J.S. Milne: Elliptic Curves

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Lecture Notes in Math. 900, Springer-Verlag, 1982, 414 pages, ISBN 3-540-11174-3 and 0-387-11174-3

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Elliptic Curves

J.S. Milne

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The Kea is a friendly intelligent parrot found only in the mountains of New Zealand.

Preface

In early 1996, I taught a course on elliptic curves. Since this was not long after Wiles had proved Fermat's Last Theorem and I promised to explain some of the ideas underlying his proof, the course attracted an unusually large and diverse audience. As a result, I attempted to make the course accessible to all students with a knowledge only of the standard first-year graduate courses.

When it was over, I collected the notes that I had handed out during the course into a single file, made a few corrections, and posted them on the Web, where they have since been downloaded tens of thousands of times.

The appearance of publishers willing to turn pdf files into books quickly and cheaply and make them available worldwide while allowing the author to retain full control of the content and appearance of the work has prompted me to rewrite the notes and make them available as a paperback.

J.S. Milne,
October 30, 2006.

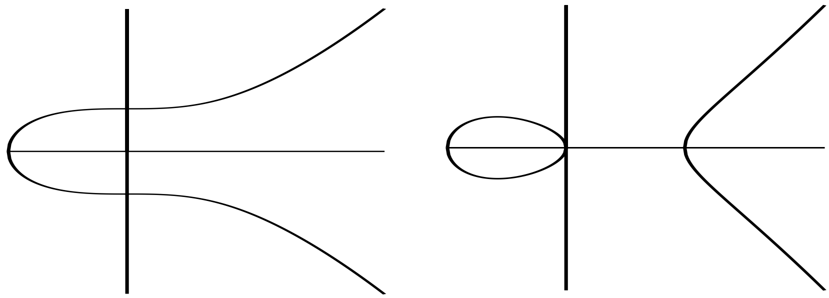
Contents

Introduction

An elliptic curve over a field k is a nonsingular complete curve of genus 1 with a distinguished point. When the characteristic of k is not 2 or 3, it can be realized as a plane projective curve

$$Y^2Z = X^3 + aXZ^2 + bZ^3, \qquad 4a^3 + 27b^2 \neq 0,$$

and every such equation defines an elliptic curve over k. The distinguished point is $(0 : 1 : 0)$. For example, the following pictures show the real points (except the point at infinity) of two elliptic curves:



$Y^2 = X^3 + 1$ $\qquad\qquad$ $Y^2 = X(X^2 - 1)$

Although the problem of computing the points on an elliptic curve E with rational numbers as coordinates has fascinated mathematicians since the time of the ancient Greeks, it was not until 1922 that it was proved that it is possible to construct all the points starting from a finite number by drawing chords and tangents. This is the famous theorem of Mordell, which shows more precisely that the rational points form a finitely generated group $E(\mathbb{Q})$. There is a simple algorithm for computing the torsion subgroup of $E(\mathbb{Q})$, but there is still no *proven* algorithm for computing the rank. In one of the earliest applications of computers to mathematics, Birch and Swinnerton-Dyer discovered experimentally a relation between the rank and the numbers N_p of the points on the curve read modulo the different prime numbers p. The problem of proving this relation (the conjecture of Birch and Swinnerton-Dyer) is one of the most important in mathematics. Chapter IV of the book proves Mordell's theorem and explains the conjecture of Birch and Swinnerton-Dyer.

In 1955, Taniyama noted that it was plausible that the N_p attached to a given elliptic curve always arise in a simple way from a modular form (in modern terminology, that the curve is modular). However, in 1985 Frey observed that this didn't appear to be true for the elliptic curve attached to a nontrivial solution of the Fermat equation $X^p + Y^p = Z^p$, $p > 2$. His observation prompted Serre to revisit some old conjectures implying this, and Ribet proved enough of his conjectures to deduce that Frey's observation is correct: the elliptic curve attached to a nontrivial solution of the Fermat equation is not modular. Finally, in 1994 Wiles (with the help of Taylor) proved that every elliptic curve in a large class is modular. Since the class would contain any curve attached to a nontrivial solution of the Fermat equation, this proves that no such solution exists. Chapter V of the book is devoted to explaining these results.

The first three chapters of the book develop the basic theory of elliptic curves.

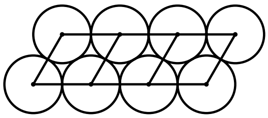
Elliptic curves have been used to shed light on some important problems that, at first sight, appear to have nothing to do with elliptic curves. I mention three such problems.

Fast factorization of integers

There is an algorithm for factoring integers that uses elliptic curves and is in many respects better than previous algorithms. People have been factoring integers for centuries, but recently the topic has become of practical significance: given an integer n that is the product $n = pq$ of two (large) primes p and q, there is a secret code for which anyone who knows n can encode a message, but only those who know p, q can decode it. The security of the code depends on no unauthorized person being able to factor n. See Koblitz 1987, VI 4, or Silverman and Tate 1992, IV 4.

Lattices and the sphere packing problem

The sphere packing problem is that of finding an arrangement of n-dimensional unit balls in Euclidean space that covers as much of the space as possible without overlaps. The arrangement is called a lattice packing if the centres of the spheres are the points of a lattice in n-space.



The best packing in the plane is a lattice packing.

Elliptic curves have been used to find lattice packings in many dimensions that are denser than any previously known (see IV, §11).

Congruent numbers

A natural number n is said to be ***congruent*** if it occurs as the area of a right triangle whose sides have rational length. If we denote the lengths of the sides of the triangle by x, y, z, then n will be congruent if and only if the equations

$$x^2 + y^2 = z^2, \qquad n = \tfrac{1}{2}xy$$

have simultaneous solutions in $\mathbb{Q}$. The problem was of interest to the ancient Greeks, and was discussed systematically by Arab scholars in the tenth century. Fibonacci showed that 5 and 6 are congruent, Fermat that $1, 2, 3$, are not congruent, and Euler proved that 7 is congruent, but it appeared hopeless to find a simple criterion for deciding whether a given n is congruent until Tunnell showed that the conjecture of Birch and Swinnerton-Dyer implies the following critertion:

> An odd square-free n is congruent if and only if the number of triples of integers (x, y, z) satisfying $2x^2 + y^2 + 8z^2 = n$ is equal to twice the number of triples satisfying $2x^2 + y^2 + 32z^2 = n$.

See Koblitz 1984.

Among the many works on the arithmetic of elliptic curves, I mention here only the survey article Cassels 1966, which gave the first modern exposition of the subject, Tate's Haverford lectures (reproduced in Silverman and Tate 1992), which remain the best elementary introduction, and the two volumes Silverman 1986, 1994, which have become the standard reference.

PREREQUISITES

A knowledge of the basic algebra, analysis, and topology usually taught in advanced undergraduate or beginning graduate courses. Some knowledge of algebraic geometry and algebraic number theory will be useful but not essential.

NOTATIONS

We use the standard notations: $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$, $\mathbb{Z}$ the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, and $\mathbb{F}_p$ the field with p elements. A number field is a finite extension of $\mathbb{Q}$.

Throughout the book, k is a field and k^{al} is an algebraic closure of k. A k-field is a field containing k, and a homomorphism of k-fields is a homomorphism of fields acting as the identity map on k.

All rings will be commutative with 1, and homomorphisms of rings are required to map 1 to 1. For a ring A, $A^\times$ is the group of units in A:

$$A^\times = \{a \in A \mid \text{there exists a } b \in A \text{ such that } ab = 1\}.$$

For an abelian group X, $X_n = \{x \in X \mid nx = 0\}$. For a finite set S, $\#S$ or (occasionally) $[S]$ denotes the number of elements of S. For an element a of a set with an equivalence relation, we sometimes use $[a]$ to denote the equivalence class of a.

$X \stackrel{\text{def}}{=} Y$	X is defined to be Y, or equals Y by definition;
$X \subset Y$	X is a subset of Y (not necessarily proper, i.e., X may equal Y);
$X \approx Y$	X and Y are isomorphic;
$X \simeq Y$	X and Y are canonically isomorphic, or there is a given or unique isomorphism from one to the other.

REFERENCES

In addition to the references listed at the end, I refer to the following of my course notes (available at www.jmilne.org/math/).

ANT Algebraic Number Theory (August 31, 1998).
AG Algebraic Geometry (February 20, 2005).
CFT Class Field Theory (May 6, 1997).
FT Fields and Galois Theory (February 19, 2005).
MF Modular Functions and Modular Forms (May 22, 1997).

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Chapter I

Plane Curves

1 Basic definitions; Bezout's theorem

In this section we review part of the theory of plane curves. Omitted details (and much more) can be found in Fulton 1969 and Walker 1950.

