frac{8}{3}$ **25** (a) $\frac{\pi}{2}$ units^3 (b) π units^3 **27** $\frac{5}{4}$
28 [Fig. A11.1; gradients both $-\frac{1}{2}$] $\frac{37\pi}{3}$ [parts give $\frac{13\pi}{4}$, $\frac{61\pi}{12}$ and 4π]

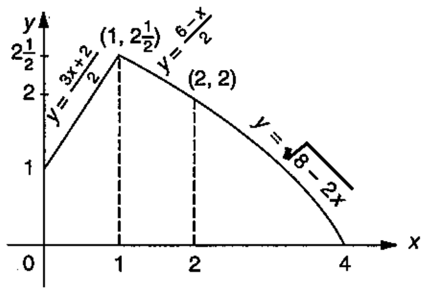


Fig. A.11.1

PAPER 1

1 (a) $2x - \frac{1}{2x^3}$ (b) 9 **2** $x = 2, y = -1$ or $x = -9, y = -\frac{25}{3}$ **4** (a) $x + 4y = 7$
(b) $y = 4x - 18$ **5** [Fig. AR.1] $x < -\frac{1}{2}$ **6** (a) $(-1,3)$ (b) 77.5°
7 (b) 103.9°, 166.1°, 283.9°, 346.1° **8** $4\frac{5}{6}$ **9** (a) 1.5 (b) 14 cm (c) 4.02 cm^2
10 $-\sqrt{5} \le k \le \sqrt{5}$ (b) $6 + 4x - 2x^2$

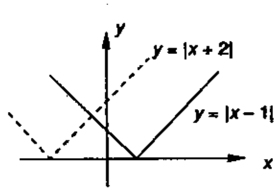


Fig. AR.1

PAPER 2

1 (a) 2 rad (b) 4 cm **2** (a) 101.8°, 168.2°, 281.8°, 348.2° (b) 45.5°, 254.5°
3 (a) -6 m s^{-2} (b) 4 m **4** (a) 6048 (b) 2, -1 **5** (4,3), $(-16,-37)$ **6** $\frac{62}{3}$units^2
7 2, -3; $2y + 25x = 28$ **8** 9% **9** (a)(i) 5 (ii) -1, 4
(b) $-\frac{5}{2}$, 4[$g^{-1}: x \longrightarrow \frac{3}{a-x}$] **10** (a) $\mathbf{i} + 7\mathbf{j}$ (b) 109.4°

PAPER 3

1 (a) $x < -1$ or $x > 3$; min at $x = 3$, max at $x = -1$ (b) $\left(\frac{a+b}{2}\right)^2$
2 $x = 2, y = 3$ or $x = \frac{18}{7}, y = \frac{19}{7}$ **3** 3.2 cm s^{-1} **4** (a) ± 2 (b) -1 **5** (a) $\frac{48\pi}{5}$ units^3
(b) $4\frac{4}{5}\pi$ units^3 **6** (a) $3y = x - 4$ (b) 6 **7** (a) $1 + 12x + 54x^2$, $16 - 32x + 24x^2$; 504
(b) $-1 \le y \le 1$ or $4 \le y \le 6$ **8** (a) max at $(-1,1)$, min at $\left(-\frac{1}{3}, \frac{23}{27}\right)$
(b) $-\frac{1}{t^2} - t^2$; $\frac{51}{26}$ ($\approx$ 1.95)% increase **9** $\frac{m}{2}(\mathbf{a} + \mathbf{b})$, $k\mathbf{a} + \left(\frac{1-k}{3}\right)\mathbf{b}$; (a) $\frac{1}{4}, \frac{1}{2}$ (b) 1:1, 1:3
10 (a) 70.5°, 180°, 289.5° (b)(i) 4 s (ii) -4 m s^{-1} (iii) -9 m

PAPER 4

1 (a)(i) $2(2x-1)(x^2 - x + 1)$ (ii) $1 - \frac{2}{(2-x)^2}$ (b) [Fig.AR.2] 4 **2** (a) $\frac{1}{2}$ (b) 221.0°
3 (a) $y = x - \frac{x^2}{2} - \frac{2x^3}{3} + 2$ (b) min at $\left(-1, \frac{7}{6}\right)$, max at $\left(\frac{1}{2}, \frac{55}{24}\right)$ (c) $6y + 12x = 23$
4 2:13 **5** (a)(i) $0 \le f(x) \le 4$ (ii) $-5 \le f(x) \le 4$ (iii) $-1 \le f(x) \le 3$
(b)(i) $\sqrt{13}$ (ii) 5 (iii) 0 **6** (a)(i) $-3 < x < 2$ (ii) $6\frac{1}{4}$ (b) $\frac{x}{4x+9}, \frac{x}{13x+27}$ (c) 1
7 (a) $\mathbf{b} - \mathbf{a}$, $\frac{2}{3}(\mathbf{b} - \mathbf{a})$, $\frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$, $\mathbf{b} - \frac{2}{3}\mathbf{a}$ (c) $\frac{4}{7}, \frac{6}{7}$ (d) 6:1, 4:3 **8** (a) ± 12 (b)(i) (5,12)
(ii) $y + 7x = 47$ **9** [Fig.AR.3] (a) $4\frac{1}{2}$ (b) (4,2), (2,4); [OA = AB = $\sqrt{20}$]; 24
10 $\frac{ax(h-x)}{h}$; $\frac{ah}{4}$

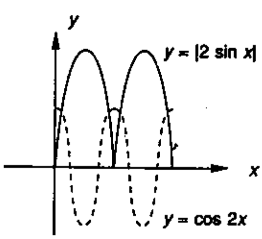


Fig. AR.2

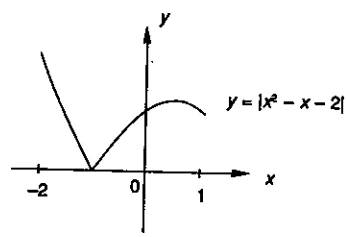


Fig. AR.3

PAPER 5

1 4 cm **2** $\frac{302\pi}{3}$ units^3 **3** (a)(i) $1 - 8x + 24x^2$ (ii) $32 - 80x + 80x^2$; 1488 (b)(i) 2, 3
(ii) $-4, 2\frac{1}{2}$ **4** (a) 0.50 units^2 (b) 2.26 cm^3 s^{-1} (c) 2 **5** (a)(i) 4 (ii) 14 (iii) 7
(b) [Fig.AR.4] 5 **6** (a)(i) 5 (ii) $k \le 5$ (b)(i) [$|\vec{OA}| = \sqrt{5}$, $|\vec{OB}| = \sqrt{20} = 2\sqrt{5}$]
(ii) [Fig.AR.5] [$\vec{OC} = 6\mathbf{i} - 5\mathbf{j}$; $\vec{OD} = -2\mathbf{i} + 3\mathbf{j}$] 163.5° **7** $PQ^2 = x^2 - 2x + 2$; $\frac{4}{3} : \frac{2}{\sqrt{3}}$ ($\approx$ 1.15)
8 (a) $\frac{7}{3}$ [Fig.AR.6] (b) $y^2 = 3x^2 + 18x + 15$; $(-1,0)$, $(-5,0)$ **9** $\approx$ 1.5, 5
10 (a) (ii) 9π units^3 (b)(i) 20.7°, 159.3°, 200.7°, 339.3° (ii) 30°, 150°

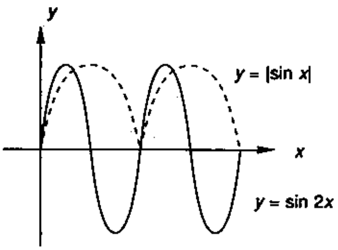


Fig. AR.4

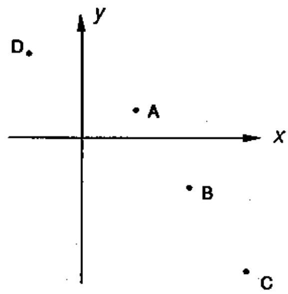


Fig. AR.5

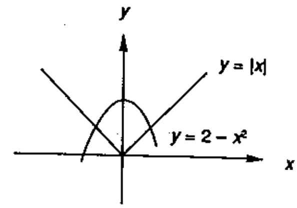


Fig. AR.6

EXERCISE 12.1

1 (a) 1 (b) 1 (c) 41 (d) -29 (e) $-\frac{7}{8}$ (f) $\frac{7}{27}$ (g) $5\frac{3}{8}$ (h) $-t^3 + 2t^2 + t - 1$
2 (a) 15 (b) 60 (c) 171 **3** (a) -15 (b) $-\frac{19}{27}$ (c) -18 (d) $3\frac{7}{9}$ (e) $-\frac{1}{2}$ (f) 33
4 -21 **5** $\frac{55}{27}$ **6** $-a + b - c + d$ **7** $9p - 53$ **8** 1 **9** -10 **10** $2, -4$ **11** $p - 2q = 6$; $4, -1$
12 $-2, -2$ **13** 3, 1 or -1, 5 **14** $-3, -2$ **15** $5, -4$ **16** $2, -4$ **17** $-3, 2$

EXERCISE 12.2

1 (a) $(x + 1)(x^2 - x + 1)$ (b) $(x - 1)(x - 1)(x - 2)$ (c) $(x + 1)(x - 2)(x - 3)$
(d) $(x + 1)(x + 2)(x + 3)$ (e) $(x - 1)(x^2 + 2)$ (f) $(x + 1)(x - 2)(x + 4)$
(g) $(x + 1)(x + 1)(2x + 3)$ (h) $(x - 1)(x + 1)(3x + 2)$ (i) $(x - 1)(x^2 + x + 1)$
(j) $(x - 2)(x + 3)(x - 3)$ (k) $(x + 2)(x - 2)(2x - 3)$ (l) $(x - 1)(2x - 1)(3x - 2)$
2 -7; $(x - 1)(x + 3)(x - 2)$ **4** $-1, -2$; $x - 1$ **5** -6; $x - 2, x - 1$
6 $-4, -3$; $(x + 1)(x^2 - x - 3)$ **7** $-4, 2$; $(x - 2)(x^2 - 2x - 2)$
8 $-4, 6$; $(x - 3)(x - 2)(x + 1)$ **9** $-1, 1$; $x + 1, x - 1$ **10** $-3, -2$; $(x - 2)(x^2 - x - 4)$ **11** $-2, 0$
12 2 **13** 3; $x - 1$ **14** $x + 1, x + 2$ **15** 3, 1; $3x^3 - 8x^2 + 15x + 26$; $(x + 1)(3x^2 - 11x + 26)$

EXERCISE 12.3

1 (a) $1, -1, -1$ (b) $1, -1, -2$ (c) $-1\ -3, -2$ (d) 1, 1, 2 (e) -3 (f) $-1, -3, 4$
(g) 2, 3, 4 (h) $-1, 2.79, -1.79$ (i) $-1, 2, 3$ (j) $2, -\frac{1}{2}, \frac{3}{2}$ (k) $2, \frac{1}{2}, -3$
2 $-2, -11$; $x - 4$; $1, -3, 4$ **3** $-1, -5$; $2x + 1, x + 1$; $-1, -\frac{1}{2}, 2$ **4** (a) $1, -2, -7$ (b) $3, -2, -5$
(c) $3, -4, -2$ (d) $4, -3, 0$ (e) $2, -5, 6$ (f) $2, -1, -3$ **5** (a) $1, -3, 2$; $3, -1, 4$
(b) $\frac{1}{2}, 2, 3$ **6** $4, -11$; $1, 1, -6$ **7** (a) $-3, -4$; $2, -2, 3$ (b) -2; $1, -1, 2$ **8** $\frac{1}{2}, 1, -2$
9 $(-1,-1)$, $(-2,-8)$, $(3,27)$ **10** [Values of B inconsistent] -6 [A = 2, B = 3, C = -4] **11** 1, -2, -2; double value at $x = -2$ **12** $x = 1$, min; $x = 2$, max; $x = 3$, min **13** (a) $y = 3x + 3$
(b) (2,9) (c) $6\frac{3}{4}$ **14** (a) $y = 3x - 2$ (b) $(-2,-8)$ (c) $6\frac{3}{4}$ units^2

REVISION EXERCISE 12

1 3, 2; 23 **2** (a) $\frac{1}{6}$ (b) $[b = \frac{2}{p}]$ **3** (a) $-2p - 38$; $p = -19$ (b) 2, 2, –4 (c) 1
4 1, 2, 3 **5** (a) –3 (b) $-1, -\frac{1}{2}, 2$ (c) $3\frac{1}{2}, 1\frac{1}{2}$ **6** (a) 12; –2, 2, 3 (b) –2, –0.78, 1.28
(c) 4, –3, –2 **7** (a)(i) 3, –6 (ii) Yes for first expression (b)(i) –33 (ii) 2 **8** 2, –3, 4
9 $y = x^3 - 6x^2 + 12x - 7$; (1, 0) **10** 1, –1, 2 **11** (a) –16 (b) 4 (c)(ii) –2, 6, 3
12 (a) $y = 2x + 1$ (ii) (–2,–3) (iii) $6\frac{3}{4}$ **13** (–1,1), (1,3), (2,10); P $2\frac{2}{3}$, Q $\frac{5}{12}$
14 1, 1, –2; –1, –1, –4 [equation is $(x + 2)^2 - 3(x + 2) + 2 = 0$] **15** [Find f(3)]
16 1, 2, –3 **17** 0°, 60°, 109.5°, 250.5°, 300°, 360° **18** $(x - 1)(x^2 - 2px - q)$ **19** $b = \frac{3a-2}{3}$
21 $x < -1$ and $2 < x < 4$; [equivalent to $x^3 - 5x^2 + 2x + 8 = (x + 1)(x - 2)(x - 4) < 0$]

EXERCISE 13.1

1 (a) 2, 4; 26, 58 (b) –3, 3; 15, 39 (c) $\frac{1}{3}, \frac{1}{6}; 1\frac{1}{3}, 2\frac{2}{3}$ (d) 1.7, –0.3; –0.1, –2.5
2 (a) 19 (b) 31 (c) $3n - 5$ **3** (a) –15 (b) –59 (c) 14th (d) $21 - 4n$
4 (a) $3n - 13$ (b) 2, –3 **5** (a) 5 (b) 61 **6** $2\frac{1}{2}$ **7** (a) $2\frac{1}{2}$ (b) $\frac{1}{6}(2n + 1)$ **8** 61
9 $14\frac{1}{2}$; $\frac{1}{4}(3n + 7)$ **10** (a) –4, 3 (b) $3n - 7$ **11** (a) 22 cm (b) $3n + 1$ cm
12 (a)(i) –37 (ii) 102 (b) –2 (c) –5 **13** (a)(i) $8\frac{1}{2}$ (ii) $18\frac{1}{2}$ (b) 2 **14** 14, –1
15 $2\frac{1}{3}$ **16** 6, 4 **17** –18, $4\frac{1}{2}$ **18** 5, 3 **19** $\frac{1}{2}(7 - n)$ **20** –3, 2, 7 or 7, 2, –3
21 $q - d, q + d$; 12, 7, 2 **22** (a) and (d) **23** (a) $a + 2d = 0$ **24** 4 **25** –1 or $1\frac{1}{2}$

EXERCISE 13.2

1 (a) –300 (b) 330 (c) $-17\frac{1}{2}$ (d) 3.3 (e) 104 (f) –710 (g) $45\frac{1}{2}$ **2** $8\frac{2}{3}$
3 –555 **4** 31 **5** 644 **6** 8 **7** 10 **8** 30; 1710 **9** $-3\frac{2}{5}, -1\frac{1}{5}, 1$
10 [$T_8 = a + 7d = S_8 = 4(2a + 7d)$ gives $a + 3d = 0$] **11** $1\frac{1}{2}$; $127\frac{1}{2}$ **12** $-\frac{1}{3}$; 208 **13** 7; 15
14 19:5 **15** 6 **16** 11 cm **17** (a) 43 cm (b) 0 [$a = 45, d = -3$] **18** 54 **19** (a) 2
(b) 12 (c) 25 **20** (a) 9 (b) 9 **21** 57:7

EXERCISE 13.3

1 (a) $\frac{5}{16}$ (b) $\frac{640}{81}$ (c) $\frac{32}{3}$ **2** 2^{n-7} **3** $\frac{243}{2}, \frac{1}{3}$ **4** $\frac{1}{8}, 2$ **5** $-\frac{27}{8}$ **6** (b), (d), (e), (f), (h)
7 (a) $\frac{3}{2}$ (b) $\frac{16}{27}$ (c) $\frac{4}{3}$ **8** (a) $\frac{2}{3}$ (b) $\frac{1}{12}$ (c) $\frac{1}{27}$ **9** ak, ak^3; 2 **10** 12, 18 **11** 27, 3
12 (a) $-1\frac{1}{2}$ (b) $\frac{81}{2}$ **14** (a) $\frac{3}{2}$ or 3 (b) –1 or 2 **15** 18 **16** 5th **17** (a) $\frac{1}{2}$ or $-\frac{3}{2}$
(b) $\frac{1}{4}$ or $\frac{81}{4}$ **18** (a) $-\frac{4}{7}$ or $\frac{4}{3}$ (b) $-\frac{343}{54}$ or $\frac{1}{2}$ **19** $\frac{2}{3}$ **20** 9 **21** 26 cm **22** 2, –3

EXERCISE 13.4

1 $\frac{1365}{16} \approx 85.3$ **2** 3.97 **3** (a) $-\frac{2}{3}$ (b) 243 (c) 133 **4** (a) 2 (c) 3 **5** (a) 8 (b) $\frac{1}{2}$
(c) $\frac{63}{4}$ **6** (a) 15 (b) $\frac{3}{4}$ (c) $\frac{1024}{27}$ (d) 116 **7** (a) 3 (b) 510 **8** (a) $\pm\frac{1}{3}$ (b) 81
(c) $\frac{364}{3}, \frac{182}{3}$ cm **9** 71 cm **10** 8 **11** (a) 2 (b) 252 **12** 27, 18, 12 or –27, 18, –12;
$105\frac{1}{2}$ or $27\frac{1}{2}$ **13** 467 cm **14** (a) $\frac{75}{8}$ (b) $\frac{5}{8}$ (c) 5 (d) $\frac{1}{2}$ **15** (a) $2^8 = 256$ (b) $\frac{1}{2}$
(c) 511 **16** 9 **17** 18; 2 400 000 barrels

EXERCISE 13.5

1 (a) $\frac{8}{3}$ (b) 36 (c) not possible (d) 216 (e) not possible (f) $6\frac{2}{3}$ (g) 36 (h) $\frac{64}{5}$ (i) not possible (j) 2 **2** $\frac{7}{8}$ **3** 486 cm **4** $\frac{20}{9}$ **5** 2 **6** 60 **7** (a) 6 or 3 (b) $\frac{1}{3}$ or $\frac{2}{3}$
8 $\frac{3}{4}$ **9** $\frac{255}{128}$; (a) $\frac{1}{128}$ (b) 0.39% (c) 11 **10** $\frac{1}{2}$ **11** $\frac{1}{3}$ **12** 12 or 6; $\frac{1}{3}$ or $\frac{2}{3}$ **13** $\frac{a-1}{a}$
14 (a) $-\frac{1}{5}$ (b) $\frac{1}{2}$ (c) $\frac{8}{5}$ **15** 5; 27 **16** 360° **17** (a) $\frac{1}{3}$ (b) 24 (c) 36
18 (a) $-\frac{1}{4}$ or $\frac{1}{3}$ (b) $\frac{2048}{405}$ or 3 **19** $\frac{1}{2}$ **20** (a) 10 055 (b) 30 000 **21** (a) 1 300 000 (b) 920 000 (c) 29% **22** 1 **23** (a) $\frac{1}{4}$[areas are 1, $\frac{1}{4}$, $\frac{1}{16}$, ... times ΔABC] (b) $\frac{1}{3}$
24 (a) $-\frac{1}{2}$, $\frac{1}{4}$ (b) $\frac{2}{3}$ **25** (a) $\frac{a^2}{1-r^2}$ (b) $a = 1 + r$

REVISION EXERCISE 13

1 (a) 14 (b) 370 **2** −8, 3 **3** 2, 2 **4** 8 **5** 1392 **6** 7, −3 or 1, 3 **7** $\frac{3}{7}$, $\frac{3}{2}$ **8** (a) 4 (b) $\frac{2}{3}$ **9** (a) 116 (b) 3, $\frac{1}{3}$ or $\frac{2}{3}$, −2; $4\frac{1}{2}$ **10** (a)(i) $\frac{3}{2}$ (ii) 20 (iii) $30\frac{1}{2}$ (b)(i) $\frac{1}{3}$ (ii) $\frac{1}{4}$ (iii) $\frac{64}{9}$ **11** (a) 15.2 cm (b)(i) $\frac{2}{3}$ (ii) 54 **12** 6, −3 **13** (i) 20 000 (ii) 17 (iii) 16 665 (iv) 16.7% **14** 89.3%, $n = 18$ **15** −1, ±2 **16** 4, 9; $1\frac{1}{2}$
17 1, 3, 5, ... **18** (a) 0.905 (b) 0.786 **19** −1, 4, 12 **20** $\frac{1}{3}$ **21** $1 < x < 2$ or $-1 < x < 0$
22 [In Fig. A13.1, $\tan\theta = \frac{ar-a}{a} = r - 1$, $\tan\theta = \frac{ar^2-ar}{ar} = r - 1$] (b) 22.5°
23 16 [$A(1.05)^{n-1} = 2A$ so $1.05^n = 2.1$] **24** 11

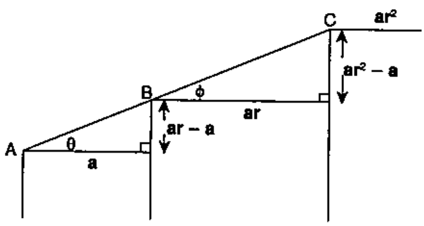


Fig. A13.1

EXERCISE 14.1

1 (a) 2 sin A sin B (b) 2 cos A sin B **3** (a) $\frac{56}{65}$ (b) $\frac{63}{65}$ (c) $\frac{56}{33}$ **4** 0 [= cos $\frac{\pi}{2}$]
5 $\cos(40° + x)$; 5.6°, 274.4° **6** (a) $-\frac{21}{221}$ (b) $-\frac{140}{221}$ (c) $-\frac{220}{21}$ **7** $\frac{1}{5}$ **8** $\frac{5}{4}$; $-\frac{17}{30}$
9 [Find tan(A + B)] **10** 0.7071 [= sin $\frac{\pi}{4}$] **11** $\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$; $\frac{b-a}{1+ab}$; $\frac{a^2b+2a-b}{1+2ab-a^2}$
12 (a) 30°, 210° (b) 18.4°, 198.4° (c) 165.0°, 345.0° (d) 169.1°, 349.1° (e) 79.1°, 259.1° **13** (a) 40° (b) 80°, 260° **14** (a) 1 (b) $\frac{24}{25}$ (c) $\frac{7}{24}$ **15** (a) $\frac{84}{85}$ (b) $\frac{36}{77}$ (c) $\frac{85}{77}$ (d) $\frac{13}{84}$ **16** (a) $-\frac{135}{377}$ (b) $\frac{152}{377}$ (c) $-\frac{135}{352}$ (d) $\frac{152}{345}$
18 $-\frac{1}{6}$, $\frac{5}{6}$, 99.6°, 33.6°; 66.6°, 33.0° **19** [Expand and divide by cos θ]; 48.5°, 228.5°
21 (b) cos θ cos ø + sin θ sin ø [|**a**| = |**b**| = 1]

EXERCISE 14.2

2 (a) $\frac{120}{169}$ (b) $-\frac{119}{169}$ (c) $\frac{120}{119}$ (d) $\frac{3}{\sqrt{13}}$ **3** (a) $\frac{3}{5}$ (b) $\frac{24}{25}$ (c) $\frac{7}{25}$ (d) $\frac{24}{7}$ (e) $\frac{1}{\sqrt{10}}$ (f) $\frac{3}{\sqrt{10}}$ **4** (a) $\frac{5}{13}$ (b) $\frac{12}{13}$ (c) $\frac{120}{119}$ **5** (a) $\frac{\sqrt{8}}{3}$ (b) $\frac{2\sqrt{8}}{9}$ (c) $\frac{7}{9}$
6 (h) [$\cos(\frac{\pi}{4} - \theta) = \frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta$, then square] **7** $\sqrt{3}$ **8** (a) $\frac{1}{2}$ or -2 (b) $\frac{1}{\sqrt{5}}$ or $\frac{2}{\sqrt{5}}$
9 $\frac{3}{4}$ **10** (a) $\sqrt{1-p^2}$ (b) $\sqrt{\frac{1-p}{2}}$ (c) $\sqrt{\frac{1+p}{2}}$ (d) $2p^2 - 1$ (e) $2p\sqrt{1-p^2}$
(f) $8p^4 - 8p^2 + 1$ **11** (a) $\frac{1}{2}\sin 2A$; $s\sqrt{1-s^2}$ (b) $\cos 2A$; $1 - 2s^2$ (c) $\tan 2A$; $\frac{2s\sqrt{1-s^2}}{1-2s^2}$
(d) $\cos 2A$; $1 - 2s^2$ **12** $\frac{1}{2}\sin 70°$ (b) $\frac{1}{2}\tan 80°$ (c) $-\sin 10°$ (d) $\tan 10°$
(e) $\frac{1}{2}\cos 60°$ **13** (a) 0°, 60°, 300°, 360° (b) 15°, 105°, 195°, 285° (c) 143.2°
14 (a) 30°, 90°, 150°, 270° (b) 60°, 300° **15** 48.2°, 120°, 240°, 311.8°
16 72.0°, 144.0°, 216.0°, 288.0° **17** 15.3° **18** 30°, 150°

EXERCISE 14.3

1 (a) $5\cos(\theta - 36.9°)$ (b) $13\sin(\theta + 67.4°)$ (c) $\sqrt{13}\cos(\theta + 56.3°)$ (d) $3\sin(\theta - 19.5°)$
(e) $3\sin(\theta + 41.8°)$ **2** (a) 5, 36.9°; –5, 216.9° (b) 13, 22.6°; –13, 202.6°
(c) $\sqrt{13}$, 303.7°; $-\sqrt{13}$, 123.7° (d) 3, 109.5°; –3, 289.5° (e) 3, 48.2°; –3, 228.2°
3 $5\cos(2\theta + 53.1°)$; max at 153.4°, 333.4°; min at 63.4°, 243.4°
4 $3\sin(\frac{\theta}{2} + 28.1°)$; max at 123.7° **5** $3\sin(\theta + 41.8°)$; 48.2° [Fig. A14.1]

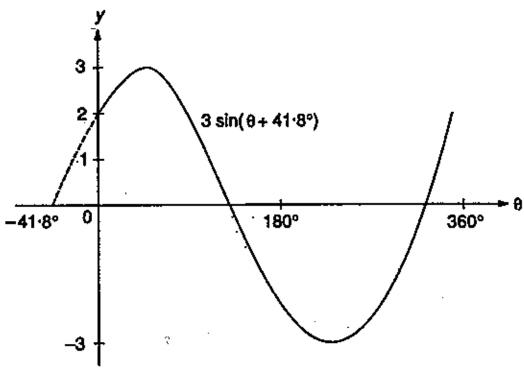


Fig. A14.1

EXERCISE 14.4

1 (a) 103.3°, 330.4° (b) 82.6°, 322.6° (c) 19.2°, 228.2° (d) 61.3°, 157.7°
(e) 118.7°, 337.7° **2** $2\sin(\frac{\theta}{2} + 30°)$; 0°, 240° **3** [$\sqrt{5}\cos(\theta - 63.4°)$] 36.9°, 90°
4 18.6°, 258.6° **5** $\sqrt{10}\cos(2x + 71.6°)$; 135°, 153.4°, 315°, 333.4° **6** 82.5°, 184.1°, 262.5°

REVISION EXERCISE 14

1 0°, 30°, 150°, 180°, 210°, 330° **3** [tan A = 1.5, tan B = –1] $\frac{1}{5}$; -5, $-\frac{12}{5}$ **4** 221.8°, 318.2°
5 20.9°, 69.1°, 200.9°, 249.1° **6** (i) $\frac{11}{3}$ (ii) $\frac{1}{\sqrt{26}}$ (iii) $\frac{12}{13}$ **7** $-\frac{1}{2}$; $\frac{\sqrt{3}}{2}$; $\frac{1}{2}$ **8** (a)(i) $-\frac{56}{65}$
(ii) $-\frac{7}{25}$ (iii) $-\frac{120}{169}$ (b) $\sqrt{10}$, 18.4°; 69.2°, 327.7° **10** (a) 41.6°, 244.7° (b)(i) $\frac{1}{\sqrt{1-c^2}}$

(ii) $\frac{c}{\sqrt{1-c^2}}$ (iii) $2c\sqrt{1-c^2}$ (iv) $\frac{c+\sqrt{1-c^2}}{c-\sqrt{1-c^2}}$ **11** 20°, 100°, 140° **12** (a) $-\frac{161}{289}$ (b) $-\frac{240}{289}$
13 (a) 103.7°, 309.5° (b) 7 tan A tan B = 1; $\frac{6}{7}$ **14** (a) max $\sqrt{3}$ at 41.8°, min $-\sqrt{3}$ at 221.8°
(b) 52.5°, 172.5°, 232.5°, 352.5° **15** (a) $3\sin(x-28.1°)$ [Fig. A14.2] (b) 69.9°, 166.3°

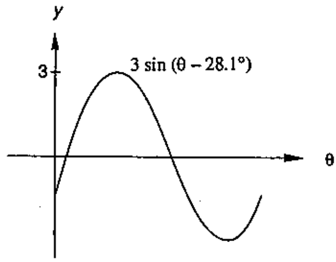


Fig. A14.2

16 26.6°, 153.4°, 206.6°, 333.4° **17** (a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ **18** (a) 2 (b) $\frac{5}{\sqrt{2}}$
(c) $\frac{3}{\sqrt{13}}$ (d) −8 **19** 1 **21** (a) $\pm x$ (b) $\frac{3x}{1-2x^2}$ (c) $\frac{x}{1+2x^2}$ (d) 63.4°, 45°
22 [Equation reduces to $\tan^2\theta + \tan\theta - 6 = 0$] 63.4°, 108.4°, 243.4°, 288.4°
23 [demominator of tan(A – B) must be 0] **24** (a) $\sqrt{2}$ (b) 54°, 160°
25 [∠TPR = 90° – θ, ∠PRT = 90°; ∠PSR = ∠PTR (angles in the same segment);
RS = $\frac{PQ}{\tan\theta} - PQ\tan\theta = PQ(\frac{1-\tan^2\theta}{\tan\theta})$] 22.5°
26 [area of sector = $\frac{1}{2}AE^2 \times 2\theta = \frac{1}{2}a^2$] 0.9 rad; 0.45 rad (≈ 25.5°)

EXERCISE 15.1

1 (a) 1 (b) $\frac{1}{16}$ (c) $-\frac{1}{27}$ (d) 2 (e) 27 (f) 4 (g) $\frac{1}{8}$ (h) $\frac{1}{x^2}$ (i) $-\frac{3}{2}$ **3** (a) 1
(b) 1 (c) 16

EXERCISE 15.2

1 (a) 6 (b) 3 (c) 0 (d) 2 (e) $-\frac{3}{4}$ (f) $-\frac{1}{2}$ (g) −3 (h) 0, 3 (i) 2, 1 (j) 1, − 1
(k) −1, 6 (l) −1 (m) 3 (n) 2, 0 (o) 1 (p) −2, 1 **3** −1, $\frac{1}{2}$ **4** 22, −14

EXERCISE 15.3

1 (a) 2.3 (b) 2.4 (c) 2.3 (d) 1.8 **3** [Gradients will equal the values of y at the points.]