Polynomial rings

We shall make frequent use of the fact that polynomial rings over fields are unique factorization domains: for the field itself, there is nothing to prove, and the general case follows by induction from the statement that if A is a unique factorization domain, then so also is $A[X]$ (AG, §1). Thus, in $k[X_1,\ldots,X_n]$, every polynomial f can be written as a product

$$f = f_1^{m_1} \cdots f_r^{m_r}$$

of powers of irreducible polynomials f_i with no f_i being a constant multiple of another, and the factorization is unique up to replacing the f_i with constant multiples. The ***repeated factors*** of f are those f_i with $m_i > 1$. If f is irreducible, then the ideal (f) that it generates is prime.

Affine plane curves

The ***affine plane*** over k is $\mathbb{A}^2(k) = k \times k$. A nonconstant polynomial $f \in k[X,Y]$, assumed to have no repeated factor in $k^{\mathrm{al}}[X,Y]$, defines an ***affine plane curve*** C_f over k whose points with coordinates in any field $K \supset k$ are the zeros of f in K^2:

$$C_f(K) = \{(x,y) \in K^2 \mid f(x,y) = 0\}.$$

For any $c \in k^{\times}$, the curves C_f and C_{cf} have the same points in every field K, and so we don't distinguish them.[1]

The curve C_f is said to be ***irreducible*** if f is irreducible in $k[X,Y]$, and it is said be ***geometrically irreducible*** if f remains irreducible over k^{al}. For any curve C_f, we can write $f = f_1 f_2 \cdots f_r$ with the f_i distinct irreducible polynomials in $k[X,Y]$, and

$$C_f(K) = C_{f_1}(K) \cup \cdots \cup C_{f_r}(K)$$

with the C_{f_i} irreducible curves. The C_{f_i} are called the ***irreducible components*** of C_f.

We often write $C\colon f = 0$ to mean that C is the curve C_f, or even $C\colon f_1 = f_2$ to mean that C is the curve $C_{f_1 - f_2}$.

EXAMPLE 1.1 Let $f_1(X,Y)$ be an irreducible polynomial in $\mathbb{Q}[\sqrt{2}][X,Y]$, no constant multiple of which lies in $\mathbb{Q}[X,Y]$, and let $\bar{f}_1(X,Y)$ be its conjugate over $\mathbb{Q}$ (obtained by replacing each $\sqrt{2}$ in f_1 with $-\sqrt{2}$). Then $f(X,Y) \overset{\text{def}}{=} f_1(X,Y)\bar{f}_1(X,Y)$ lies in $\mathbb{Q}[X,Y]$ because it is fixed by the Galois group of $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$. The curve C_f is irreducible but not geometrically irreducible. For example, the curve $X^2 - 2Y^2 = 0$ is irreducible over $\mathbb{Q}$, but becomes the pair of lines $(X - \sqrt{2}Y)(X + \sqrt{2}Y) = 0$ over $\mathbb{Q}[\sqrt{2}]$.

EXAMPLE 1.2 Assume that k has characteristic $p \neq 0$ and that it is not perfect, so that there exists an $a \in k$ that is not a pth power in k. Consider

$$f(X,Y) = X^p + aY^p.$$

Then f is irreducible in $k[X,Y]$, but in $k^{\mathrm{al}}[X,Y]$ it equals $(X + \alpha Y)^p$ where $\alpha^p = a$ (recall that the binomial theorem takes on a particularly simple form for pth powers in characteristic p). Thus f does not define an affine plane curve. This problem only occurs for non perfect k: when k is perfect, a polynomial with no repeated factor in $k[X,Y]$ will not acquire a repeated factor in $k^{\mathrm{al}}[X,Y]$.

We define the partial derivatives of a polynomial by the usual formulas. Let $P = (a,b) \in C_f(K)$, some $K \supset k$. If at least one of the partial derivatives $\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}$ is nonzero at P, then P is said to be ***nonsingular***, and the ***tangent line*** to C at P is defined to be

$$\left(\frac{\partial f}{\partial X}\right)_P (X-a) + \left(\frac{\partial f}{\partial Y}\right)_P (Y-b) = 0.$$

[1]More formally, an affine plane curve is an equivalence class of nonconstant polynomials in $k[X,Y]$ having no repeated factor in $k^{\mathrm{al}}[X,Y]$, where two polynomials are defined to be equivalent if one is a nonzero constant multiple of the other; then C_f denotes the equivalence class containing f.

A curve C is said to be ***nonsingular*** if all the points in $C(k^{\mathrm{al}})$ are nonsingular.[2] A curve or point that is not nonsingular said to be ***singular.***

ASIDE 1.3 Let $f(x,y)$ be a real-valued function on $\mathbb{R}^2$. In advanced calculus one learns that $\nabla f \overset{\mathrm{def}}{=} \left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right)$ is a vector field on $\mathbb{R}^2$ which, at any point $P = (a,b) \in \mathbb{R}^2$, points in the direction in which $f(x,y)$ increases most rapidly (i.e., has the most positive directional derivative). Hence $(\nabla f)_P$ is normal to any level curve $f(x,y) = c$ through P, and so the line

$$(\nabla f)_P \cdot (X - a, Y - b) = 0$$

is normal to the normal to the level curve. Since it also passes through (a,b), it is the tangent line to the level curve.

REMARK 1.4 A point on the intersection of two irreducible components of a curve is always singular. Consider, for example, C_f where $f = f_1 f_2$ and $f_1(0,0) = 0 = f_2(0,0)$. Then

$$\begin{aligned}\frac{\partial f}{\partial X}(0,0) &= \left(f_1\frac{\partial f_2}{\partial X} + f_2\frac{\partial f_1}{\partial X}\right)(0,0) = 0\\ \frac{\partial f}{\partial Y}(0,0) &= \left(f_1\frac{\partial f_2}{\partial Y} + f_2\frac{\partial f_1}{\partial Y}\right)(0,0) = 0,\end{aligned}$$

which shows that $P = (0,0)$ is singular on C_f.

EXAMPLE 1.5 Consider the curve

$$C: \quad Y^2 = X^3 + aX + b.$$

At a singular point (x,y) of C,

$$2Y = 0, \quad 3X^2 + a = 0, \quad Y^2 = X^3 + aX + b.$$

Assume $\mathrm{char}(k) \neq 2$. Then $y = 0$ and x is a common root of $X^3 + aX + b$ and its derivative, i.e., it is a double root of $X^3 + aX + b$. It follows that

$$\begin{aligned} C \text{ is nonsingular} &\iff X^3 + aX + b \text{ has no multiple root (in } k^{\mathrm{al}})\\ &\iff \text{its discriminant}^3\, \Delta = 4a^3 + 27b^2 \text{ is nonzero.}\end{aligned}$$

Assume $\mathrm{char}(k) = 2$. Then C always has a singular point in k^{al}, namely, (α, β) where $3\alpha^2 + a = 0$ and $\beta^2 = \alpha^3 + a\alpha + b$.

[2] The curve will then have no singular point with coordinates in any field $K \supset k$. For if C has a singular point in K, then the ideal generated by the polynomials $f, \frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}$ will not be the whole of $k[X,Y]$, and so the polynomials will have a common zero in k^{al} by Hilbert's Nullstellensatz (Fulton 1969, p. 20).

[3] According to the usual definition, this is actually the negative of the discriminant.

Let $P = (a, b) \in C_f(K)$. We can write f as a polynomial in $X - a$ and $Y - b$ with coefficients in K, say,

$$f(X, Y) = f_1(X - a, Y - b) + \cdots + f_n(X - a, Y - b)$$

where f_i is homogeneous of degree i in $X - a$ and $Y - b$ (this is the Taylor expansion of f). The point P is nonsingular if and only if $f_1 \neq 0$, in which case the tangent line to C_f at P has equation $f_1 = 0$.

Suppose that P is singular, so that

$$f(X, Y) = f_m(X - a, Y - b) + \text{ terms of higher degree,}$$

with $f_m \neq 0$, $m \geq 2$. Then P is said to have ***multiplicity*** m on C, denoted $m_P(C)$. If $m = 2$, then P is called a ***double point.*** For simplicity, take $(a, b) = (0, 0)$. Then (over k^{al})

$$f_m(X, Y) = \prod L_i^{r_i}$$

where each L_i is a homogeneous polynomial $c_i X + d_i Y$ of degree one with coefficients in k^{al}. The lines $L_i = 0$ (assumed to be distinct) are called the ***tangent lines*** to C_f at P, and r_i is called ***multiplicity*** of L_i. The point P is said to be an ***ordinary singularity*** if the tangent lines are all distinct, i.e., $r_i = 1$ for all i. An ordinary double point is called a ***node***.

EXAMPLE 1.6 The curve $Y^2 = X^3 + aX^2$ has a singularity at $(0, 0)$. If $a \neq 0$, it is a node, and the tangent lines at $(0, 0)$ are $Y = \pm\sqrt{a}X$. They are defined over k if and only if a is a square in k. If $a = 0$, the singularity is a cusp (see 1.12 below).