EXERCISE 15.4

1 (a) $\log_4 16 = 2$ (b) $\log_3 81 = 3$ (c) $\log_{10} 1000 = 3$ (d) $\log_{10} 0.001 = -3$ (e) $\log_4 2 = \frac{1}{2}$
(f) $\log_x 2 = 3$ (g) $\log_7 21 = x$ (h) $\log_x 16 = -4$ (i) $\log_{10} 0.1 = -1$ (j) $\log_8 64 = 2$
(k) $\log_4 9 = x$ (l) $\log_x 0.3 = -3$ **2** (a) $2^x = 16$; 4 (b) $3^x = 27$; 3 (c) $4^x = 64$; 3
(d) $2^x = \frac{1}{8}$; −3 (e) $10^x = 0.001$; −3 (f) $64^x = 4$; $\frac{1}{3}$ (g) $7^x = \frac{1}{49}$; −2 (h) $5^x = 625$; 4
(i) $3^x = \frac{1}{27}$; − 3 (j) $13^x = 169$; 2 (k) $169^x = 13$; $\frac{1}{2}$ **3** (a) 1 (b) 2 (c) 0 (d) 0
(e) 2 (f) 5 (g) $\frac{1}{2}$ (h) 2 (i) −2 (j) $1\frac{1}{2}$ (k) $\frac{3}{4}$ (l) 2 (m) 1 (n) −1 (o) −2
4 (a) 3 (b) $\frac{1}{8}$ (c) 5 (d) 2 (e) 66 (f) 3 (g) 3 (h) $\frac{1}{3}$ (i) 5 (j) 83 (k) 2

EXERCISE 15.5

1 (a) 0.6 (b) 1.6 **2** (a) $\frac{1}{2}\log_7 2$ (b) 2 (c) $3\log_5 3$ (d) $2\log_2 5$ (e) $3\log_3 5$ (f) 4 (g) $2\log_6 11$ (h) 4 (i) 3 (j) 0 (k) $2\log_3 6 = 2 + 2\log_3 2$ (l) 2 (m) $\frac{5}{2}\log_5 3 + \frac{1}{2}\log_5 2$ (n) $\frac{3}{2}$ (o) $\frac{3}{2}$ (p) 2 (q) $3\log_3 5$ (r) x (s) $\frac{1}{2}\log_3 10$ (t) $1 + \log_5 2$ **3** (a) 2.727 (b) –0.203 (c) 0.203 (d) 4.192 (e) 3.786 (f) 3.989 (g) 1.668 (h) 2.465 (i) –1.262 **4** (a) 0.921 (b) 1.13 (c) 1.486 (d) 1.633 (e) 0.774 (f) –0.209 (g) 0.283 (h) 1.356 (i) 1.921 (j) –0.435 (k) 0.791 **5** 4 **6** (a) 5^{a+3b} (b) 5^{1+a-b} **7** (a) $2p + q$ (b) $\frac{1}{2}(p + q)$ (c) $3p - q$ (d) $3 + p - 2q$ **8** 49 **9** 100 **10** 2 **11** (a) 3^{3p+2q} (b) 3^{p-q} **12** (a) 100 (b) $\frac{144}{5}$ **13** $2^{2a+2b}, 2^{3a-2b}$; $\frac{2}{5}$, $\frac{21}{10}$ **14** (a) 28 (b) $\frac{80}{7}$ **15** $\frac{1}{2}(1 + a - 3b)$
16 $\frac{1}{5}(2p + q)$; $\frac{1}{5}(p - 2q)$; $\frac{1}{25}(2p + q)(p - 2q)$ **17** [$x = 4^p = 2^{2p}$] (a) $\frac{1}{16}$ (b) 64 **18** (a) 3 (b) 1 (c) 4, –1 (d) 3, 5 (e) $-\frac{1}{2}$, 3 (f) 8 (g) 1 (h) 5 **19** (a) [$a^x = b$ so $\log_b a^x = 1$] (b) 3; $\frac{1}{3}$ **20** [$2\log_a y = \log_a x + \log_a z$ so $y^2 = xz$]

EXERCISE 15.6

1 (a) 1.46 (b) 3.81 (c) 0.161 (d) 0.136 (e) 0.163 (f) 0.585 (g) 2.40 (h) 0.754 (i) 1.16 (j) 6.13 (k) 1.79 (l) –0.161 **2** 1.46, 1.21 **3** 1.07 **4** 9 **5** 8 **6** (a) 8 (b) 7 **7** 10.5 **8** 18.1 **9** (a) 15.8 (b) 5.59 **10** 2006 **11** 3.47 **11** 3.47 **12** 4 **13** 4 **14** 5 **15** $y = 10x^{-3}$; 10, –3 **16** [$\lg \frac{1}{10} = \lg 1 - \lg 10$] 5.7°, 174.3° **17** (a) 1.12 (b) 2.05 **18** 1.64 **19** $\frac{8}{3}$, 1.58

REVISION EXERCISE 15

1 (a) 32 (b) $\frac{1}{9}$ (c) 64 **2** (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) ±5 (d) 3, 2 (e) 3,2 (f) $\frac{1}{2}$ (g) 2.78 (h) 1.81 **3** $a^{\frac{2u-v}{5}}$ **4** (a) 4 (b) 64 (c) $2\sqrt{2} \approx 2.83$ **5** $\frac{9}{4}, \frac{3}{4}$ **6** (a) $\frac{1}{7}$ (b) $\frac{3}{2}$ (c) 0.166 **7** (a) 2 (b) 6 **8** 2, 1 **9** (a) 0.113 (b) 1 (c) $\frac{1}{2}(3 + 3a - b)$ **10** 4
11 (a) 1.10 (b) 3, 2 (c) $\frac{1}{25}(u + v)(2u - 3v)$ **12** (a) 5.68 (b) 3.85 (c) 1.55
13 2.20 **14** 15 **15** $\sqrt{a} = 9b$ **16** 11.5 **17** ≈ 5 **18** [line is $y = 2 - \frac{x}{2}$] 0.7
19 [Fig. A15.1; $\lg 10x = 1 + \lg x$] (1,0), (0,10) **20** (b) 2.58
(c) $\log_2 x = -x + \log_2 6 = -x + 2.6$; –1, 2.6 (d) $y = -x + 2.6$ (e) 1.8 [intersection of line in (d) and $y = \log_2 x$] **21** (a)(i) $\frac{1}{2}$ (ii) 20, –4 (b) 0.845, 0.2 **22** 84.3°, 95.7°, 264.3°, 275.7°
23 [Sequence is $\lg k$, $1 + \lg k$, $2 + \lg k$] $5(\lg k^2 + 9)$ **24** 64, 16 **25** 3, 2 or $-\frac{3}{4}, -\frac{1}{2}$
26 $\frac{2a + b}{1 + a + b}$ [$78^x = 52$ take logs base 3: then $x = \frac{\log_3 52}{\log_3 78} = \frac{\log_3(13 \times 2^2)}{\log_3(2 \times 3 \times 13)}$] **27** $x > 3$ [$\log_3 x > 1$]
28 [$q(p + 2) = 2^{3r-1}$, $\frac{p-2}{q} = 2^{2r+1}$; multiply] 6, $\frac{1}{2}$

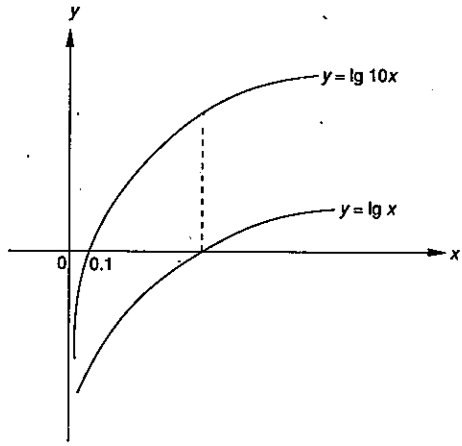


Fig. A15.1

EXERCISE 16.1 *(Answers from graphs are approximate)*

1 3.0, 0.22 **2** $y = 20x^3$ **3** 100, –0.4 **4** $\frac{y}{x} = \frac{a}{b}x + \frac{1}{b}$, $Y = \frac{y}{x}$, $X = x$; 4.6, 2 **5** 15, 2.8; 330
6 $y = \frac{3}{x+1} + 2$ **7** 2, 10 **8** $y\sqrt{x} = ax + b$; take $Y = y\sqrt{x}$, $X = x$; 2, 4.5
9 Plot Y against X where $Y = \frac{1}{y}$, $X = \frac{1}{x^2}$; 20, 80 **10** $y = \frac{p}{x^2} + q$; take $Y = y$, $X = \frac{1}{x^2}$; 30, –2

REVISION EXERCISE 16 *(Answers from graphs are approximate)*

1 Take $Y = y^2$, plot Y against x; $a = -1$, $b = 2$; 0.5 **2** (a) $y = \frac{16}{x^2} + 5$ (b) (i) $y = 10x^{-3}$
(c) convert to $\frac{y^2}{x} = ax - b$ and plot $\frac{y^2}{x}$ against x. **3** $\frac{y}{x} = -\frac{a}{b}x + \frac{1}{b}$; plot $Y = \frac{y}{x}$ against x
gradient $= -\frac{a}{b}$, intercept $= \frac{1}{b}$; $a = 2$, $b = -5$ **4** (a) $y = \frac{1}{q}xy - \frac{p}{q}$; plot y against $X = xy$.
Gradient $= \frac{1}{q}$, intercept $= -\frac{p}{q}$ (b) [$xy = ax^2 + b$; gradient $= a$, intercept $= b$]
(i) $a = -0.7$, $b = 7.5$ (ii) 5.4 **5** (a) 1.44, 0.55 (b) (i) ± 3 (ii) $y = \frac{1}{4}(3x^2 - 4x - 7)$
(iii) -1, $\frac{7}{3}$ **6** (a) 0.1, $\frac{1}{2}$ [$y = \frac{x^{\frac{1}{2}}}{10}$] 0.224 (b) $\frac{1}{y} = (-p)\frac{1}{x} + q$; plot $\frac{1}{y}$ against $\frac{1}{x}$, gradient $= -p$,
$\frac{1}{y}$ – intercept $= q$ **7** 5.7, 20 **8** (a) 2.5, 1 (b) [Convert to $y^2 = -\frac{q^2}{2p^2}x^2 + \frac{q^2}{2}$] 5, 3
9 [$\lg a + \lg y = b \lg(\sqrt{x} - 1)$; plot $Y = \lg y$ against $X = \lg(\sqrt{x} - 1)$]; 4, 0.7 **10** 1.5, 10

EXERCISE 17.1

1 (a) $-\frac{3}{4}x^{-\frac{7}{4}}$ (b) $2x^{-\frac{1}{2}}$ (c) $2(4x - 3)^{-\frac{1}{2}}$ (d) $x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$ (e) $(x^2 - 4x)(x^3 - 6x^2)^{-\frac{2}{3}}$ (f) $3x^{\frac{1}{2}}$
(g) $-2x^{-\frac{3}{2}}$ (h) $6x^2(4x^2 - 3)^{-\frac{1}{2}}$ (i) $(4x - 1)(1 - 2x + 4x^2)^{-\frac{1}{2}}$ **2** (a) $\frac{4}{7}x^{\frac{7}{4}} + c$ (b) $2x^{\frac{1}{2}} + c$
(c) $\frac{3}{5}x^{\frac{5}{3}} - \frac{3}{2}x^{\frac{2}{3}} + c$ (d) $\frac{3}{2}x^{\frac{1}{3}} + c$ (e) $\frac{3}{4}x^{\frac{4}{3}} - \frac{6}{7}x^{\frac{7}{6}} + c$ (f) $3x^{\frac{5}{3}} + c$ (g) $6x^{\frac{1}{2}} + c$ (h) $\frac{1}{2}x^{\frac{2}{3}} + c$
(i) $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + c$ **3** (a) $\frac{7}{3}$ (b) $-\frac{45}{4}$ (c) $\frac{5}{6}$ (d) 1 (e) 6 **4** (a) $\frac{(2x-3)^3}{6} + c$
(b) $\frac{1}{10}(2x + 5)^5 + c$ (c) $-\frac{1}{2}(x - 2)^{-2} + c$ (d) $\frac{2}{3}(x - 3)^{\frac{3}{2}} + c$ (e) $-\frac{1}{3}(3x - 2)^{-1} + c$
(f) $(2x + 3)^{\frac{1}{2}} + c$ (g) $(2x + 3)^{\frac{1}{2}} + c$ (h) $-\frac{1}{16}(3 - 4x)^4 + c$ (i) $-(3 - 2x)^{\frac{1}{2}} + c$
(j) $\frac{1}{15}(3x + 2)^5 + c$ (k) $\frac{3}{16}(4x - 1)^{\frac{4}{3}} + c$ (l) $(2x - 5)^{\frac{1}{2}} + c$ (m) $\frac{3}{16}(4x - 1)^{\frac{4}{3}} + c$

(n) $\frac{1}{2}(1-2x)^{-1}+c$ **5** (a) 1 (b) $\frac{112}{9}$ (c) 2 (d) 14 (e) $-\frac{4}{3}$ (f) $8\frac{2}{3}$ (g) 2 (h) $\frac{85}{4}$
6 $\frac{1}{8}$ **7** $\frac{\pi}{3}$ **8** 0.0075 **9** –0.024 **10** 0.24 **11** 2% **12** –0.06