ASIDE 1.7 (For the experts.) Essentially, we have defined an affine (resp. projective) plane curve to be a geometrically reduced closed subscheme of $\mathbb{A}^2_k$ (resp. $\mathbb{P}^2_k$) of dimension 1: such a scheme corresponds to an ideal of height one, which is principal, because polynomial rings are unique factorization domains; the polynomial generating the ideal is uniquely determined by the scheme up to multiplication by a nonzero constant.

Intersection numbers

Let $\mathcal{F}(k)$ be the set of pairs of polynomials $f, g \in k[X, Y]$ having no common factor h in $k[X, Y]$ with $h(0, 0) = 0$. For $(f, g) \in \mathcal{F}(k)$, we wish to define the intersection number of the curves C_f and C_g at the origin. The next proposition shows that there is exactly one reasonable way of doing this.

PROPOSITION 1.8 *There is a unique map $I\colon \mathcal{F}(k) \to \mathbb{N}$ such that*

(a) $I(X, Y) = 1$;
(b) $I(f, g) = I(g, f)$ *all* $(f, g) \in \mathcal{F}(k)$;
(c) $I(f, gh) = I(f, g) + I(f, h)$ *all* $(f, g), (f, h) \in \mathcal{F}(k)$;

(d) $I(f, g + hf) = I(f, g)$ *all* $(f, g) \in \mathcal{F}(k)$, $h \in k[X, Y]$;
(e) $I(f, g) = 0$ *if* $g(0, 0) \neq 0$.

PROOF. We first prove the uniqueness. The theory of resultants (see 1.24) gives polynomials $a(X, Y)$ and $b(X, Y)$ such that $af + bg = r \in k[X]$ and $\deg_Y(b) < \deg_Y(f)$, $\deg_Y(a) < \deg_Y(g)$. If $\deg_Y(f) \leq \deg_Y(g)$, write

$$I(f, g) \stackrel{(c)}{=} I(f, bg) - I(f, b) \stackrel{(d)}{=} I(f, r) - I(f, b),$$

and otherwise, write

$$I(f, g) = I(r, g) - I(a, g).$$

Continue in this fashion until Y is eliminated from one of the polynomials, say, from g, so that $g = g(X) \in k[X]$. Write $g(X) = X^m g_0(X)$ where $g_0(0) \neq 0$. Then

$$I(f, g) \stackrel{(c,e)}{=} mI(f, X).$$

After subtracting a multiple of X from $f(X, Y)$, we can assume that it is a polynomial in Y alone. Write $f(Y) = Y^n f_0(Y)$ where $f_0(0) \neq 0$. Then

$$I(f, X) \stackrel{(a,b,c,e)}{=} n.$$

This completes the proof of uniqueness.

The theory of resultants shows that, for any field K containing k, $\mathcal{F}(k)$ is contained in $\mathcal{F}(K)$ (see 1.25), and so, in proving the existence, we may replace k with its algebraic closure. Let

$$k[X, Y]_{(0,0)} = \{h_1/h_2 \mid h_1, h_2 \in k(X, Y), h_2(0, 0) \neq 0\}$$

(local ring at the maximal ideal (X, Y)). For $f, g \in \mathcal{F}(k)$, the quotient ring $k[X, Y]_{(0,0)}/(f, g)$ is finite dimensional as a k-vector space , and when we set $I(f, g)$ equal to its dimension we obtain a map with the required properties (Fulton 1969, III 3). □

REMARK 1.9 As noted in the proof, $I(f, g)$ doesn't depend on whether f and g are regarded as polynomials with coefficients in k or in a bigger field K. In fact, the formula $I(f, g) = \dim_k k[X, Y]_{(0,0)}/(f, g)$ holds even when k is not algebraically closed.

EXAMPLE 1.10 Applying (1.8d), we find that

$$I(Y^2 - X^2(X + 1), X) = I(Y^2, X) = 2.$$

Although the Y-axis is not tangent to the curve $Y^2 = X^2(X + 1)$, the intersection number is > 1 because the origin is singular.

The argument in the proof is a practical algorithm for computing $I(f, g)$, but if the polynomials are monic when regarded as polynomials in Y, the following method is faster. If $\deg_Y(g) \geq \deg_Y(f)$, we can divide f into g (as polynomials in Y) and obtain

$$g = fh + r, \quad \deg_Y r < \deg_Y f \text{ or } r = 0.$$

By property (d),

$$I(f, g) = I(f, r).$$

Continue in this fashion until one of the polynomials has degree 1 in Y, and apply the following lemma.

LEMMA 1.11 *If $f(0) = 0$, then $I(Y - f(X), g(X, Y)) = m$ where X^m is the power of X dividing $g(X, f(X))$.*

PROOF. Divide $Y - f(X)$ into $g(X, Y)$ (as polynomials in Y) to obtain

$$g(X, Y) = (Y - f(X))h(X, Y) + g(X, f(X)),$$

from which it follows that

$$I(Y - f(X), g(X, Y)) = I(Y - f(X), g(X, f(X)) = mI(Y - f(X), X).$$

Finally, since we are assuming $f(0) = 0$, $f(X) = Xh(X)$, and so

$$I(Y - f(X), X) = I(Y, X) = 1. \qquad \square$$

Consider two curves C_f and C_g in $\mathbb{A}^2(k)$, and let $P \in C_f(K) \cap C_g(K)$, some $K \supset k$. We say that P is an ***isolated point*** of $C_f \cap C_g$ if C_f and C_g do not have a common irreducible component passing through P. Then f and g have no common factor h with $h(a, b) = 0$, and so we can define the ***intersection number*** of C_f and C_g at P to be

$$I(P, C_f \cap C_g) \stackrel{\text{def}}{=} I(f(X + a, Y + b), g(X + a, Y + b)).$$

For example, if $P = (0, 0)$, then $I(P, C_f \cap C_g) = I(f, g)$.

EXAMPLE 1.12 Let C be the curve $Y^2 = X^3$, and let $L : Y = 0$ be its tangent line at $P = (0, 0)$. Then

$$I(P, L \cap C) \stackrel{\text{def}}{=} I(Y^2 - X^3, Y) \stackrel{\text{1.8d}}{=} I(X^3, Y) = 3.$$

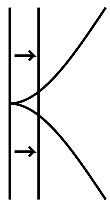
Thus P is a ***cusp***, i.e., it is a double point with only one tangent line L and $I(P, L \cap C) = 3$.

REMARK 1.13 As one would hope, $I(P, C \cap D) = 1$ if and only if P is nonsingular on both C and D and the tangent lines to C and D at P are distinct. More generally,

$$I(P, C \cap D) \geq m_P(C) \cdot m_P(D), \tag{1}$$

with equality if and only if C and D have no tangent line in common at P (Fulton 1969, p. 75).

ASIDE 1.14 Intuitively, the intersection number of two curves at a point is the actual number of intersection points after one of the curves has been moved slightly. For example, the intersection number at $(0, 0)$ of the Y-axis with the curve $Y^2 = X^2(X + 1)$ should be 2 because, after the Y-axis has been moved slightly, the single point of intersection becomes two points. Of course, this picture is complicated by the fact that the intersection points may become visible only when we use complex numbers — consider, for example, the intersection number at $(0, 0)$ of the Y-axis with the curve $Y^2 = X^3$.



Projective plane curves

The ***projective plane*** over k is

$$\mathbb{P}^2(k) = \{(x, y, z) \in k^3 \mid (x, y, z) \neq (0, 0, 0)\}/\sim$$

where $(x, y, z) \sim (x', y', z')$ if and only if there exists a $c \neq 0$ in k such that $(x', y', z') = (cx, cy, cz)$. We write $(x : y : z)$ for the equivalence class of (x, y, z) — the colon is meant to suggest that only the ratios matter. Let $P \in \mathbb{P}^2(k)$; the triples (x, y, z) representing P lie on a single line $L(P)$ through the origin in k^3, and $P \mapsto L(P)$ is a bijection from $\mathbb{P}^2(k)$ to the set of all such lines.

Projective n-space $\mathbb{P}^n(k)$ can be defined similarly for any $n \geq 0$.

Let $U_2 = \{(x : y : z) \mid z \neq 0\}$, and let $L_\infty(k) = \{(x : y : z) \mid z = 0\}$. Then

$$(x, y) \mapsto (x : y : 1) : \mathbb{A}^2(k) \to U_2$$
$$(x : y) \mapsto (x : y : 0) : \mathbb{P}^1(k) \to L_\infty(k)$$

are bijections. Moreover, $\mathbb{P}^2(k)$ is the disjoint union

$$\mathbb{P}^2(k) = U_2 \sqcup L_\infty(k)$$

of the "affine plane" U_2 with the "line at infinity" L_∞. The line

$$aX + bY + cZ = 0 \quad (a \text{ and } b \text{ not both zero})$$

meets L_∞ at the point $(-b : a : 0)$, which depends only on the slope of the affine line $aX + bY + 1 = 0$. We can think of $\mathbb{P}^2(k)$ as being the affine plane with exactly one point added for each family of parallel lines.