EXERCISE 17.2

1 (a) $(x-2)(3x-2)$ (b) $2x(2x^2-1)$ (c) $x(5x^3+3x-2)$ (d) $(x+1)(x-2)^2(5x-1)$
(e) $x^4(1-2x)(5-14x)$ (f) $-(1-x)(3-x)^2(9-5x)$ (g) $x(x^2-x-1)^2(8x^2-5x-2)$
(h) $2x(x^2-3)^2(4x^2-3)$ (i) $2(3x-2)(12x^2-4x-3)$ (j) $2(x^2+1)(2x-1)^2(7x-2x+3)$
(k) $\frac{1}{2}x^{-\frac{1}{2}}(x^3-1)(13x^3-1)$ (l) $(\sqrt{x}-1)(2\sqrt{x}-1)$ (m) $2(1-2x)^2(1-8x)$
(n) $\frac{1}{2}(x-1)^{-\frac{1}{2}}(x+1)^3(9x-7)$ (o) $(x+1)^2(5x^2-2x-7)=(x+1)^3(5x-7)$
(p) $6(3x-1)(2x+3)^2(5x+2)$ **2** $y=5x-7$ **3** 2 **4** $(4x+1)(x+1)^2$; $6(x+1)(2x+1)$
5 $(2x-1)^3(10x-1)$; $16(2x-1)^2(5x-1)$ **6** 0, 1, $\frac{2}{5}$ **7** $2(x+1)(x+2)^2(3x^2+5x+1)$

EXERCISE 17.3

1 (a) $\frac{2}{(x+2)^2}$ (b) $\frac{1}{(x+2)^2}$ (c) $\frac{5}{(2x+1)^2}$ (d) $\frac{(-3x^2+4x+3)}{(x^2+1)^2}$ (e) $\frac{x^2-2x-2}{(x-1)^2}$ (f) $\frac{(-x^2+2x)}{(1-x)^2}$
(g) $\frac{(x-4)}{2(x-2)^{\frac{3}{2}}}$ (h) $\frac{2(x+1)}{(2x+1)^{\frac{3}{2}}}$ (i) $\frac{x^2(x^2+3)}{(x^2+1)^2}$ (j) $\frac{-2}{(x^2-2)^{\frac{3}{2}}}$ (k) $\frac{3x^2+2x}{(3x+1)^2}$ (l) $\frac{1}{(2-x)^2}$ (m) $\frac{3+8x-3x^2}{(x^2+1)^2}$
(n) $\frac{(1-2x-x^2)}{(x^2+1)^2}$ **2** $\frac{1}{(x+1)^2}$; $-2(x+1)^{-3}$ **3** $\frac{4}{9}$ **4** $\frac{3}{(2x+3)^2}$; $\frac{1}{35}$ **5** $\frac{1}{2x^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$; 2 **6** $\frac{3}{(x^2+3)^{\frac{3}{2}}}$; $\frac{1}{3}$
7 $\frac{-1}{(x+1)^{\frac{1}{2}}(x-1)^{\frac{3}{2}}}$ **8** 0, 4 **9** (a) 3 (b) 1 or –3 **10** $\frac{(x+2)}{2(x^2+x+1)^{\frac{3}{2}}}$; $x=-2$

EXERCISE 17.4

1 (a) $-\frac{4x+y}{x}$ (b) $-\frac{y}{x}$ (c) $\frac{-(2xy+2)}{x^2}$ (d) $\frac{-y^2}{(2xy+2)}$ (e) $\frac{(3-y)}{(x-2y)}$ (f) $\frac{-(1+y)}{(1+x)}$ (g) $-\frac{x}{y}$
(h) $\frac{(2x-y)}{(x+8y)}$ (i) $\frac{3x}{2y}$ (j) $\frac{-\sqrt{y}}{\sqrt{x}}$ (k) $\frac{1}{y}$ (l) $\frac{(2-2x)}{(2y+5)}$ (m) $\frac{(1-2xy-y^2)}{(x^2+2xy+1)}$ (n) $\frac{4x-2}{3(1+2y)}$ (o) $\frac{(1-y)}{(x+2y)}$
(p) $\frac{(x-y)}{(x+y)}$ (q) $\frac{9x^2-2y}{2(x+y)}$ (r) $\frac{x}{12y^2}$ (s) $\frac{\sqrt{y}}{\sqrt{x}}$ **2** (a) $-\frac{3}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) –4 (e) $\frac{2}{11}$ (f) 1
3 $y=x+1$; $x+y=3$ **4** $3y=x+2$, $3y+x=-5$ **5** $4y=x+14$ **6** $\pm\frac{1}{5}$ **7** 3, –1
8 $5y=4x+12$

EXERCISE 17.5

1 (a) $15x^4+\frac{4}{3}x^{-\frac{5}{3}}+\frac{1}{x^2}$ (b) $\frac{3}{2}x^{\frac{1}{2}}$ (c) $18x^2(2x^3-1)^2$ (d) $\frac{(1-x^2)}{(1+x^2)^2}$ (e) $(2x-2)(2x^2-4x+3)^{-\frac{1}{2}}$
(f) $\frac{(1-x)}{2\sqrt{x}(1+x)^2}$ (g) $-2(4x-3)^{-\frac{3}{2}}$ (h) $1+\frac{1}{2}x^{-\frac{3}{2}}$ (i) $2(x-\frac{2}{x})(1+\frac{2}{x^2})=2(x-\frac{4}{x^3})$ (j) $\frac{3}{(x+1)^2}$
(k) $\frac{(x-1)}{2x^{\frac{3}{2}}}$ (l) $(2x^3+1)(x^4+2x+1)^{-\frac{1}{2}}$ (m) $-\frac{3}{2}x^{-\frac{5}{2}}$ (n) $(x-1)^2(3-2x)(13-10x)$
(o) $\frac{1}{3}+\frac{10}{3}x^{-3}-3x^{-4}$ (p) $6(x-1)(x^2-2x-1)^2$ (q) $2(1-x)(2x^2-4x+5)^{-\frac{3}{2}}$
(r) $9x(3x^2-4)^{\frac{1}{2}}$ **2** (a) $4x^{\frac{1}{4}}+c$ (b) $\frac{1}{6}(4x+3)^{\frac{3}{2}}$ (c) $-\frac{1}{2}(1-4x)^{\frac{1}{2}}+c$ (d) $-\frac{1}{6}(3x+4)^{-2}$
(e) $(4x-3)^{\frac{1}{2}}+c$ **3** (a) $\frac{2}{3}$ (b) $\frac{62}{3}$ (c) $\frac{1}{21}$ **4** $x+4y=13$ **5** ± 2
6 $y+5x=12$, $y=8x-12$ **7** $\frac{1}{(1-2x)^2}$; $\frac{4}{15}$ **8** $\frac{x(x+4)}{(x+2)^2}$; $\frac{22}{15}$ **9** $\frac{-3x}{2\sqrt{1-x}}$; $\frac{3(x-2)}{4(1-x)^{\frac{3}{2}}}$; max at $x=0$
10 $15y=108x-422$ **12** $-\sqrt{2}$

EXERCISE 17.6

1 (a) $3\cos 3x$ (b) $\frac{1}{2}\cos\frac{x}{2}$ (c) $-\frac{1}{4}\sin\frac{x}{4}$ (d) $3\sec^2 3x$ (e) $-\cos x\operatorname{cosec}^2 x$
(f) $\sin x + x\cos x$ (g) $2x\cos 2x - 2x^2\sin 2x$ (h) $-4x\sin(2x^2-1)$ (i) $-\cos(\frac{\pi}{3}-x)$
(j) $\frac{1}{2}\sec^2\frac{x}{2}$ (k) $x\cos x$ (l) $\frac{1-2\sin x}{(2-\sin x)^2}$ (m) $-6\sin 2x\cos^2 2x$ (n) $\cos x - x\sin x - 2\cos 2x$
(o) $3\cos 3x\cos 2x - 2\sin 3x\sin 2x$ (p) $2\sin 2x\cos 2x(4+\sin^2 2x)^{-\frac{1}{2}}$
(q) $18x\cos^2(1-3x^2)\sin(1-3x^2)$ (r) $\sec^2 2x(\tan 2x)^{-\frac{1}{2}}$ **2** (a) $-3\sin 3x$ (b) $\frac{1}{3}\cos\frac{x}{3}$
(c) $-4x\sin(2x^2-1)$ (d) $6\sin^2 2x\cos 2x$ (e) $\frac{1}{3}\sec^2(\frac{x}{3}-2)$
(f) $\frac{1}{2}\cos\frac{x}{2}\cos 2x - 2\sin\frac{x}{2}\sin 2x$ (g) $\frac{-2\cos x}{(1+\sin x)^2}$ (h) $2x\tan\frac{x}{2}+\frac{1}{2}x^2\sec^2\frac{x}{2}$
(i) $-2x\sin x^2$ (j) $\cos 2x - \sin x - x(2\sin 2x + \cos x)$ **3** $2\cos 2x, -4\sin 2x\ (=4y)$ **4** $-\pi$
5 $[\frac{d^2y}{dx^2} = -4(A\cos 2x + B\sin 2x)]$; $-3, 2$ **6** 1.11, 4.25 **7** $2(\sin x+\cos 2x)(\cos x - 2\sin 2x)$
8 $\frac{\pi}{6}, \frac{5\pi}{6}$ (0.52, 2.62) **9** $\frac{1}{(1+\cos x)}$; 1 **10** (a) 2.03, 5.18 (b) $\frac{\pi}{6}, \frac{5\pi}{6}$; $\sqrt{3}, -\sqrt{3}$
11 $y = x, x+y=\pi$ **12** $x + y = \frac{x}{2}$

EXERCISE 17.7

1 (a) $-\frac{1}{2}\cos 2x + c$ (b) $\frac{1}{4}\sin 4x + c$ (c) $-2\cos\frac{x}{2}+c$ (d) $-\cos 3x + c$ (e) $\frac{1}{3}\tan 3x + c$
(f) $\frac{1}{2}\sin 2x + \cos x + c$ (g) $-\cos x + \sin x + c$ (h) $\frac{x+\sin x}{2}+c$ (i) $\frac{1}{5}\sin 5x + c$
(j) $\cos(\frac{\pi}{4}-x)+c$ (k) $2\tan\frac{x}{2}+c$ (l) $\frac{1}{2}\sin 2x + \cos x + c$ (m) $x + \frac{1}{2}\cos 2x + c$
(n) $-2\cos x - \frac{1}{4}\cos 2x + c$ **2** (a) 1 (b) 1 (c) $\frac{\pi}{4}$ (d) 1 (e) 0 (f) $1+\frac{\pi}{2}$ (g) 0.47
(h) $\frac{2}{3}$ (i) 2 (j) 0 (k) $\frac{(5\pi+3)}{24} = (0.78)$ **3** $-\frac{1}{\theta}+\frac{1}{4}\sin 2\theta + 1 + \frac{2}{\pi}$ **4** (a) 2 (b) $\frac{\pi^2}{2}$
5 $\frac{\sin x}{(1+\cos x)^2}$; $\frac{1}{2}$ **6** $2\pi^2$ **7** (a) $\frac{\pi}{4}$ (b) 0.59 (c) $\frac{\pi(\pi-2)}{4}$ **8** $\frac{1}{2}$; 1.50 **9** $\frac{x}{2}+\frac{1}{8}\sin 4x + c$
10 (a) $\frac{\pi}{6}$ (b) $\frac{3}{4}$ **11** (b) $1-\frac{\pi}{4}$ **12** $\frac{2}{3}$

REVISION EXERCISE 17

1 $\frac{(1-x^2)}{(1+x^2)^2}$; $\frac{2}{5}$ **2** $2\cos x - x\sin x$ **3** $2\pi(3\pi+8)$ **4** $\frac{3}{5}$ **5** (a)(i) $12(4x+1)^2$
(ii) $\tan 3x + 3x\sec^2 3x$ (b) $\frac{(3x^2-y^3)}{(3xy^2+1)}$ (c) $3y = 8x - 29$ **6** $\frac{\pi^2}{8}-1$ **8** (a) $\frac{(\cos x + x\sin x)}{\cos^2 x}$
(b) $\frac{2\cos x}{(1-\sin x)^2}$ (c) $\frac{1}{(\sin x+\cos x)^2}$ $[=\frac{1}{1+\sin 2x}]$ (d) $2(5x-1)(2x-1)^{\frac{1}{2}}$
9 $y = -3x + 10, y = -3x - 10$ **10** (a) $\frac{1}{2}(1-\cos 2x)$; $\pi + 2$ (b) 2 (c) $\frac{2\pi}{3}$ **11** -0.0058
12 $\frac{4}{7}$ **13** 4 **15** (a) $[\frac{dy}{dx} = \sin 2x(1-4\sin^2 x)]$ $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$ (b) $0, \frac{3\pi}{2}$; 1, 0 **16** 0.01
17 $[\frac{dy}{dx} = \frac{(-1-x^2)}{(x^2-1)^2} < 0]$ **18** $\frac{dy}{dx} = -\frac{2y^2+2}{4xy+1}$; $-\frac{13}{14}, \frac{10}{7}$ **19** 1 **21** $0, \frac{4\pi}{3}$ **22** $-\frac{y}{x}, x(k^2+x^2)^{-\frac{1}{2}} = \frac{x}{y}$;
[product $= -1$] **23** (a) $[v = 8\cos 2t = 8\sqrt{(1-\frac{s^2}{16})}, a = -16\sin 2t = -4s]$
(b) ± 4 units from O [max and min values of $4\sin 2t$]
24 (a) 450 min [max h when $t = 0$, min h when $\frac{\pi t}{450} = \pi$] (b) $-\frac{\pi}{450}$ $[\frac{dh}{dt} = -\frac{\pi}{225}\sin\frac{\pi\times 75}{450}]$
(c) 150 min $[2 + 2\cos\frac{\pi t}{450} = 3]$ **25** (a) $-\operatorname{cosec}^2\theta$ (b) $[V = -\frac{1}{3}\pi r^2 h = -\frac{1}{3}\pi r^2 r\cot\theta]$
(c) $2\pi\%$ $[\frac{\delta v}{v}\times 100\% \approx \frac{-\operatorname{cosec}^2\theta}{\cot\theta}\times 0.04\times\frac{\pi}{4}\times 100\%]$

EXERCISE 18.1

1 (a) $\frac{1}{x}$ (b) $\frac{2}{x}$ (c) $\frac{3}{3x-1}$ (d) $\cot x$ (e) $\frac{1+\sec^2 x}{x+\tan x}$ (f) $-2\tan 2x$ (g) $-2\tan x$
(h) $\frac{1}{2}\cot\frac{x}{2}$ (i) $\frac{4(x-1)}{2x^2-4x-1}$ (j) $\frac{1}{2x-5}$ (k) $-\frac{1}{2x}$ (l) $1+\ln x$ (m) $\frac{1-2\ln x}{x^3}$
(n) $\frac{\cos x - x\sin x}{x\cos x} = \frac{1}{x} - \tan x$ (o) $-\frac{1}{x}\sin(\ln x)$ (p) $\frac{1}{x}\ln 3x^2$ (q) $2x\ln(x-1) + \frac{x^2+1}{x-1}$
(r) $\frac{2}{x}\ln x$ (s) $\frac{-3\sin 3x}{\cos 3x} = -3\tan 3x$ (t) $\ln 2x + \frac{x-1}{x}$ (u) $\frac{1+\cos x}{x+\sin x}$ (v) $\frac{4x+5}{2x^2+5x-3}$
(w) $\frac{7}{(x-4)(x+3)}$ **3** $\frac{12x-1}{(3x+1)(2x-1)}$ **4** $\frac{3}{x^2-x-2}$ **5** (a) $y=e^{2x+1}$, $y=e^{5-2x}$ (b) $-\frac{1}{2}, 2\frac{1}{2}$
6 For each, plot $\ln y$ against x; (a) $\ln y = 3x-2$, (b) $\ln y = -2x + \ln 3$ **7** ≈ 1.3 **8** 3, 4
9 $\frac{1-\cos x}{x-\sin x}$; $\frac{2}{\pi}$ **10** $\frac{-1}{1-\cos x}$ **11** $\frac{-1}{1+\sin x}$