A nonconstant homogeneous polynomial $F \in k[X,Y,Z]$, assumed to have no repeated factor in k^{al}, defines a ***projective plane curve*** C_F over k whose points in any field $K \supset k$ are the zeros of F in $\mathbb{P}^2(K)$:

$$C_F(K) = \{(x:y:z) \in \mathbb{P}^2(k) \mid F(x,y,z) = 0\}.$$

Note that, because F is homogeneous,

$$F(cx,cy,cz) = c^{\deg F} F(x,y,z),$$

and so, although it doesn't make sense to speak of the value of F at a point P of $\mathbb{P}^2$, it does make sense to say whether or not F is zero at P. Again, we don't distinguish C_F from C_G if $G = cF$ with $c \in k^\times$. A plane projective curve is (uniquely) a union of irreducible projective plane curves (those defined by irreducible polynomials). The degree of F is called the ***degree*** of the curve C_F.

EXAMPLE 1.15 The curve

$$Y^2Z = X^3 + aXZ^2 + bZ^3$$

intersects the line at infinity at the point $(0:1:0)$, i.e., at the same point as all the vertical lines do. This is plausible geometrically, because, as you go out the affine curve

$$Y^2 = X^3 + aX + b$$

in $\mathbb{R} \times \mathbb{R}$ with increasing x and y, the slope of the tangent line tends to ∞.

Let $U_1 = \{(x:y:z) \mid y \neq 0\}$, and let $U_0 = \{(x:y:z) \mid x \neq 0\}$. Then U_1 and U_0 are again, in a natural way, affine planes; for example, we can identify U_1 with $\mathbb{A}^2(k)$ via

$$(x:1:z) \leftrightarrow (x,z).$$

Since at least one of x, y, or z is nonzero,

$$\mathbb{P}^2(k) = U_0 \cup U_1 \cup U_2.$$

A projective plane curve $C = C_F$ is the union of three affine plane curves,

$$C = C_0 \cup C_1 \cup C_2, \quad C_i = C \cap U_i.$$

When we identify each U_i with $\mathbb{A}^2(k)$ in the natural way, then C_0, C_1, and C_2 become identified with the affine curves defined by the polynomials $F(1,Y,Z)$, $F(X,1,Z)$, and $F(X,Y,1)$ respectively.

EXAMPLE 1.16 The curve

$$C: \quad Y^2Z = X^3 + aXZ^2 + bZ^3$$

is unusual, in that it is covered by two (rather than 3) affine curves, namely,

$$C_2 : Y^2 = X^3 + aX + b \quad \text{and}$$
$$C_1 : Z = X^3 + aXZ^2 + bZ^3.$$

The notions of tangent line, multiplicity, intersection number, etc. can be extended to projective curves by noting that each point P of a projective curve C will lie on at least one of the affine curves C_i.

EXERCISE 1.17 Let P be a point on a projective plane curve $C = C_F$. Show that P is singular on the affine plane curve C_i for some i if and only if

$$F(P) = 0 = \left(\frac{\partial F}{\partial X}\right)_P = \left(\frac{\partial F}{\partial Y}\right)_P = \left(\frac{\partial F}{\partial Z}\right)_P.$$

When P is nonsingular, show that the projective line

$$L: \quad \left(\frac{\partial F}{\partial X}\right)_P X + \left(\frac{\partial F}{\partial Y}\right)_P Y \quad + \left(\frac{\partial F}{\partial Z}\right)_P Z = 0$$

has the property that $L \cap U_i$ is the tangent line at P to the affine curve C_i for all i such that P lies in U_i.

Bezout's theorem

THEOREM 1.18 (BEZOUT) *Let C and D be projective plane curves over k of degrees m and n respectively having no irreducible component in common. Then C and D intersect over k^{al} in exactly mn points counting multiplicities, i.e.,*

$$\sum_{P \in C(k^{\mathrm{al}}) \cap D(k^{\mathrm{al}})} I(P, C \cap D) = mn.$$

PROOF. For elementary proofs, see Fulton 1969, Chap. 5, or Silverman and Tate 1992 , Appendix A. □

ASIDE 1.19 Over $\mathbb{C}$, one can show that the map

$$F, G \mapsto \sum_{P \in C_F(\mathbb{C}) \cap C_G(\mathbb{C})} I(P, C_F \cap C_G)$$

is continuous in the coefficients of F and G (for the discrete topology on $\mathbb{N}$). Thus, the total intersection number is constant on continuous families of homogeneous polynomials, which allows us, in proving the theorem, to take F and G to be X^m and Y^n respectively. Then $P = (0:0:0)$ is the only common point of the curves, and

$$I(P, C_{X^m} \cap C_{Y^n}) = I(X^m, Y^n) = mn,$$

which proves the theorem. A similar argument works over any field once one has shown that the total intersection number is constant on *algebraic* families (Shafarevich 1994, Chap. III, 2.2).

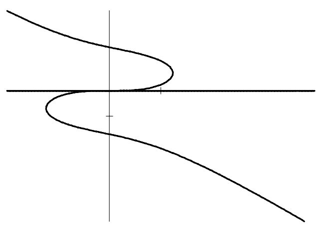
EXAMPLE 1.20 According to Bezout's theorem, a curve of degree m will meet the line at infinity in exactly m points counting multiplicities. Our favourite curve

$$C: \quad Y^2Z = X^3 + aXZ^2 + bZ^3$$

meets L_∞ at a single point $P = (0:1:0)$, but

$$\begin{aligned} I(P, L_\infty \cap C) &= I(Z, Z - X^3 - aXZ - bZ^3) \\ &= I(Z, X^3) \\ &= 3. \end{aligned}$$

In general, a nonsingular point P on a curve C is called a ***point of inflection*** (or ***flex***) if the intersection multiplicity of the tangent line and C at P is ≥ 3.



$$Z = X^3 + XZ^2 + Z^3$$

REMARK 1.21 According to (1.18), any two irreducible components of a projective plane curve have a common point (possibly in a finite extension of k), which will be singular (1.4). Therefore, every nonsingular projective plane curve is geometrically irreducible.

Appendix: resultants

Let $f(X) = s_0X^m + s_1X^{m-1} + \cdots + s_m$ and $g(X) = t_0X^n + t_1X^{n-1} + \cdots + t_n$ be polynomials with coefficients in k. The ***resultant*** $\mathrm{Res}(f, g)$ of f and g is defined to be the determinant

$$\begin{array}{ll} \left|\begin{array}{cccccc} s_0 & s_1 & \dots & s_m & & \\ & s_0 & \dots & & s_m & \\ & & \dots & & & \dots \\ t_0 & t_1 & \dots & t_n & & \\ & t_0 & \dots & & t_n & \\ & & \dots & & & \dots \end{array}\right| & \begin{array}{l} n \text{ rows} \\ \\ \\ m \text{ rows} \end{array} \end{array}$$

There are n rows of ss and m rows of ts, so that the matrix is $(m+n)\times(m+n)$; all blank spaces are to be filled with zeros. The resultant is a polynomial in the coefficients of f and g.

PROPOSITION 1.22 *The resultant* $\mathrm{Res}(f,g)=0$ *if and only if*

(a) *both* s_0 *and* t_0 *are zero; or*
(b) *the two polynomials have a common root in* k^{al} *(equivalently, a common factor in* $k[X]$*).*

PROOF. If (a) holds, then the first column of the determinant is zero, and so certainly $\mathrm{Res}(f,g)=0$. Suppose that α is a common root of f and g, so that there exist polynomials f_1 and g_1 in $k^{\mathrm{al}}[X]$ of degrees $m-1$ and $n-1$ respectively such that

$$f(X)=(X-\alpha)f_1(X),\qquad g(X)=(X-\alpha)g_1(X).$$

From these equalities we find that

$$f(X)g_1(X)-g(X)f_1(X)=0. \tag{2}$$

On equating the coefficients of $X^{m+n-1},\dots,X,1$ in (2) to zero, we find that the coefficients of f_1 and g_1 are the solutions of a system of $m+n$ linear equations in $m+n$ unknowns. The matrix of coefficients of the system is the transpose of the matrix

$$\begin{pmatrix} s_0 & s_1 & \dots & s_m & & \\ & s_0 & \dots & & s_m & \\ & & \dots & & & \dots \\ t_0 & t_1 & \dots & t_n & & \\ & t_0 & \dots & & t_n & \\ & & \dots & & & \dots \end{pmatrix} \tag{3}$$

The existence of the solution shows that this matrix has determinant zero, which implies that $\mathrm{Res}(f,g)=0$.

Conversely, suppose that $\mathrm{Res}(f,g)=0$ but neither s_0 nor t_0 is zero. Because the above matrix has determinant zero, we can solve the linear equations to find polynomials f_1 and g_1 satisfying (2). A root α of f must also be a root of f_1 or of g. If the former, cancel $X-\alpha$ from the left hand side of (2), and consider a root of $f_1/(X-\alpha)$. As $\deg f_1<\deg f$, this argument eventually leads to a root of f that is not a root of f_1, and so must be a root of g.

The parenthetical version of (b) is equivalent to the nonparenthetical version because Euclid's algorithm shows that the greatest common divisor of two polynomials doesn't change when the base field is extended. □

EXAMPLE 1.23 For a monic polynomial $f(X)=X^m+\cdots+s_m$, the resultant of $f(X)$ and $f'(X)$ is $(-1)^{m(m-1)/2}\mathrm{disc}(f)$.

Let $\mathbf{c}_1, \ldots, \mathbf{c}_{m+n}$ be the columns of the matrix (3). Then

$$\begin{pmatrix} X^{m-1}f(X) \\ X^{m-2}f \\ \vdots \\ f(X) \\ X^{n-1}g(X) \\ \vdots \\ g(X) \end{pmatrix} = X^{m+n-1}\mathbf{c}_0 + \cdots + 1\mathbf{c}_{m+n},$$

and so

$$\operatorname{Res}(f,g) \overset{\text{def}}{=} \det(\mathbf{c}_0, \ldots, \mathbf{c}_{m+n}) = \det(\mathbf{c}_0, \cdots, \mathbf{c}_{m+n-1}, \mathbf{c})$$

where $\mathbf{c}$ is the vector on the left of the above equation. On expanding out this last determinant, we find that

$$\operatorname{Res}(f,g) = a(X)f(X) + b(X)g(X)$$

where $a(X)$ and $b(X)$ are polynomials of degrees $\leq n-1$ and $\leq m-1$ respectively.

If $f(X)$ and $g(X)$ have coefficients in an integral domain R, for example, $\mathbb{Z}$ or $k[Y]$, then $\operatorname{Res}(f,g) \in R$, and the polynomials $a(X)$ and $b(X)$ have coefficients in R.

PROPOSITION 1.24 *Let $f(X,Y), g(X,Y) \in k[X,Y]$, and let $r(X) \in k[X]$ be the resultant of f and g regarded as polynomials in Y with coefficients in $k[X]$.*

(a) *There exist $a(X,Y), b(X,Y) \in k[X,Y]$ such that*

$$af + bg = r(X) \in k[X]$$

and $\deg_Y(a) < \deg_Y(g)$, $\deg_Y(b) < \deg_Y(f)$.

(b) *The polynomial $r = 0$ if and only if f and g have a common factor in $k[X,Y]$.*

PROOF. (a) Immediate from the above discussion.

(b) We know that $r = 0$ if and only if f and g have a common factor in $k(X)[Y]$, but by Gauss's lemma, this is equivalent to their having a common factor in $k[X,Y]$. □

COROLLARY 1.25 *If f and g have no common factor in $k[X,Y]$, then they have no common factor in $K[X,Y]$ for any field $K \supset k$.*

PROOF. Since the resultant of f and g is the same whether we work over k or K, this follows from (b) of the proposition. □

The resultant of homogeneous polynomials

$$F(X,Y) = s_0 X^m + s_1 X^{m-1} Y + \cdots + s_m Y^m$$
$$G(X,Y) = t_0 X^n + t_1 X^{n-1} Y + \cdots + t_n Y^n$$

is defined as for inhomogeneous polynomials.

PROPOSITION 1.26 *The resultant* $\mathrm{Res}(F,G) = 0$ *if and only if* F *and* G *have a nontrivial zero in* $\mathbb{P}^1(k^{\mathrm{al}})$.

PROOF. The nontrivial zeros of $F(X,Y)$ in $\mathbb{P}^1(k^{\mathrm{al}})$ are of the form:

(a) $(a:1)$ with a a root of $F(X,1)$, or
(b) $(1:0)$ in the case that $s_0 = 0$.

Since a similar statement is true for $G(X,Y)$, this proposition is a restatement of the previous proposition. □

Clearly, the statement is more pleasant in the homogeneous case.

Computer algebra programs can find the resultants of two polynomials in one variable: for example, entering "`resultant`$((x+a)^5, (x+b)^5, x)$" into Maple gives the answer $(-a+b)^{25}$. What this means is that the polynomials have a common root if and only if $a = b$, and this can happen in 25 ways.

ASIDE 1.27 There is a geometric interpretation of the last proposition. Take k to be algebraically closed, and regard the coefficients of F and G as indeterminates. Let V be the subset of $\mathbb{A}^{m+n+2} \times \mathbb{P}^1$ where both $F(s_0,\ldots,s_m;X,Y)$ and $G(t_0,\ldots,t_n;X,Y)$ vanish. The proposition says that the projection of V on $\mathbb{A}^{m+n+2}$ is the set where $\mathrm{Res}(F,G)$, regarded as a polynomial in the s_i and t_i, vanishes. In other words, the proposition tells us that the projection of the Zariski-closed set V is the Zariski-closed set defined by the resultant of F and G.

Elimination theory does this more generally. Given polynomials $P_i(T_1,\ldots,T_m;X_0,\ldots,X_n)$, homogeneous in the X_i, it provides an algorithm for finding polynomials $R_j(T_1,\ldots,T_m)$ such that the $P_i(a_1,\ldots,a_m;X_0,\ldots,X_n)$ have a common zero if and only if $R_j(a_1,\ldots,a_m) = 0$ for all j. See, for example, Cox et al. 1992, Chap. 8, §5.

2 Rational points on plane curves

For a projective plane curve C_F over $\mathbb{Q}$ (or some other field with an interesting arithmetic), the two fundamental questions in diophantine geometry are:

(a) Does C have a point with coordinates in $\mathbb{Q}$, i.e., does $F(X,Y,Z)$ have a nontrivial zero in $\mathbb{Q}$?
(b) If the answer to (a) is yes, can we describe the set of points?

There is also the question of whether there are algorithms to answer these questions. For example, we may know that a curve has only finitely many points without having an algorithm to actually find the points or even their number.

For simplicity, in the remainder of this section, I'll assume that C is absolutely irreducible, i.e., that $F(X,Y,Z)$ is irreducible and remains irreducible over $\mathbb{Q}^{\mathrm{al}}$.

Curves of degree one

In this case, the curve is a line

$$C : aX + bY + cZ = 0, \quad a, b, c \in \mathbb{Q}, \text{ not all zero.}$$

It always has points, and it is possible to parametrize the points — for example, if $c \neq 0$, then the map

$$(s : t) \mapsto (s : t : -\tfrac{a}{c}s - \tfrac{b}{c}t)$$

is a bijection from $\mathbb{P}^1(K)$ onto $C(K)$ for all fields $K \supset \mathbb{Q}$.

Curves of degree two

In this case $F(X,Y,Z)$ is a quadratic form in 3 variables, and C is a conic. Note that C can't be singular, because, if P has multiplicity $m \geq 2$, then a line L through P and a second point Q on the curve will have

$$I(P, L \cap C) + I(Q, L \cap C) \overset{(1.13)}{\geq} m + 1 \geq 3,$$

which violates Bezout's theorem.

Sometimes it is easy to see that $C(\mathbb{Q}) = \emptyset$. For example,

$$X^2 + Y^2 + Z^2$$

has no nontrivial zero in $\mathbb{Q}$ because it has no nontrivial real zero. Similarly,

$$X^2 + Y^2 - 3Z^2$$

has no nontrivial zero, because if it did it would have a zero (x, y, z) with $x, y, z \in \mathbb{Z}$ and $\gcd(x, y, z) = 1$. The only squares in $\mathbb{Z}/3\mathbb{Z}$ are 0 and 1, and so

$$x^2 + y^2 \equiv 0 \mod 3 \implies x \equiv 0 \equiv y \mod 3.$$

But then 3 must divide z, which contradicts our assumption that $\gcd(x, y, z) = 1$.

A theorem of Legendre says that these arguments can be turned into an effective procedure for deciding whether $C_F(\mathbb{Q})$ is nonempty. He first shows that, by an elementary change of variables, F can be put in diagonal form

$$F = aX^2 + bY^2 + cZ^2, \quad a, b, c \in \mathbb{Z}, \quad a, b, c \text{ square free;}$$

he then shows that if $abc \neq 0$ and a, b, c are not all of the same sign, then there is a nontrivial solution to $F(X, Y, Z) = 0$ in $\mathbb{Q}$ if and only if $-bc$, $-ca$, $-ab$ are quadratic residues modulo a, b, c respectively. By the quadratic reciprocity law, this implies that there exists an integer m, depending in a simple way on the coefficients of F, such that $C_F(\mathbb{Q}) \neq \emptyset$ if and only if $F(X, Y, Z) \equiv 0 \mod m$ has a solution in integers relatively prime to m.