EXERCISE 18.2

1 (a) $4e^{4x}$ (b) $5e^{5x-1}$ (c) $-3e^{5-3x}$ (d) $2xe^{x^2}$ (e) $-\sin x\, e^{\cos x}$ (f) $3^x + xe^x$
(g) $2e^{-\frac{x}{2}} - (x-2)e^{-\frac{x}{2}} = (4-x)e^{-\frac{x}{2}}$ (h) ae^{ax-b} (i) $2(x+1)e^{x^2+2x-1}$ (j) $e^x(\cos x + \sin x)$
(k) $-e^{-x}$ (l) $\frac{xe^x}{(x+1)^2}$ (m) $xe^{-x}(2-x)$ (n) $e^x + e^{-x}$ (o) $-2e^{-x}\cos x$ (p) $e^{-2x}(-1-6x)$
(q) $e^{2x}(2\ln x + \frac{1}{x})$ (r) $\frac{-e^x}{(e^x-1)^2}$ (s) $\frac{e^x(x^2-x-1)}{(x-1)^2}$ (t) $2e^{2x}(\cos 2x - \sin 2x)$ (u) $\frac{e^x(x-1)}{x^2}$
(v) $x^2e^{2x}(2x+3)$ (w) $2e^{2x} - 2e^{-2x}$ **2** $(1, e^3)$; $5e^3$, $-e^3$ **3** $x < 3\frac{1}{2}$ **4** $e^{3x}(1+3x)$; $3e^{3x}(2+3x)$;
min at $x = -\frac{1}{3}$ **5** 0, -1 **6** min at $x=-1$, max at $x=3$ **7** $e^x(\cos x - \sin x)$; $-2e^x\sin x$;
max at $x=\frac{\pi}{4}$; min at $x=\frac{5\pi}{4}$ **8** $2e^x\cos x$; $2e^x(\cos x - \sin x)$; max at $x=\frac{\pi}{2}$, min at $x=\frac{3\pi}{2}$
9 [$\frac{dy}{dx} = e^x(\sin x + \cos x)$; $\frac{d^2y}{dx^2} = 2e^x \cos x$] **10** 2 **11** $a^x \ln a$ **12** $\cos x - \sin x$

EXERCISE 18.3

1 (a) $\frac{1}{3}e^{3x}+c$ (b) $-e^{-x}+c$ (c) $-e^{2-x}+c$ (d) $\frac{1}{2}\ln|2x+3|+c$ (e) $-\frac{1}{2}\ln|3-2x|+c$
(f) $-\ln|4-x|+c$ (g) $x+\ln|x|+c$ (h) $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+c$ (i) $-\frac{1}{3}e^{-3x}+c$
(j) $-\frac{1}{2}e^{1-2x}+c$ (k) $\frac{1}{3}\ln|3x+1|+c$ (l) $\ln x - \frac{2}{x} + c$ **2** (a) $\frac{1}{4}(e^4-1)$ (b) $e-1$
(c) e^4-e^2 (d) $\frac{1}{2}(e^5-e)$ (e) e^4-1 (f) $\frac{1}{2}(e^{-2}-e^{-4})$ (g) $2(e-1)$ (h) e^3-e **3** (a) $-\ln 2$
(b) $\ln\frac{5}{4}$ (c) $\frac{1}{2}\ln\frac{7}{3}$ (d) $\ln 2$ (e) $\frac{1}{3}\ln 2$ (f) $\frac{9}{4}+\ln 2$ (g) $\ln\frac{7}{5}$ (h) $-\frac{1}{4}\ln 5$ (i) $\ln 3$
4 $\frac{1}{2}\ln\frac{3}{2}$ **5** $\frac{1}{2}\ln\frac{3}{2}$ **6** $\frac{1}{3}(10-e^{-3t})$ **7** $\frac{1}{2}(e^2-2e+1)$ **8** $x-e^{-x}+c$ **9** $\pi\ln\frac{5}{3}$
10 (a) $y=\frac{1}{2}e^{2x}$ (b) $\frac{1}{4}(e^2-1)$ **11** (b) $\ln x$; $x\ln x - x + c$ (c) $e^2 - 2\ln 2$
12 (a) $y=x-1$ (b) $\frac{5}{3} - 2\ln 2$ (≈ 0.28)

REVISION EXERCISE 18

1 (a) $\frac{1}{2}(e-e^{-1})$ (b) $\ln\frac{3}{4}$ (c) $2e(e-1)$ (d) $\frac{1}{k}(e^{kb}-e^{ka})$ (e) e^5-e^3 (f) $\ln\frac{6}{5}$
(g) $\frac{1}{2}(e-\frac{1}{e})$ **2** $-\frac{2}{\pi}$ **3** $\frac{1}{2}(1-e^{-2})$ units^2 **4** (a) $(1, e^{-1})$ (b) $2(e^{-1}-e^{-2})$ (≈ 0.47)
5 Min at $x=-1$, max at $x=2$ **6** $2x-x^2-1$ **8** (a) $\frac{3x+2}{2x(x+1)}$ **9** e; max **10** 0.005
11 $-\frac{1}{3}\ln 4$ **13** -1 **14** -1 **16** $4xe^{2x}$; $\frac{1}{4}(e^2+1)$ **17** [Limits for y are 0 and 1; $x=e^y$]
$\frac{\pi}{2}(e^2-1)$ units^3 **19** $-\frac{1}{2}\tan\frac{x}{2}$; $\ln 2 = 0.69$ **20** $R=\sqrt{13}$, ($\alpha = 33.7°$); $13e^{3x}\cos(2x+2\alpha)$
21 [$\frac{dy}{dx} = -x(2\ln x + 1)$ but $x=0$ is not possible; only value is $\ln x = -\frac{1}{2}$] max; $\frac{1}{2e}$
22 (a) $1+\ln x$ (b) $x^x(1+\ln x)$ **23** $\frac{1-e^y}{xe^y}$ **25** [Fig. A18.1] $2\ln 3$ units^2 **26** $\frac{1}{\sqrt{x^2-1}}$;
$\ln\frac{3+\sqrt{8}}{2+\sqrt{3}} \approx 0.446$ **27** (a), (b) [Fig. A18.2] (c) $\frac{1}{2}(e^x-e^{-x})$; $\frac{1}{4}(e^{2x}-2+e^{-2x})$ (e) $e-\frac{1}{e}$

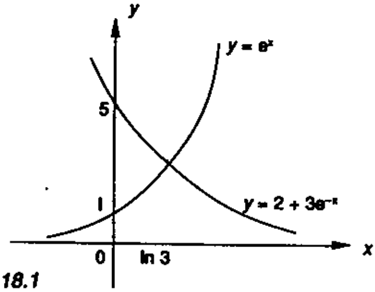


Fig. A18.1

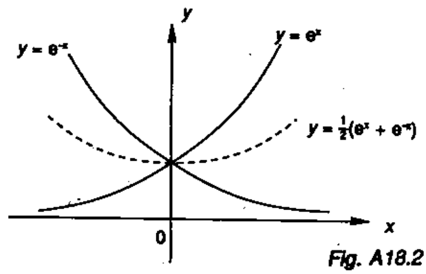


Fig. A18.2

EXERCISE 19.1

1 (a) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ i.e. $4x^2 + 9y^2 = 36$ (b) $(\frac{x-1}{2})^2 + (\frac{y-1}{-3})^2 = 1$ i.e. $9x^2 - 18x + 4y^2 - 8y = 23$
(c) $x = y^{\frac{3}{2}}$ i.e. $x^2 = y^3$ (d) $y = (x-1)^2 - 1$ i.e. $y = x(x-2)$
(e) $x = y + 1 + \frac{1}{y+1}$ i.e. $x(y+1) = y^2 + 2y + 2$ (f) $x = 2y\sqrt{(1-y^2)}$ i.e. $x^2 = 4y^2(1-y^2)$
(g) $x^2 + y^2 = y(2x+1)$ [$x = t^2 - t = y - t$ so $t = y - x$] (h) $x + y = 1$ [Add]
(i) $x = 1 - 4y + 2y^2$ (j) $x = 4y^2 - 8y + 2$ (k) $x^2 = y^2 - 4$ (l) $y^2 = x - 1$ (m) $xy = 2 - 5x$
(n) $x^2y = 3$ **2** 2 **3** $\frac{-2}{2t+t^2}$; $4y + x = 0$; $3y = 12x - 17$ **4** $32y = 2x + 31$
5 $x^2 + y^2 = 1$ $(y \geq 0)$; semicircle centre O, radius 1 **6** $\frac{1}{2}$, $(-\frac{3}{4}, \frac{3}{2})$; 1, (0,3); $\frac{1}{4}$, 1 **7** (a) $-\frac{1}{3}$
(b) $-\frac{1}{2}$ **8** (a) $\frac{1-2t}{-2} = t - \frac{1}{2}$ (b) $4y + 4x + 3 = 0$ **9** (a) $2y = 4x - 3t$ (b) $(0, -\frac{3t}{2})$
(c) Both $= \sqrt{5}\ t$ **10** ±1; (2,1) (0,–3) **11** $\frac{2y-x}{3}$, $\frac{x+y}{3}$; $2x^2 + 5y^2 - 2xy = 9$ **12** $x = \frac{t}{t-2}$, $y = \frac{t^2}{t-2}$
13 (a) $y = tx - 2t^2 + t$ (b) $(0, -2t^2 + t)$, $(2t - 1, 0)$ (c) $x = \frac{2t-1}{2}$, $y = \frac{-2t^2+t}{2}$
(d) $2x^2 + x + 2y = 0$ **14** $(x-2)^2 + (y+1)^2 = 25$; radius 5, centre (2,–1) **15** $\frac{1}{2}t(1+t)$
16 (a) $y = x + 1$ [$\frac{y-2}{x-1} = \tan 45° = 1$] Straight line through (1,2) gradient 1; t is the distance along line from (1,2). (b) $x = -2 + 3t$, $y = 3 + 2t$ [Fig. A19.1] **17** $x = t - 1$, $y = t(t+1)$
18 (–5,–12) [Substitute in equation; equal roots] **19** (8,2) **20** (a) 3, –2 (b) $\frac{6}{t^2+4t-2}$; –5, 1
21 (a) $y = 2tx - t^2 + 2$ (b) 0, 6 **22** (a) $(x+y)(x-y)^2 = 8$ [Find x + y and $x - y$]
(b) $3y = x + 6$ **23** (a) $(0, 2 - 4m)$, $(-\frac{2}{m} + 4, 0)$ [equation of line is $y - 2 = m(x-4)$]
(b) $x = -\frac{1}{m} + 2$, $y = 1 - 2m$ (c) $x + 2y = xy$ [Multiply $\frac{1}{m}$ by $2m$]
24 (a) $y - 3 = t(x-4)$, $t(y-3) = x - 4$ (b) $(0, 3-4t)$, $(4-3t, 0)$
(c) $2x = 4 - 3t$, $2y = 3 - 4t$; $8x - 6y = 7$

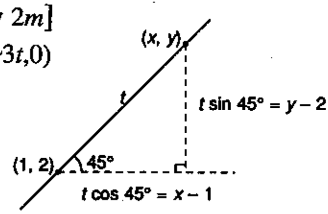


Fig. A19.1

REVISION EXERCISE 19

1 (a) $y + 2x = 8$, $2y = x - 44$ (b) $4x^3 = 27y^2$ **2** (a) $4y = 3(4 - x^2)$ (b) $x = 4 - 6y + 2y^2$
(c) $(1-x)y = x^2 - 2x + 3$ (d) $xy^2 = \frac{4}{e}$ (e) $x^2y^3 = 4$ **3** $\frac{x+2y}{7}$, $\frac{3x-y}{7}$; $8x^2 - 10xy - 3y^2 + 49 = 0$
4 (a)(i) $\frac{10}{9}$ (ii) $12t$, $\frac{12}{t}$; $4x^2 - 9y^2 = 144$ (b)(i) 3, $-\frac{1}{3}$ (ii) [gradients are –3, $\frac{1}{3}$] **5** (i) 1
(ii) $3y + x = -12$ **6** (a) $\frac{2t-8}{4t-3}$ (b) 4; $y = -15$ (c) $\frac{7}{5}$ **7** (a) ±2
(b) $y = 1 + t$ [$y^2 - 2y = t^2 - 1$; solve for y] (c) $t - 1$; $y^3 = x^2 + 3xy + 2y^2$
8 (a) $(\frac{t}{1+t^2}, \frac{t^2}{1+t^2})$ (b) $x^2 + y^2 = y$ **9** (i) $3x + 8y = 18$ (ii) $y + 4x = 24$; $(\frac{3}{2}, \frac{3}{2})$ when $t = 1$, (6,6) when $t = -2$ **10** 1, $-\frac{1}{2}$ **11** (a) $y(x-1)^2 = 2x$ (b) $2\cos\theta$

(c) $y = 2x + 1$, $y + 2x = -1$; $(-\frac{1}{2}, 0)$

12 $y + 3x = 12$, $y = 5x - 4$

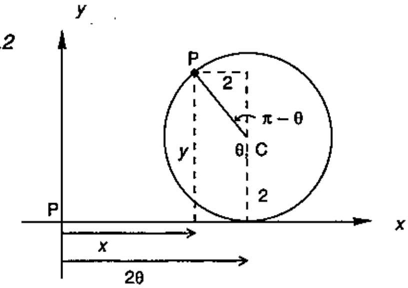
13 (a) $\frac{4}{3}$; $y = \frac{4}{3}x - 2$ i.e. $3y = 4x - 6$ [$\frac{dy}{dx} = \frac{4}{3}$; integrate and substitute values of x and y for say $t = 1$: otherwise solve x for t and substitute in y]

(b) $y + tx = t$; $\frac{t}{t-1}$, $\frac{-t}{t-1}$; 0

14 $\frac{\sin t - \cos t}{\sin t + \cos t}$ **15** (a) $x = t - 1$, $y = t^2 + 2$; $y = x^2 + 2x + 3$

(b) $-4\mathbf{i} + 11\mathbf{j}$; $3\mathbf{j}$ **16** (a) [Fig. A19.2] (b) 2π (d) 16

Fig. A19.2



17 $x = \frac{3t}{1+t^3}$, $y = \frac{3t}{1+t^3}$; $x + y = 3$ **18** (a) $-\tan t$ (b) [OP = $2\cos p$, OQ = $2\sin p$, PQ = 2]

(c) $\left(\frac{x}{2}\right)^{\frac{2}{3}} + \left(\frac{y}{2}\right)^{\frac{2}{3}} = 1$