Now suppose C has a point P_0 with coordinates in $\mathbb{Q}$. Can we describe all the points? Yes, because each line through P_0 will (by Bezout's theorem, or more elementary arguments) meet the curve in exactly one other point, except for the tangent line. Since the lines through P_0 in $\mathbb{P}^2$ form a "$\mathbb{P}^1$", we obtain in this way a bijection between $C(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$. For example, take P_0 to be the point $(-1 : 0 : 1)$ on the curve $C : X^2 + Y^2 = Z^2$. The line $bX - aY + bZ$, $a, b \in \mathbb{Q}$, of slope $\frac{b}{a}$ through P_0 meets C at the point $(a^2 - b^2 : 2ab : a^2 + b^2)$. In this way, we obtain a parametrization $(a : b) \mapsto (a^2 - b^2 : 2ab : a^2 + b^2)$ of the points of C with coordinates in $\mathbb{Q}$.

Curves of degree 3

We first make an observation that will be useful throughout this book. Let

$$f(X, Y) = \sum a_{ij} X^i Y^j$$

be a polynomial with coefficients $a_{ij} \in k$, and let K be a Galois extension of k (possibly infinite). If $(a, b) \in K \times K$ is a zero of $f(X, Y)$, then so also is $(\sigma a, \sigma b)$ for any $\sigma \in \mathrm{Gal}(K/k)$ because

$$0 = \sigma(f(a, b)) = \sigma(\sum a_{ij} a^i b^j) = \sum a_{ij} (\sigma a)^i (\sigma b)^j = f(\sigma a, \sigma b).$$

Thus, $\mathrm{Gal}(K/k)$ acts on $C_f(K)$. More generally, if $C_1, C_2, \ldots$ are affine plane curves over k, then $\mathrm{Gal}(K/k)$ stabilizes the subset $C_1(K) \cap C_2(K) \cap \ldots$ of $K \times K$. On applying this remark to the curves $f = 0$, $\frac{\partial f}{\partial X} = 0$, $\frac{\partial f}{\partial Y} = 0$, we see that $\mathrm{Gal}(K/\mathbb{Q})$ stabilizes the set of singular points of C_f. Similar remarks apply to projective plane curves.

Let $C : F(X, Y, Z) = 0$ be a projective plane curve over $\mathbb{Q}$ of degree 3. If it has a singular point, then Bezout's theorem shows that it has only one, and that it is a double point. A priori the singular point P_0 may have coordinates in some finite extension K of $\mathbb{Q}$, which we may take to be Galois over $\mathbb{Q}$, but $\mathrm{Gal}(K/\mathbb{Q})$ stabilizes the set of singular points in $C(K)$, hence fixes P_0, and so $P_0 \in C(\mathbb{Q})$. Now a line through P_0 will meet the curve in exactly one other point (unless it is a tangent line), and so we again get a parametrization of the points.

Nonsingular cubics will be the topic of the rest of the book. We shall see that Legendre's theorem fails for nonsingular cubic curves. For example,

$$3X^3 + 4Y^3 + 5Z^3 = 0$$

has nontrivial solutions modulo m for all integers m but has no nontrivial solution in $\mathbb{Q}$.

Let C be a nonsingular cubic curve over $\mathbb{Q}$. As we discussed in §2, from any point P in $C(\mathbb{Q})$, we can construct a second point in $C(\mathbb{Q})$ as the point of intersection of the tangent line at P with C, and from any pair of points P, Q in $C(\mathbb{Q})$, we can construct a third point in $C(\mathbb{Q})$ as the point of intersection of the chord through P, Q with C.

In a famous paper, published in 1922, Mordell proved the following theorem:

THEOREM 2.1 (FINITE BASIS THEOREM) *Let C be a nonsingular cubic curve over $\mathbb{Q}$. Then there exists a finite set of points on C with coordinates in $\mathbb{Q}$ from which every other such point can be obtained by successive chord and tangent constructions.*

In fact, $C(\mathbb{Q})$, if nonempty, has a natural structure of a commutative group (see the next section), and the finite basis theorem says that $C(\mathbb{Q})$ is finitely generated. There is as yet no *proven* algorithm for finding the rank of the group.

REMARK 2.2 For a singular cubic curve C over $\mathbb{Q}$, the nonsingular points form a group which, for example, may be isomorphic to $(\mathbb{Q}, +)$ or $(\mathbb{Q}^{\times}, \times)$ (see II, §3), neither of which is finitely generated.[4] Thus, the finite basis theorem fails for singular cubics.

CURVES OF GENUS > 1

The genus of a nonsingular projective curve C over $\mathbb{Q}$ is the genus of the Riemann surface $C(\mathbb{C})$. More generally, the genus of an arbitrary curve over $\mathbb{Q}$ is the genus of the nonsingular projective curve attached to its function field (Fulton 1969, p. 180). Mordell conjectured in his 1922 paper, and Faltings (1983) proved, that every curve of genus > 1 has only finitely many points coordinates in $\mathbb{Q}$. This applies to any nonsingular projective plane curve of degree ≥ 4, or to any singular projective plane curve provided the degree of the curve is sufficiently large compared to the multiplicities of its singularities (see 4.8 below).

REMARK 2.3 Let $P \in \mathbb{P}^2(\mathbb{Q})$. Choose a representative $(a : b : c)$ for P with a, b, c integers having no common factor, and define the height $H(P)$ of P to be $\max(|a|, |b|, |c|)$. For a curve C of genus > 1, there is an effective bound for the number of points $P \in C(\mathbb{Q})$ but no known effective bound $H(C)$, in terms of the polynomial defining C, for the heights of the points $P \in C(\mathbb{Q})$. With such an upper bound $H(C)$, one could find all the points on C with coordinates in $\mathbb{Q}$ by a finite search. See Hindry and Silverman 2000, F.4.2, for a discussion of this problem.

[4] The elements of any finitely generated subgroup of $\mathbb{Q}$ have a common denominator, and a finitely generated subgroup of $\mathbb{Q}^{\times}$ can contain only finitely many prime numbers.

ASIDE 2.4 There is a heuristic explanation for Mordell's conjecture. Let C be a curve of genus $g \geq 1$ over $\mathbb{Q}$, and assume that $C(\mathbb{Q}) \neq \emptyset$. It is possible to embed C into another projective variety J of dimension g (its ***jacobian variety***). The jacobian variety J is an abelian variety, i.e., it has a group structure, and a generalization of Mordell's theorem (due to Weil) says that $J(\mathbb{Q})$ is finitely generated. Hence, inside the g-dimensional set $J(\mathbb{C})$ we have the countable set $J(\mathbb{Q})$ and the (apparently unrelated) one-dimensional set $C(\mathbb{C})$. If $g > 1$, it would be an extraordinary accident if the second set contained more than a finite number of elements from the first set.

A brief introduction to the p-adic numbers

Let p be a prime number. Any nonzero rational number a can be expressed $a = p^r \frac{m}{n}$ with $m, n \in \mathbb{Z}$ and not divisible by p. We then write $\mathrm{ord}_p(a) = r$, and $|a|_p = \frac{1}{p^r}$. We define $|0|_p = 0$. Then:

(a) $|a|_p = 0$ if and only if $a = 0$.
(b) $|ab|_p = |a|_p |b|_p$.
(c) $|a + b|_p \leq \max\{|a|_p, |b|_p\} \quad (\leq |a|_p + |b_p|)$.

These conditions imply that

$$d_p(a, b) \stackrel{\text{def}}{=} |a - b|_p$$

is a translation-invariant metric on $\mathbb{Q}$. Note that, according to this definition, to say that a and b are close means that their difference is divisible by a high power of p. The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ for this metric. We now explain what this means.

A sequence (a_n) is said to be a ***Cauchy sequence*** (for the p-adic metric) if, for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$|a_m - a_n|_p < \varepsilon \quad \text{whenever} \quad m, n > N(\varepsilon).$$

The sequence (a_n) ***converges*** to a if for any $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that

$$|a_n - a|_p < \varepsilon \quad \text{whenever } n > N(\varepsilon).$$

Let R be the set of all Cauchy sequences in $\mathbb{Q}$ (for the p-adic metric). It becomes a ring with the obvious operations. An element of R is said to be a ***null sequence*** if it converges to zero. The set of null sequences is an ideal I in R, and $\mathbb{Q}_p$ is defined to be the quotient R/I.