PAPER 6

1 (a) 0.57 (b) $2\frac{1}{2}$ **2** (a) $\frac{1}{10}$ (b) ln 2 (c) $2\sqrt{10} - 2\sqrt{5} \approx 1.85$ **3** $-3, -11$; $x - 3$
4 (b) $y = 1 + ex$ (c) (0,1) (d) $\frac{e}{2} - 1$ **5** $\sqrt{5}\sin(x + 26.6°)$; 140.5°, 346.4° **6** (a) $\frac{4}{x^2 - 4}$
(b) $-\frac{1}{4}\ln 3$ **7** 4.7, 1.5 **8** (a) $\frac{\pi}{6}$ (b) 0.20 **9** (a) $\frac{1}{2}n(13 + 3n)$; 8 (b) 5, 2 or $\frac{80}{3}$, $-\frac{5}{4}$
10 (1,2), (3,–2)

PAPER 7

1 (a) $x\cos x$; $\frac{\pi}{2} - 1$ (b) $y = -2(2 - x)^{\frac{1}{2}}$ **2** (a) $\left(1 - \frac{1}{m}, 1 - m\right)$ (b) $x + y = xy$
3 (a) min at $x = -1$, max at $x = 2$ (b) 5, 3 (c) $(x + 1)^2(8x - 7)$
4 (b)(i) 132.8° (ii) 120, 180°, 240° **5** (a) 2, 2, $\frac{1}{2}$ (b) 3 **6** 0.4, 1.5
7 (a) [tangent is $y = e^2x$] $\frac{e(e-2)}{2}$ (b)(i) $\frac{1 - 2\ln x}{x^3}$ (ii) $\frac{-e^{-x}(x+2)}{(x+1)^2}$ (c) -3 **8** (a) 3000
(b) $\frac{3}{2}$, 5 [$422 = \frac{a(r^n - 1)}{r-1}$ so $422(r - 1) + a = ar^{n-1} \times r$] **9** (a) $-\frac{y^2 + 3}{2xy + 2}$ (b)(i) $4y = (x + 3)^2$
(ii) 1, 4 **10** (a) [Simplify $\cot 2\theta + \operatorname{cosec} 2\theta$] 9.7°, 80.3°, 189.7°, 260.3° (b) $\frac{1}{2}$

PAPER 8

1 (a)(i) $2(e - 1)$ (ii) $\ln\frac{2}{3}$ (b) $\frac{2x^2 + 1}{x(x^2 + 1)}$ (c) 2, –1; ln 3 **2** $\frac{5}{2}\cos(x - 36.9°)$; max 5, $x = 36.9°$; min –5, $x = 216.9°$ **3** (a) 0.0067 (b) [Fig. AR8] $\frac{2\pi}{3} + 4\sqrt{3}$ **4** (a)(i) $\frac{4}{5}$ (ii) $-\frac{7}{25}$ (iii) $\frac{1}{\sqrt{5}}$
(b) 120°, 300° [Note, that cos 60° = sin 30° etc] **5** (a) $y = e^{5-2x}$ (b) 11 (c) 2, 2, –3

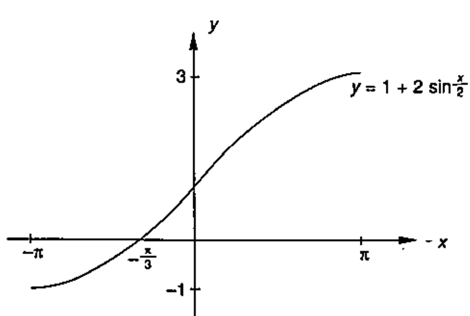


Fig. AR.8

6 (a) $\frac{3}{4}$; 16 (b) 165 **7** Plot y^2 against x; 30, –4; 7.5 **8** (a) 0, –2 (b)(i) 1.24
(ii) 1.65 [If $y = \ln(1 + x)$ then $e^y = 1 + x$. Find value of x where $y = 0.5$] **9** (a) $\frac{1}{2}\ln(\tan x) + c$
(b) $\frac{1}{2}\ln 5$ **10** (a) $\frac{1-t^2}{2t}$ (b) $(1,\frac{2}{3}), (-1,-\frac{2}{3})$ (c) $\frac{1}{3}, -3$ (d) $y = \sqrt{x}(1 - \frac{x}{3})$ i.e. $9y^2 = x(3-x)^2$

PAPER 9

1 (a)(i) $\frac{-5}{(x+2)^2}$ (ii) $2(x-2)(2x-3)^2(5x-9)$ (b)(i) $\frac{1}{3}\ln 4$ (ii) $-\frac{2}{3}$ **2** (a)(i) 15 (ii) 64
(b) 10 (c) 3, 7, 11,... **3** (a) π units^2 (b)(i) $y = 1 - x$ (ii) (1,0) (iii) $\frac{1}{2} - \frac{1}{e^2}$ units^2
4 (a)(i) $\frac{10}{e^2} \approx 1.35$ (ii) $10 + \frac{10}{e^2} \approx 11.35$ (c) $\frac{1}{6}$ units^2 **5** (a) $2\frac{1}{2}$ (b) 1, 2
(c) 4.42 **6** (a) $\frac{1}{2}, (1,1\frac{1}{2})$; –2, (6,–1) (b) 1, –5; $(x-1)(x-1)(x+3)$ **7** (a) (1,1)
(b) $\frac{14}{3} - \ln 4$ units^2 (c) $\frac{27}{4}\pi$ units^3 **8** (a) $\frac{3t^2}{1-t^2}$ (b) $16y + 3x = -4$ **9** 1.2, 3.5
10 (a)(i) 0°, 146.4°, 213.6°, 360° (ii) 278.0°

PAPER 10

1 (a) 10 (b) 2, 5, 8,... **2** (a) –13, 6; $(x-2)(2x-1)(x+3)$ (b) 1 (c) 2
3 (a)(i) $\frac{1}{2}\sec^2\frac{x}{2}$ (ii) $4\sin 2x\cos 2x = 2\sin 4x$ (iii) $\frac{\cos x}{2\sqrt{1+\sin x}}$ (b)(i) $y = 3x$ (ii) (–1, –3)
(iii) $\left(4\ln 2 - \frac{13}{6}\right)$ units$^2 \approx 0.606$ units^2 **4** (a)(i) $\frac{1}{\sqrt{1+t^2}}$ (iii) 270°, 323.1°
(b)(i) $3\tan^2 x\sec^2 x$ (ii) [$\tan^4 x = (\sec^2 x - 1)^2 = \sec^4 x - 2\sec^2 x + 1 = [\sec^2(1 + \tan^2 x)$ ...]
(iii) $\frac{\pi}{4} - \frac{2}{3}$ [$\int \tan^4 x\,dx = \int(\sec^2 x\tan^2 x - \sec^2 x + 1)dx = \frac{1}{3}\tan^3 x - \tan x + x + c$]
5 (a) $\frac{4-x}{2(2-x)^{\frac{3}{2}}}$; 2 (b) $\frac{1}{2}, -1, -\frac{1}{7}$ (c) $-\frac{y^2+2xy}{x^2+2xy+1}$ **6** (a)(i) 12 (ii) $-\frac{4}{e}$ (iii) $36(1 - \frac{1}{e})$
(b)(i) $\ln 2 - \frac{3}{8}$ (ii) $-\frac{1}{e} + \frac{1}{2e^2} + \frac{1}{2} = \frac{(e-1)^2}{2e^2}$ **7** (a) [Fig. AR9] $\ln x = -x + \ln 7.39$; $a = -1$,
$b = \ln 7.39 = 2$ (b) $3^{r+4s}, 3^{2r-2s}$; $\frac{2}{5}, \frac{9}{10}$ **8** (a) $y + 4x = 16$ (b) $-\frac{1}{4}$ (c) $x = (y+2)^2$
9 $\ln(x+1)$; $(x+1)\ln(x+1) - x + c$; 1 [Area $= \int_0^{e-1}\ln(y+1)dy$] **10** (a) $\frac{6-t}{2}$ (b) $3 - t$ (c) 4, 5

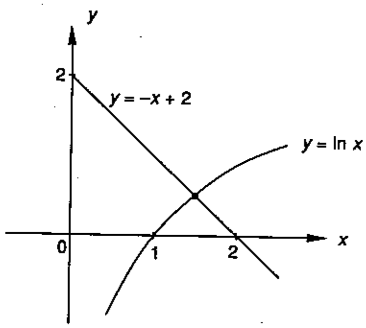


Fig. AR.9

EXERCISE 20.1

1 [Fig.A20.1] 200 m **2** (a) 275 m (b) $\frac{55}{7}$ m s^{-1} (c) 18.75 s **3** $\frac{55}{3}$ s, $\frac{400}{3}$ m
4 1 km min^{-1} = $\frac{50}{3}$ m s^{-1} **5** (a) 6300 m (b) 0.07 m s^{-2} (c) 666 m [Fig. A20.2]
6 [Fig.A20.3] 45 s, 900 m from start **7** 2.5 m s^{-2}, $\frac{5}{6}$ m s^{-2}, 5 m s^{-1} **8** 18 s, 150 s

9 [Fig.A20.4] (a) 6 min (b) 0.5 m s^{-2} **10** [Fig.A20.5] (a) 16 s (b) 4 s (c) 320 s **11** (a) [Fig.A20.6] (b) 4 (c) 26 **12** (a) [Fig.A20.7] (b) 30 **13** (a) [Fig.A20.8] (b) 20 s, 80 s **14** (a) $\frac{5}{6}$ (b) 5.375

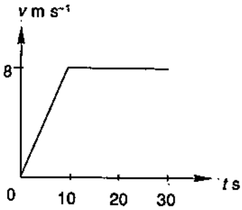


Fig. A20.1

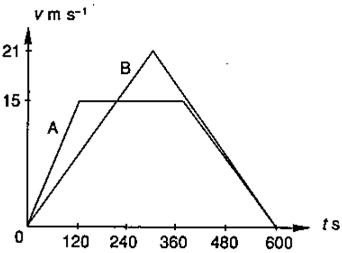


Fig. A20.2

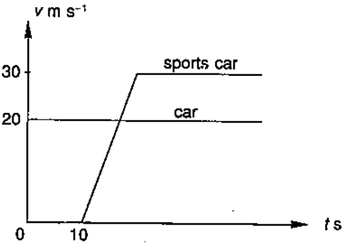


Fig. A20.3

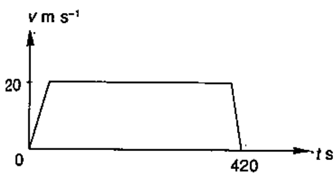


Fig. A20.4

Fig. A20.5

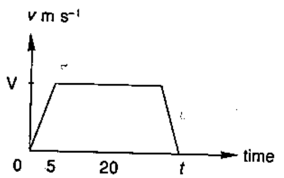


Fig. A20.6

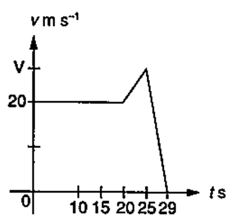


Fig. A20.7

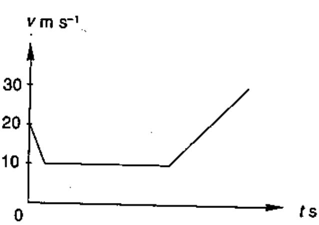


Fig. A20.8

EXERCISE 20.2

1 (a) 4.5 m s^{-1}; 11.25 m (b) 8 m s^{-1}; 55 m (c) $3 + 0.5t$ m s^{-1}; $3t + 0.25t^2$ m **2** 22 m s^{-1}, 260 m, 302.5 m **3** 20 s; 20 m s^{-1} **4** 16 m; 4 s **5** (a) $\frac{5}{3}$m s^{-2} (b) $\frac{71}{6}$ m s^{-1} (c) 20 m **6** $\frac{8}{3}$ m s^{-1}; $\frac{160}{3}$ m; [Fig.A20.9] **7** 10 m s^{-1}, 15 m **8** $\frac{12}{7}$ **9** $u = 5.5$ m s^{-1}, $a = 1$ m s^{-2} **10** 2 m s^{-2}, 4 m s^{-1}; 24 m s^{-1}, 10 s **11** (a) $\frac{10}{3}$ (b) $\frac{5}{3}$ m **12** (a) 4 s (b) 24 m **13** $10u - 16a$ (i) 2, $\frac{1}{2}$ (ii) 5 m (iii) 3 m s^{-1} **14** (a) 12 m (b) 4 s (c) 10 s **15** (b) $(100 - \frac{50}{a})$ m; $\frac{2}{3}$m s^{-2}

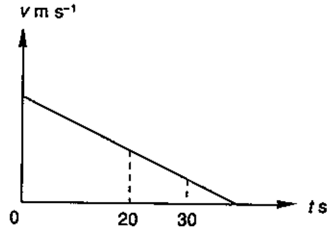


Fig. A20.9

EXERCISE 20.3

1 7.2 m, 1.2 s **2** 10 m s^{-1} **3** 5 s, 35 m s^{-1} **4** 20 m **5** 0.2 s, 4.8 s **6** 3.2 m **7** $\frac{75}{16}$ m **8** $(3+2\sqrt{3}) = 4.5$ s **9** (a) 2 (b) −14.2 m s^{-1} **10** (a) 25 m s^{-1} (b) 31.25 m (c) downward **11** (a) 20 m s^{-1} (b) 2 s (c) 20 m (d) 2 s **12** 170 m; 58 m s^{-1} **13** (a) 60, 20 m s^{-1} (b) 160 m (c) 100 m **14** $\frac{80}{3}$ **15** (i) 10 m s^{-1} (ii) 8 s (iv) 11 s

REVISION EXERCISE 20

1 (a) 10 (b) 6 s (c) 50 m s^{-1} **2** 8 s; $\frac{16}{3}$ s **3** (a) 1.2 s (b) 6 m s^{-1} (c) 7.2 s **4** 24 m s^{-1}; 0.7, 26.35 **5** (a) 9 m (b) 4 m s^{-1} **6** 320 m, 55 **7** [Fig.A20.10] (a) 4 (b) $2\frac{1}{2}$ m s^{-2} (c) 20 m **8** $S_1 = 4.8V$, $S_2 = \frac{625 - V^2}{5}$; $V = 9, 15$; $t = 16$ s **9** (a) [Fig. A20.11] (b) $t_3 = 3t_2$ (c) 19.2, 5.2, 15.6 **10** [Fig.A20.12] (a) 80 s (b) 15 m s^{-1} (c) 17.5 m s^{-1} **11** (a) 45 (b) 101.25 m **12** (a) 45 m (b) 8 s (c) 4 s **13** [Fig.A20.13] (a) 16 (b) 14.2 **14** (a) 20 (b) 20 m **15** $\frac{2V}{3}$, $\frac{3V}{4}$; second train **16** 30; $\frac{1}{8}$ **17** 6 s **18** 11.25 **19** (a) 10 s (b) 22, 24 m s^{-1} (c) $\frac{20}{3}$ s **20** (a) 10 m/s (b) 8 s (d) 10.8 s

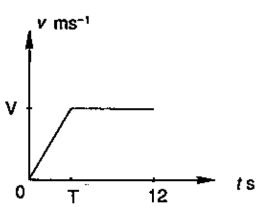


Fig. A20.10

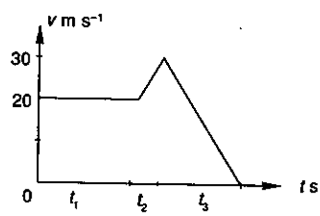


Fig. A20.11

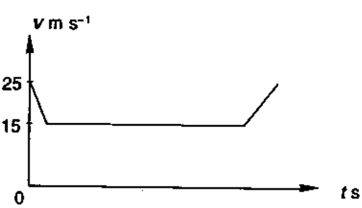


Fig. A20.12

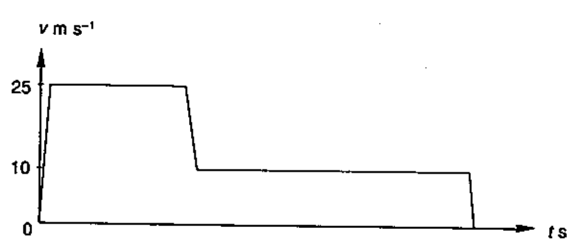


Fig. A20.13

EXERCISE 21.1

1 (a) 5.83 km h^{-1}, 031° (b) 9.4 m s^{-1}, 013° (c) 13.6 m s^{-1}, 068.5° (d) 69.3 km h^{-1}, 080° **2** 036.8° **3** 10.8 m s^{-1} at 21.8° to line of ship **4** 155.2 km h^{-1} at 015° **5** (a) 2.2 km h^{-1} at 63.5° to bank (b) 0.2 km **6** 229.1 km h^{-1} at 341° **7** 53.2° to bank, 6 min **8** (a) 081.9° (b) 170 km h^{-1} (c) 1.76 h (d) 278.1° (e) 1.34 h **9** A to B, 153 km h^{-1}; B to A 238 km h^{-1} **10** 056.4°; 42 min **11** (a) 20 km h^{-1} (b) 200 km h^{-1} (c) 354.3° (d) 90 min **12** (a) 103.9 m (b) 69.3 s **13** (a) 006.6° (b) 43.2 min (c) 173.4°, 37.7 min **14** (a) 120 (b) 70.2 km

EXERCISE 21.2

1 250 km h^{-1}, 126.9° **2** 72.1 km h^{-1}, 303.7° **3** 32.8 km h^{-1}, 077.6° **4** 108.2 km h^{-1}, 236.3° **5** 33.3 km h^{-1} in direction 064.7° **6** 832 km h^{-1}, 018.7° **7** 018.8°; 16.5 min **8** 19.2 km h^{-1}, 231.3° **9** (a) 15.7 km h^{-1}, direction 174.1° (b) 1 h 55 min **10** 284.5° **11** 17.4 km h^{-1}, 036.7° **12** (a) 100.5° (b) 1326 h **13** 3.6 m s^{-1}, 77°; 3.5 m s^{-1}, 83.8°

REVISION EXERCISE 21

1 (a) 96 m (b) 28° **2** 165.5°, 194 km h^{-1}; 85 km h^{-1} from 087.8° **3** 1 h 46 min **4** 079.8° **5** 103°; 1 hr 52 min **6** [Fig.A21.1] (i) 041° (ii) 1 hr 59 min **7** (i) 60° with bank (against current) (ii) directly across (iii) $t_A = \frac{80\sqrt{3}}{3}$ s, $t_B = 48$ s (iv) 144 m **8** (i) 148 km h^{-1} (ii) 112.1°, 337.9° or 157.9°, 292.1° **9** (i) 024.3° (ii) 11 04 h **10** 19.5 km h^{-1} from 081.2°; 25.7 km h^{-1} **11** 061.3°; 5.41 min **12** (i) 339.7° (ii) 1.46 h (iii) 200.3° **13** (i) 038.2° (ii) 108 km **14** (a)(i) 093.8° (ii) 36 min (b) 26 km h^{-1}, 67.4° **15** $4\sqrt{2}$ (= 5.66) km h^{-1} from the SE **16** (a) 22.9 km h^{-1} on 109.1° (b) 109.1° (c) 45.8 km **17** (a) 36.9° ($\sin^{-1} \frac{3}{5}$) (b) $\frac{7V}{20}$ **18** (i) 482.7 m s^{-1} (ii) 124.9° (iii) 325.1° (iv) 3 h 35 min **19** (a) (i) 50.3° (ii) 310 km h^{-1} (b) (i) 2.8 m s^{-1} (ii) 21.8° **20** (a) (i) 15 m s^{-1}, 066.9° (ii) 4.355 (b) 23.02 km h^{-1}, 356.8°

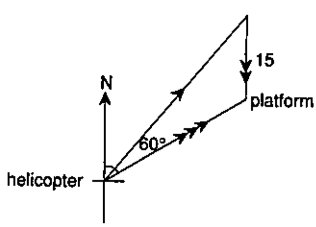


Fig. A21.1

EXERCISE 22.1

1

	(a)	(b)	(c)
	s	m	m
(i)	2	20	5
(ii)	1.2	12.5	1.8
(iii)	3.6	86.4	16.2
(iv)	2	48	5

2 (a) 4.8 s (b) 24.1 m s^{-1} downward at 41.6° **3** 0.2 and 1 s; 1.8 m **4** 4.3 m

5 0.55 and 2.05 s **6** 0.31 m **7** 1.62 s, 14 m; 52.2° **8** (a) 10 m s^{-1} (b) 2.75 m
9 (a) 30, 7 m (b) 41.4° **10** (a) Ball at angle θ (b) Neither — they have the same range
11 16 m [Initial components of velocity are 8 m s^{-1} and 0] **12** 20.4 m, 0.96 s
13 9 m **14** 120 m; [angle ø to vertical given by tan ø = $\frac{4}{3}$ so ø = θ]
15 (b) [$y = 4 = 25t \sin\theta - \frac{1}{2}gt^2$; substitute for t from (a)]; $\frac{1}{2}$, 12

REVISION EXERCISE 22

1 (a) 53.1° (b) 15.8 m s^{-1} (c) 2.53 s (d) 24 m **2** 45°, 15°, 262.6 m **3** (b) 140 m (c) −40 m s^{-1} (d) −48.8° **4** (b) 14 s (c) 1400 m (d) −140 m s^{-1} **5** (a) 19.7° (b) 5.66 s (c) 447.4 m **6** (i) 11.55 m (ii) 37 m s^{-1}, 26.6° (iii) 4 s (iv) 20 m (v) $80\sqrt{3}$ m **7** (i) 4 s (ii) 20 m (iii) $80\sqrt{3}$ m (iv) 35 m s^{-1}, −8.2°, 60 m s^{-1}
8 (a)(i) 0.8 m (ii) 4.16 m (b) 25 m s^{-1}, 53.1° **9** (a) 200 m (b)(i) 528.7 m (ii) 122.6 m s^{-1}, 66.2° (iii) 1505 m **10** 32 **11** $3\sqrt{5}$ (≈ 6.7) **12** 24.1 m; no [as the striking speed is only 22.2 m s^{-1}] **13** 37.3 m s^{-1} at 28.9° to the horizontal **14** $\frac{\sqrt{3}}{2}$ (≈ 0.87) m
16 (i) 51.2 m (ii) 2 s (iii) 32.56 m s^{-1}, yes **17** (i) 45 m s^{-1}, 30 ms^{-1} (ii) 54.1 m s^{-1} (iii) 33.7° (iv) 45 m (v) 6 s (vi) 270 m
18 $V_x = 4.8$ m s^{-1} $V_y = 3.6$ m s^{-1} (i) 15.3 m s^{-1} (ii) 16 m s^{-1} (iii) 72.5° (iv) 1.173 (v) 5.62 m **19** 80 m, 4 s (i) 4 s (ii) 120 m (iii) 36.1 m s^{-1}, −33.7°
20 (i) 26.9, 42° (ii) 1.8 s (iii) 3 s (iv) 60 m

EXERCISE 23.1

1 (a) P cos 30°, P sin 30°; $-Q$ cos 60°, Q sin 60°; $-R$ sin 10°, $-R$ cos 10°; S cos 20°, $-S$ sin 20° (b) A cos 50°, A sin 50°; B cos 20°, $-B$ sin 20°; $-W$ cos 70°, $-W$ sin 70°
2 (a) 15.6 N at 39.8° to 12 N (b) 50 N at 36.9° to 40 N (c) 9.7 N at 11.9° to 6 N (d) 13.9 N at 27.5° to 20 N (e) 17.3 N bisecting angle **3** (a) 36, 12.3 N; 38 N at 18.9° to horizontal (b) 5, 27.3 N; 27.8 N at 79.6° to horizontal (c) 0, −24.6 N; 24.6 N opposite 10 N force **4** 5 N, 3 N **5** 16.5 N in direction 132.9°
6 5 N, 60° **7** 112.6°; 12 N **8** $4\sqrt{3}$; $6\sqrt{3}$ **9** $\sqrt{45}$ N, 53.1° **10** 3, 5 **11** 0.6 N; 099.6°

EXERCISE 23.2

1 (a) $P = 25\sqrt{3}$ N, $Q = 25$ N (b) $R = 67.13$ N, $S = 104.4$ N (c) $P = 50$ N, $T = 50\sqrt{5}$ N at 116.6° to P (d) $T_1 = 48.8$ N, $T_2 = 33.3$ N, angle between T_1 and T_2 = 88°
2 14.6 N, 10.4 N **3** 36.9°; 30 **5** 8.685; 154.3° **6** (a) 41.85 N, 23.4° (b) 60.7 N, 19.2° (c) $R = 47.95$ N, $S = 53.75$ N **7** 37.71 N, 201.8° **8** 125 N, 16.3° **9** 50.04 N, 122.6°
10 53.1°; $\frac{40}{7}$ **11** 18, 35.7° **12** (a) 120° (b) 8 **13** 3 N, 2 N
14 (a) [Fig.A23.1] 5.1, 51° or 19.5, 9° (b) 12.3, 30°; 18.7

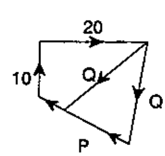


Fig. A23.1

EXERCISE 23.3

1 16 N **2** 0.5 **3** 772 **4** 7.87 N **5** 100 N **6** $\frac{1}{3}$ **7** 9.8 **9** (i) 14 (ii) 46
10 [Fig.A23.2] (ii) $\frac{1}{4}$ (ii) 2 N **11** $\frac{2}{11}$; $3\frac{8}{11}$ N **12** 40; $\frac{1}{4}$ **13** [Fig.A23.3] 14 **14** (a) 59.1 (b) 326 **15** 30 N in direction 048.1°; $\frac{3}{8}$

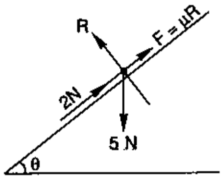


Fig. A23.2

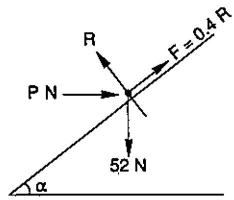


Fig. A23.3

REVISION EXERCISE 23

1 24 N, 32 N **2** 4.28; 17.85 **3** 8.83 N, 316.6°; 8.83; 0.24 **4** 32.6, 43.9 N **5** 10, 090°; 223.9°, 18 N **6** 7 N at 38.2° to AC **7** (a) 128.7° (b)(i) 8.48 (ii) 3.59 N **8** 7, 129.2°
9 $\sqrt{375}$ ($\approx$ 19.4) N, 63.4° **10** 0.8; 49.6 **11** 461, 692 N, 0.865 **12** [Fig.A23.4] 24
13 2, 0.225 **14** (i) 82.5 N (ii) 80 N **15** 22.8 N **16** 88.4 N **17** 0.095 N
18 5.36 N parallel to plane **19** (a) 25.6 N in direction 329.6° (b) 22.1 N in direction 180°; 7.1
20 8.3, 4.4; 4.7, 129.3°

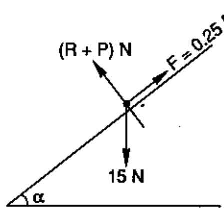


Fig. A23.4

EXERCISE 24.1

1 10 m s^{-2} **2** $6\frac{2}{3}$ m s^{-2} **3** $1\frac{9}{11}$ m s^{-2} **4** 0.4 N **5** 1.2 N **6** 8 m s^{-1} **7** 3.4 m s^{-2}
8 44.72 m s^{-2} at 26.6° to 20 N **9** 2.5 m s^{-2} **10** 3 s **11** 8 N **12** 6.16 m s^{-2}
13 $\frac{13}{70}$ (0.186) m s^{-2} **14** (a) 800 N (b) 840 N (c) 768 N **15** 1232 N **16** (a) 0.67 m s
(b) 6.7 N (c) 93.3 N (d) 98.3 N **17** 931.9 N **18** 0.25 m s^{-2} **19** 0.375 **20** 0.5

EXERCISE 24.2

1 (a) 2.5 m s^{-2} (b) 37.5 N (c) 75 N **2** 4 m s^{-2}; 12 N **3** 1.07 m s^{-2} **4** $\frac{76}{21}$, $\frac{84}{19}$ **5** $\frac{4}{3}$ m s^{-2}
6 (a) 1800 N, 5800 N (b) 2070 N, 6670 N **7** (a) 1705 N, 465 N (b) 1925 N, 525 N

8 (a) $0.4\ \mathrm{m\,s^{-2}}$ (b) $2\frac{2}{15}$ (≈ 3.13) m **9** $0.41\ \mathrm{m\,s^{-2}}$ **10** $1\ \mathrm{m\,s^{-2}}$ **11** $\frac{2}{3\sqrt{3}}$ (0.38)
12 (a) 12 N (b) $5\ \mathrm{m\,s^{-2}}$ (c) 9 N **13** (a) 40 N, 10 N, 5 N (b) 40.3 N, 10.1 N, 5.05 N
14 (a) $\frac{5}{6}\ \mathrm{m\,s^{-2}}$ (b) $18\frac{1}{3}$ N, 17.5 N

REVISION EXERCISE 24

1 (i) 16.8 (ii) down the plane, $0.8\ \mathrm{m\,s^{-2}}$ **2** (i) 4200 N (ii) $6\ \mathrm{m\,s^{-1}}$ (iii) 3400 N
3 42.4; $0.714\ \mathrm{m\,s^{-2}}$ **4** (a) $\frac{10}{9}$ m (b) $3.46\ \mathrm{m\,s^{-1}}$ **5** (i) 3100 N, 10 110 N (ii) 600 N, 2600 N
(iii) 300 N, 1300 N **6** (i) 1350 N, 150 N (ii) 6090 N, 390 N **7** $\frac{5}{3}\mathrm{m\,s^{-2}}$ (i) $\frac{9}{8}$m
(ii) $5\ \mathrm{m\,s^{-2}}$ **8** (i) $2\ \mathrm{m\,s^{-2}}$ (ii) 16 N (iii) $16\sqrt{2}$ N **9** (i) $1\frac{2}{3}\ \mathrm{m\,s^{-2}}$ (ii) 8.75 m **10** 0.257
11 (a) $2.5\ \mathrm{m\,s^{-2}}$ (b) 0.6 s (c) 0.4 s **12** (a) 0.4 (b) 3.5 m **13** 392 N, 98 N **14** $6\frac{1}{2}$, 1
15 $\frac{5\sqrt{2}}{7} \approx 1.01\ \mathrm{m\,s^{-2}}$ **16** (i) $\frac{2}{3}\ \mathrm{m\,s^{-2}}$ (ii) 9.33 N, $\frac{5}{12}$ m **17** (i) 9.6 N
(ii) $0.6\ \mathrm{m\,s^{-1}}$ (iii) 0.198 m **18** (i) 24 N, 0.5 (ii) 2 kg **19** 1.08 N
(i) $0.8\ \mathrm{m\,s^{-1}}$ (ii) $4\ \mathrm{m\,s^{-1}}$, 2.8 m **20** (i) $1.25\ \mathrm{m\,s^{-2}}$ (ii) 380 N (iii) 14.4 kW
(iv) $0.625\ \mathrm{m\,s^{-2}}$