If $\alpha = (a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, then one shows that $|a_n|_p$ becomes constant for large n, and we set this constant value equal to $|\alpha|_p$. The map $\alpha \mapsto |\alpha|_p \colon R \to \mathbb{Q}$ factors through $\mathbb{Q}_p$, and has the properties (a), (b), (c) listed above. We can therefore talk about Cauchy sequences and so on in $\mathbb{Q}_p$.

THEOREM 2.5 *(a) $\mathbb{Q}_p$ is a field, and it is complete, i.e., every Cauchy sequence in $\mathbb{Q}_p$ has a unique limit in $\mathbb{Q}_p$.*

(b) The map sending $a \in \mathbb{Q}$ to the equivalence class of the constant Cauchy sequence $\alpha(a) = a, a, a, \ldots$ is an injective homomorphism $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, and every element of $\mathbb{Q}_p$ is a limit of a sequence in $\mathbb{Q}$.

REMARK 2.6 (a) The same construction as above, but with $|\cdot|_p$ replaced by the usual absolute value, yields $\mathbb{R}$ instead of $\mathbb{Q}_p$.

(b) Just as real numbers can be represented by decimals, p-adic numbers can be represented by infinite series of the form

$$a_{-n}p^{-n} + \cdots + a_0 + a_1 p + \cdots + a_m p^m + \cdots \quad 0 \le a_i \le p-1.$$

The ring of p-adic integers $\mathbb{Z}_p$ can be variously defined as:

(a) the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$;
(b) the set of elements $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p \le 1$;
(c) the set of elements of $\mathbb{Q}_p$ that can be represented in the form

$$a_0 + a_1 p + \cdots + a_m p^m + \cdots, \quad 0 \le a_i \le p-1.$$

(d) the inverse limit $\varprojlim \mathbb{Z}/p^m\mathbb{Z}$.

NOTATION 2.7 Recall that, for $a \in \mathbb{Q}^\times$, $\mathrm{ord}_p(a) = m$ if $a = p^m \frac{r}{s}$ with r and s not divisible by p. The following rule is obvious: $\mathrm{ord}_p(a+b) \ge \min\{\mathrm{ord}_p(a), \mathrm{ord}_p(b)\}$, with equality unless $\mathrm{ord}_p(a) = \mathrm{ord}_p(b)$.
Similarly, for an $a \in \mathbb{Q}_p$, we set $\mathrm{ord}_p(a) = m$ if $a \in p^m\mathbb{Z}_p \smallsetminus p^{m+1}\mathbb{Z}_p$. The same rule holds, and the two definitions of ord_p agree on $\mathbb{Q}$. In both cases, we set $\mathrm{ord}_p(0) = \infty$. Note that ord_p is a homomorphism $\mathbb{Q}_p^\times \to \mathbb{Z}$.

NOTES More detailed descriptions of the field of p-adic numbers can be found in Koblitz 1977 and in most books on algebraic number theory. It is worth noting that a theorem of Ostrowski says that $\mathbb{R}$ and the p-adic fields $\mathbb{Q}_p$ are the *only* completions of $\mathbb{Q}$ with respect to valuations.

Curves of degree 2 and 3 continued

Clearly, a necessary condition for a curve to have a point with coordinates in $\mathbb{Q}$ is that it have points with coordinates in $\mathbb{R}$ and all the fields $\mathbb{Q}_p$. Indeed, we observed above that $X^2 + Y^2 + Z^2 = 0$ has no nontrivial zero in $\mathbb{Q}$ because it has no nontrivial real zero, and our argument for $X^2 + Y^2 - 3Z^2 = 0$ shows that it has no nontrivial zero in the field $\mathbb{Q}_3$ of 3-adic numbers. A modern interpretation of Legendre's theorem is that, for curves of degree 2, the condition is also sufficient:

THEOREM 2.8 (LEGENDRE) *A quadratic form $F(X, Y, Z)$ with coefficients in $\mathbb{Q}$ has a nontrivial zero in $\mathbb{Q}$ if and only if it has a nontrivial zero in $\mathbb{R}$ and in $\mathbb{Q}_p$ for all p.*

PROOF. See Cassels 1991, pp. 13–22. □

In fact, the proof shows that F has a nontrivial zero in $\mathbb{Q}$ if it has a nontrivial zero on $\mathbb{Q}_p$ for all p ($\mathbb{R}$ is not needed). Legendre's original statement can be recovered from this statement by the arguments of the next subsection.

REMARK 2.9 Theorem 2.8 is true for quadratic forms $F(X_0, X_2, \ldots, X_n)$ in any number of variables over any number field K (Hasse-Minkowski theorem). A good exposition of the proof for forms over $\mathbb{Q}$ in any number of variables is to be found in Serre 1996, Chap. IV. The key cases are 3 and 4 variables (2 is easy, and for ≥ 5 variables, one uses induction on n), and the key result needed for its proof is the quadratic reciprocity law.

If for a class of polynomials (better algebraic varieties) it is known that each polynomial (or variety) has a zero in $\mathbb{Q}$ if and only if it has zeros in $\mathbb{R}$ and all $\mathbb{Q}_p$, then one says that the ***Hasse***, or ***local-global, principle*** holds for the class.

Hensel's lemma

LEMMA 2.10 *Let $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$, and let $\underline{a} \in \mathbb{Z}^n$ have the property that, for some $m \geq 0$,*

$$f(\underline{a}) \equiv 0 \mod p^{2m+1}$$

but, for some i,

$$\left(\frac{\partial f}{\partial X_i}\right)(\underline{a}) \not\equiv 0 \mod p^{m+1}.$$

Then there exists a $\underline{b} \in \mathbb{Z}^n$ such that

$$\underline{b} \equiv \underline{a} \mod p^{m+1}$$

and

$$f(\underline{b}) \equiv 0 \mod p^{2m+2}.$$

PROOF. Consider the (trivial) Taylor expansion

$$f(X_1, \ldots, X_n) = f(a_1, \ldots, a_n) + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial X_i}\right)_{\underline{a}} (X_i - a_i)$$
$$+ \text{ terms of higher degree.}$$

Set $b_i = a_i + h_i p^{m+1}$, $h_i \in \mathbb{Z}$. Then

$$f(b_1, \ldots, b_n) = f(a_1, \ldots, a_n) + \sum \left(\frac{\partial f}{\partial X_i}\right)_{\underline{a}} h_i p^{m+1}$$
$$+ \text{ terms divisible by } p^{2m+2}.$$

We have to choose the h_i so that

$$f(a_1,\ldots,a_n)+\sum\left(\frac{\partial f}{\partial X_i}\right)_{\underline{a}} h_i\, p^{m+1}$$

is divisible by p^{2m+2}. From the assumption, we know that there is a $k \le m$ such that p^k divides $\left(\frac{\partial f}{\partial X_i}\right)_{\underline{a}}$ for all i but p^{k+1} doesn't divide all of them. Any h_i's satisfying the following equation will suffice:

$$\frac{f(a_1,\ldots,a_n)}{p^{k+m+1}}+\sum\frac{\left(\frac{\partial f}{\partial X_i}\right)_{\underline{a}}}{p^k}h_i \equiv 0 \mod p.$$

□

REMARK 2.11 If, in the lemma, $\underline{a}$ satisfies the condition

$$f(\underline{a}) \equiv 0 \mod p^{2m+r}$$

for some $r \ge 1$, then the construction in the proof gives a $\underline{b}$ such that

$$\underline{b} \equiv \underline{a} \mod p^{m+r}$$

and

$$f(\underline{b}) \equiv 0 \mod p^{2m+r+1}.$$

THEOREM 2.12 (HENSEL'S LEMMA) *Under the hypotheses of the lemma, there exists a* $\underline{b} \in \mathbb{Z}_p^n$ *such that* $f(\underline{b}) = 0$ *and* $\underline{b} \equiv \underline{a} \mod p^{m+1}$.