EXERCISE 25.1

1 5000 J **2** 40 J **3** 160 J **4** $13\frac{1}{3}$ N **5** $1\frac{9}{16}$ N; 0.16 **6** $1\frac{1}{8}$ N **7** 87.5 m **8** $6\frac{2}{3}$ N
9 333.9 N **10** 81 N **11** 4.17×10^4 N **12** 1.67×10^3 N **13** (a) 0.35 m (b) 0.08 m
14 1.67×10^4 N **15** $2.45\ \mathrm{m\,s^{-1}}$ **16** 19.3° **17** $2.6\ \mathrm{m\,s^{-1}}$ **18** (a) $8.6\ \mathrm{m\,s^{-1}}$ (b) 514 J
19 $10.25\ \mathrm{m\,s^{-1}}$ **20** 49 J **21** 15 120 N **22** $\sqrt{\frac{12}{13}} \approx 0.96\ \mathrm{m\,s^{-1}}$ **23** (a) $\sqrt{7} \approx 2.65\ \mathrm{m\,s^{-1}}$
(b) 5 cm **24** (a) $4\ \mathrm{m\,s^{-1}}$ (b) $4 \times \sqrt{\frac{2}{5}} \approx 2.53$ m **25** (a) 9.12 (b) $6.69\ \mathrm{m\,s^{-1}}$

EXERCISE 25.2

1 3 kW **2** 352 W **3** 300 W **4** $28\ \mathrm{m\,s^{-1}}$ **5** 7.5 kW **6** 600 kW **7** 5.7 kW **8** 400 N
9 20 kW **10** $50\ \mathrm{m\,s^{-1}}$; $40.4\ \mathrm{m\,s^{-1}}$; $0.26\ \mathrm{m\,s^{-2}}$ **11** 343 N; $35\ \mathrm{m\,s^{-1}}$ **12** 9×10^4 N; 0.04 m s^{-}
13 386 N; 15.43 kW **14** $30.9\ \mathrm{m\,s^{-1}}$ **15** 114 N; 257 N; 11.13 kW **16** (a) 78 000 N
(b) 780 kW **17** $153\frac{1}{3}$ W **18** 1.8 m; $6\ \mathrm{m\,s^{-1}}$ **19** 3 kW **20** 7200 N; (a) $20.7\ \mathrm{m\,s^{-1}}$
(b) $1.1\ \mathrm{m\,s^{-2}}$

REVISION EXERCISE 25

1 (i) 604 kJ (ii) 1208 m **2** (i) 2 s (ii) 75 J (iii) 25 J **3** (i) 20 J (ii) 48 J (iii) 28 J
(iv) $2\frac{1}{3}$ N; 5 m **4** 5.4 kW; 6 kW **5** (i) $30\ \mathrm{m\,s^{-1}}$ (ii) $22.5\ \mathrm{m\,s^{-1}}$ (iii) $1\frac{2}{3}\ \mathrm{m\,s^{-1}}$ **6** 800 N
(i) $16\ \mathrm{m\,s^{-1}}$ (ii) $1\frac{9}{16}\ \mathrm{m\,s^{-2}}$ **7** (i) 96 J (ii) 30 J (iii) 126 J (iv) 24.1 N **8** (i) 102 m
(ii) 181.5 m **9** (a) 4 **10** 2.7°; $\frac{1}{8}\ \mathrm{m\,s^{-2}}$ **11** (i) 208 000 N (ii) 12 480 kW; $0.43\ \mathrm{m\,s^{-2}}$
12 11.25 kW; 14.25kW; $0.24\ \mathrm{m\,s^{-2}}$ **13** (i) 75 J (ii) 45 J (iii) 0.5 (iv) $\sqrt{5} \approx 2.23\ \mathrm{m\,s^{-1}}$
14 10 **15** (i) 4 J (ii) 32 J (iii) $6\ \mathrm{m\,s^{-1}}$ (iv) 0.6 **16** $\sqrt{85} \approx 9.2\ \mathrm{m\,s^{-1}}$; $\sqrt{135} \approx 11.6$
17 $\sqrt{20} \approx 4.5$ **18** 7500 W; $0.5\ \mathrm{m\,s^{-2}}$ **19** 5, 1.8 **20** (a) 3 (b)(i) 270 000 J
(ii) $375\,(v^2 - 144)$ (iii) 108 000 J (iv) 210 000 J (v) 66 000 J (vi) 86 400 J; 30 720 W

EXERCISE 26.1

1 (a) 10 N s (b) 0.5 N s (c) 0.12 N s (d) $\frac{5}{3}\times 10^3$ N s **2** 5 N s; 7.5 m s^{-1}; 3.125 m **3** 4 N **4** 2.25 N s **5** 0.36 N s **6** 20 N s; 40 N **7** 16.8 N **8** 9.38 N **9** 4.17 N; 18 s **10** 37.5 N **11** 51.4 N **12** 250 N s; 250 N **13** 3.77 kg; 45.2 N **14** 5 m s^{-1}, 3 m s^{-1} **15** $3\sqrt{3}$, 2 m s^{-1}; $\sqrt{31}\approx 5.6$ m s^{-1}, 21.1° to floor

EXERCISE 26.2

1 2.8 m s^{-1} **2** $1\frac{1}{11}$ m s^{-1}; $\frac{3}{11}$ N s **3** $1\frac{1}{3}$ m s^{-1}; 60 N **4** 0.1 m s^{-1} **5** $\frac{10}{3}$ m s^{-1} **6** 1 m s^{-1} in opposite direction; 10.4 N s **7** $13\frac{1}{3}$ m s^{-1} **8** New speed = $2933\frac{1}{3}$ m s^{-1} **9** 6.25 m s^{-1} **10** 5.7 m s^{-1}; -4.3 m s^{-2} **11** 4.3 m s^{-2}; 8.6 m s^{-1}; 3.7 m; 2.5 m s^{-1} **12** (a) 3.75 m (b) A 5 m s^{-1} downwards; B 5 m s^{-1} upwards (c) 0.25 m s^{-1} upwards (d) $\frac{1}{320}$ m **13** (a) $\frac{5}{3}$ m s^{-1} north (b) 20 N s **14** 0.36 m **15** 25 m s^{-1}; $8\frac{1}{3}$ m s^{-1} **16** 100 m s^{-1} in opposite directon; 100 kJ; 250 kJ **17** 3.97 m s^{-1}; 476.8 N **18** 1 m s^{-1}; 84% **19** 0.44 m **20** 32.5° **21** (a) $8\frac{1}{3}$ m s^{-1} (b) $416\frac{2}{3}$ J (c) $4166\frac{2}{3}$ N **22** (a) 8702.5 J (b) 4.8 m **23** (a) 6 m s^{-1}, 4 m s^{-1} (b) 12 m **24** (a) 7.2 m (b) 4 m s^{-1} (c) $4\frac{4}{5}$ m

REVISION EXERCISE 26

1 0.4 m s^{-1}; 1200 N **2** 3.75, 5.75 m s^{-1} **3** $\frac{1}{3}\sqrt{39}$, $\frac{2}{3}\sqrt{39}$ **4** 1.4 m s^{-1}; 150 **5** 20 m s^{-1}; 10.4 m s^{-1} **6** (i) 6 m s^{-1} (ii) $\frac{4}{3}$ m s^{-1} (iii) 6.3 J **7** (i) 25 m s^{-1} (ii) downwards (iii) $21\frac{2}{3}$ m s^{-2} **8** $1\frac{1}{2}$ m s^{-1} (i) 25 s, 30 s (iii) $6\frac{7}{8}$ m **9** $u = 10$ (a) 3 (b) 189 **10** 0.24; 0.16 **11** 0.54 N s; 13.5 N **12** 0.8 m; 4 m s^{-1}, 2000 N **13** 8 m s^{-1} (a) 5 (b) 39 J (c) 1300 N **14** 3:16 **15** (a) 7.2 m (b) 16, 12 m s^{-1} (c) 4.8 m **16** (a) 4 m s^{-1} (b) 3 m s^{-1} (c) 0.45 m (d) 56.6° **17** (a) 5 kg (b) 96 N, 40 N (c) 2 m s^{-1} (d) $1\frac{2}{3}$ m s^{-1} **18** 6 **19** 2; $13\frac{1}{3}$ J **20** (a) 24 N s (b) 6 m s^{-1} (c) 9 N s (d) 43.2 J (e) $\frac{1}{2}$ s (f) 5 m s^{-1}

PAPER 11

1 (a) 6 m s^{-2} (b) 325 m **2** 36.1 N in direction 046.1° **3** (a) 2.5 m s^{-2} (b) 8 sec **4** $\frac{3}{8}$ **5** 1.4 **6** (a) 7.2 m (b) 1.2 sec **7** $53\frac{1}{3}$ m s^{-1} **8** 3.2 m s^{-1} **9** 280 N **10** $3\frac{1}{3}$ kg or 7.5 kg

PAPER 12

1 13.7 km h^{-1} in direction 229.1° **2** 20 **3** 20 kW; $\frac{5}{8}$ m s^{-2} **4** (a) 0.4 (b) 36 **5** 50 N **6** 15 m s^{-1}; 1 sec **7** (a) 10 m (b) 10 m s^{-1}, 7 m s^{-1} (c) $\frac{3}{2}$ m s^{-1} **8** $4\sqrt{2}$ (≈ 5.7) N; 45° **9** (a) 4000 J (b) 400 J (c) 4.4 kW **10** $\frac{5}{4}(2\sqrt{2}+\sqrt{6})$ (≈ 6.6)

PAPER 13

1 $\frac{1}{3}$; 26 N **2** (a) 021.1° (b) 65.6 km h^{-1} **3** 0.4, 10 **4** (a) 25 (b) 11.25 m **5** (a) 60 N, 80 N (b) 37°, 7.4 N **6** (a) 6.2 kg (b) 1.4 kg **7** (a) $3V$ (b) $\frac{40}{3}MV^2$ (c) $18V$ **8** (a) 2.5 m s^{-2} (b) 15 N (c) 27.5 N **9** (a) 20 m s^{-1} (b) 4 m s^{-2} **10** (a) 5 sec (b) $3\frac{1}{3}$ sec (c) $11\frac{1}{9}$ m

PAPER 14

1 95.3°; 8.7 **2** (a) $\frac{1}{2}$ m s^{-2} (b) 54 sec **3** 14.2 km h^{-1}; (from) 148° **4** (a) 098.1° (b) $\frac{3}{4}$ hr
5 20, 60 m s^{-1} **6** 2.5 m s^{-1}; 1.5 N s **7** 7 **8** (a) 1400 W (b) 113.9 N
9 (a) $\frac{4}{\sqrt{3}}$ ($\approx$ 2.3) m s^{-1} (b) $2\frac{4}{15}$ ($\approx$ 2.27) m **10** (a) 45 m s^{-1}, 60 m s^{-1} (b) 133.2 m

PAPER 15

1 (a) 30 m s^{-1} (b) 45 m (c) 2 sec **2** (a) 36 N (b) $\frac{4}{5}(2\sqrt{3}-3)$ ($\approx$ 0.38) m s^{-2} **3** 3.6, 2.9
4 (a) 10.3 km h^{-1} in direction 283.1° (b) 8.1 km h^{-1} (c) 1 hr 44 min **5** 1, $\sqrt{78}$ ($\approx$ 8.8) N
6 2, 2 m s^{-1} **7** (a) 1 m s^{-2} (b) 9 m s^{-1} **8** (a) 2 m s^{-1} (b) 0.71 m **9** 60, 25 N
10 (a) 6 sec (b) 18 m s^{-1}, 26 m s^{-1} (c) 416 m

PAPER 16

1 6 m s^{-1}; 28.9 m upstream, 28.9 sec **2** (a) 6 m s^{-1} (b) 0.7 (c) $1\frac{1}{7}$ m
3 (a) 13.2 N, 069.1^{-1} (b) 5, 60° (c) $2\sqrt{3}$ ($\approx$ 3.5) N; 4 N **4** (a) 300 N (b) 525 N
(c) 21.1 m s^{-1} (d) 1 m s^{-2} **5** (a) 1 m s^{-1} vertically downwards, 12 m s^{-1} horizontally;
(b) 4 m (c) 3 m **6** 336.8 km h^{-1} in direction 032.9°; 217.2 km h^{-1} in direction 122.7°
7 (a) 1 m s^{-1}, 0.004 sec (b)(i) 225 J (ii) 7.5 **8** (a) 660 m (b) 2.4 m s^{-2} (c) 18.75 sec
9 (a) 3.3 N s; 660 N (b) 9.71 m s^{-1}, 3237 N **10** (a) 6 m s^{-1} (b) 4.5 m (c) 1 m s^{-1}
(d) $2\frac{5}{8}$ sec

PAPER 17

1 (a) 2 m s^{-2} (b) 1.4, 1.2 m s^{-1} (c) $\frac{6}{7}$ m s^{-2}, $6\frac{6}{7}$ N **2** (a) 163 km in direction 024.8°
(b) 209.6° (c) 41.5 min **3** (a) 2 m s^{-1} (b) $\frac{2}{3}$ m s^{-1} (c) $2\frac{2}{9}$ cm
4 (a) 2 m s^{-1} (direction B $\rightarrow$ A), 4 m s^{-1} (direction A $\rightarrow$ B); 24 J (b)(i) 2 m s^{-2} (ii) 2 m s^{-1}
(iii) 20 cm (iv) 0.4 m s^{-1} **5** (a)(i) 14 m s^{-1} (ii) 9.8 m (iii) 1.2 sec (b) 30; $\frac{4}{3}$
6 (a) $\frac{1}{2}$, 5; 74.75 m (b) 2 and 8 sec; 150 m **7** (a) 2 s (b) 40 m (c) 10, 20 m s^{-1}
(d) 3.3 s **8** (i) 180 N (ii) 680 N (iii) 16.32 kW (iv) 330 N (v) 31.92 kW
9 (a) Both $\frac{20}{\sqrt{3}}$ ($\approx$ 11.6) N (b) 40, $60\sqrt{3}$ ($\approx$ 103.9) (c) $\frac{1}{2}\sqrt{34}$ ($\approx$ 2.9), 61.9° ($\cos\theta = \frac{8}{17}$)
10 15.3 km h^{-1} from direction 130.9°

PAPER 18

1 21.9° **2** 27.1 N **3** (a) 6 m s^{-1} (b) 4.8 m s^{-1} (c) 1736 N
4 (a) 15.7 km h^{-1} in direction 262.9° (b) 58 km h^{-1} (c) 21.4 min **5** (a) 4 m s^{-2}
(b) 4 m s^{-1} (c) 1.2 sec (d) 0.95 m **6** (a) 3 N s (b) 1.25 m s^{-1} (c) $\frac{9}{8}$(=1.125) N s
(d) 0.5 m **7** (a) 90° to the bank (b) 48.2° upstream to the bank (c) 50 s, 67.1 s
(d) 57.7 s **8** (a) 39.2° (b) 19.3° (c) 11.4 m horizontally **9** 0.4, 20
10 35 000 N, $\frac{1}{30}$ m s^{-2}