PROOF. On applying the lemma, we obtain an $\underline{a}_{2m+2} \in \mathbb{Z}^n$ such that $\underline{a}_{2m+2} \equiv \underline{a} \mod p^{m+1}$ and $f(\underline{a}_{2m+2}) \equiv 0 \mod p^{2m+2}$. The first congruence implies that

$$\left(\frac{\partial f}{\partial X_i}\right)(\underline{a}_{2m+2}) \equiv \left(\frac{\partial f}{\partial X_i}\right)(\underline{a}) \mod p^{m+1},$$

and so $\left(\frac{\partial f}{\partial X_i}\right)(\underline{a}_{2m+2}) \not\equiv 0 \mod p^{m+1}$ for some i. On applying the remark following the lemma, we obtain an $\underline{a}_{2m+3} \in \mathbb{Z}^n$ such that $\underline{a}_{2m+3} \equiv \underline{a}_{2m+2} \mod p^{m+2}$ and $f(\underline{a}_{2m+3}) \equiv 0 \mod p^{2m+3}$. Continuing in this fashion, we obtain a sequence $\underline{a}, \underline{a}_{2m+2}, \underline{a}_{2m+3}, \ldots$ of n-tuples of Cauchy sequences. Let $\underline{b}$ be the limit in $\mathbb{Z}_p^n$. The map $f\colon \mathbb{Z}^n \to \mathbb{Z}$ is continuous for the p-adic topologies, and so

$$f(\underline{b}) = f(\lim_r \underline{a}_{2m+r}) = \lim_r f(\underline{a}_{2m+r}) = 0.$$

□

EXAMPLE 2.13 Let $f(X) \in \mathbb{Z}[X]$, and let $\bar{f}(X) \in \mathbb{F}_p[X]$ be its reduction mod p. Here $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $a \in \mathbb{Z}$ be such that $\bar{a} \in \mathbb{F}_p$ is a simple root of $\bar{f}(X)$. Then $\frac{d\bar{f}}{dX}(\bar{a}) \neq 0$, and so the theorem shows that $\bar{a}$ lifts to a root of $f(X)$ in $\mathbb{Z}_p$.

EXAMPLE 2.14 Let $f(X,Y,Z)$ be a homogeneous polynomial in $\mathbb{Z}[X,Y,Z]$, and let $(a,b,c) \in \mathbb{Z}^3$ be such that $(\bar{a},\bar{b},\bar{c}) \in \mathbb{F}_p^3$ is a nonsingular point of the curve $\bar{C} : \bar{f}(X,Y,Z) = 0$ over $\mathbb{F}_p$. Then, as in the previous example, $(\bar{a},\bar{b},\bar{c})$ lifts to a point on the curve $C : f(X,Y,Z) = 0$ with coordinates in $\mathbb{Z}_p$.

EXAMPLE 2.15 Let $f(X,Y,Z)$ be a quadratic form with coefficients in $\mathbb{Z}$, and let $D \neq 0$ be its discriminant. If p does not divide D, then $\bar{f}(X,Y,Z)$ is a nondegenerate quadratic form over $\mathbb{F}_p$, and it is known that it has a nontrivial zero in $\mathbb{F}_p$. Therefore $f(X,Y,Z)$ has a nontrivial zero in $\mathbb{Q}_p$ for all such p. If p divides D, then Hensel's lemma shows that $f(X,Y,Z)$ will have a nontrivial zero in $\mathbb{Q}_p$ if and only if it has an "approximate" zero.

EXERCISE 2.16 Let

$$F(X,Y,Z) = 5X^2 + 3Y^2 + 8Z^2 + 6(YZ + ZX + XY).$$

Find $(a,b,c) \in \mathbb{Z}^3$, not all divisible by 13, such that

$$F(a,b,c) \equiv 0 \bmod 13^2.$$

EXERCISE 2.17 Consider the plane affine curve $C : Y^2 = X^3 + p$. Prove that the point $(0,0)$ on the reduced curve over $\mathbb{F}_p$ does not lift to $\mathbb{Z}_p^2$. Why doesn't this violate Hensel's lemma?

NOTES The tangent process for constructing new rational points goes back to Diophantus (c250 A.D.) and was "much loved by Fermat"; the chord process was known to Newton. Hilbert and Hurwitz showed (in 1890) that, if a curve of genus zero has one rational point, then it has infinitely many, all given by rational values of a parameter. In 1901, Poincaré published a long article on rational points on curves in which he attempted to rescue the subject from being merely a collection of ad hoc results about individual equations. Although he is usually credited with conjecturing the finite basis theorem, he rather simply assumed it. Beppo Levi was the first to ask explicitly whether the finite basis theorem was true. In his remarkable 1922 paper, Mordell proved the finite basis theorem, and, in a rather off-handed way, conjectured that all curves of genus > 1 over $\mathbb{Q}$ have only finitely many rational points (the Mordell conjecture). Both were tremendously important, the first as the first general theorem in diophantine geometry, and the second as one of the most important open questions in the subject until it was proved by Faltings in 1983 (that year's "theorem of the century").

For more on the history of these topics, see Cassels 1986, Schappacher 1990, and Schappacher and Schoof 1996, from which the above notes have largely been drawn.

3 The group law on a cubic curve

Let C be a nonsingular projective plane curve of degree 3 over a field k, which, for simplicity, we assume to be perfect. Of course, we shall be especially interested in the case $k = \mathbb{Q}$. Let $C(k)$ be the set of points on C with coordinates in

k. If $C(k)$ is empty, then it is not a group, but otherwise we shall show that, once an identity element $O \in C(k)$ has been chosen, $C(k)$ has a natural structure of a commutative group.

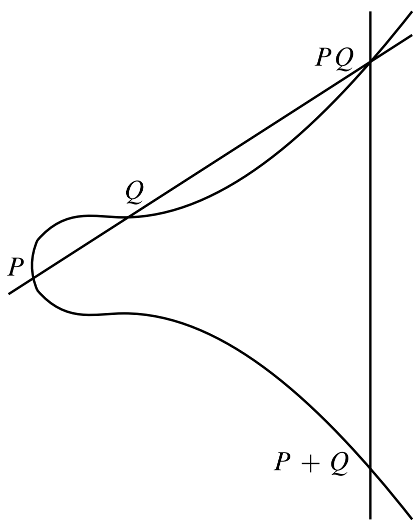
Return to our nonsingular cubic projective plane curve C and chosen point $O \in C(k)$. From two points $P_1, P_2 \in C(k)$, neither lying on the tangent line at the other, we can construct a third point as the point of intersection of C with the chord through P_1 and P_2. By Bezout's theorem, there exists exactly one such point, possibly with coordinates in a Galois extension K of k, but, by the observation p. 19, the point will be fixed by $\mathrm{Gal}(K/\mathbb{Q})$ and so lie in $C(k)$.[5] Similarly, the tangent line at a point $P \in C(k)$ will meet C at exactly one other point (unless P is a point of inflection), which lies in $C(k)$.

We now write PQ for the third point of intersection of a line through $P, Q \in C(k)$; when $P = Q$ then PQ is to be the point of intersection of the tangent line at P with C; when the line through P and Q is tangent to C at Q then $PQ = Q$; and when P is a point of inflection, then $PP = P$.

For any pair $P, Q \in C(k)$, we define

$$P + Q = O(PQ),$$

i.e., if the line through P and Q intersects C again at PQ, then $P + Q$ is the third point of intersection with C of the line through O and PQ.



THEOREM 3.1 *The above construction makes $C(k)$ into a commutative group.*

[5]This can also be proved by a more elementary argument using that a cubic polynomial $h(X) \in k[X]$ with two roots in k has all of its roots in k.

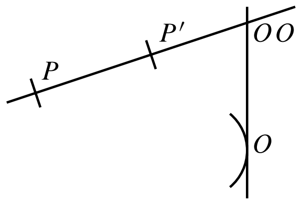
First note that the definition doesn't depend on the order of P and Q; thus

$$P + Q = Q + P.$$

Next note that

$$O + P \stackrel{\text{def}}{=} O(OP) = P.$$

Given $P \in C(k)$, define $P' = P(OO)$, i.e., if the tangent line at O intersects C at OO, then P' is the third point of intersection of the line through P and OO. Then $PP' = OO$, and $O(PP') = O(OO) = O$, i.e., $P + P' = O$.



Thus the law of composition is commutative, has a zero element, and every element has a negative. It remains to check that it is associative, i.e., that

$$(P + Q) + R = P + (Q + R).$$

GEOMETRIC PROOF OF ASSOCIATIVITY

In the next section, we shall see that associativity follows directly from the Riemann-Roch theorem. Here I sketch the elegant geometry proof of associativity. Clearly, in proving associativity, we may replace k with a larger field, and so we may assume k to be algebraic closed.

PROPOSITION 3.2 *If two cubic curves in $\mathbb{P}^2$ intersect in exactly nine points, then every cubic curve passing through eight of the points also passes through the ninth.*

PROOF. A cubic form

$$F(X,Y,Z) = a_1X^3 + a_2X^2Y + \cdots + a_{10}Z^3$$

has 10 coefficients $a_1, \ldots, a_{10}$. The condition that C_F pass through a point $P = (x : y : z)$ is a linear condition on $a_1, \ldots, a_{10}$, namely,

$$a_1x^3 + a_2x^2y + \cdots + a_{10}z^3 = 0.$$

If the eight points $P_1 = (x_1 : y_1 : z_1), \ldots, P_8$ are in "general position", specifically, if the vectors $(x_i^3, x_i^2y_i, \ldots, z_i^3)$, $i = 1, \ldots, 8$, are linearly independent, then the cubic forms having $P_1, \ldots, P_8$ as zeros form a 2-dimensional space, and so there exist two such forms F and G such that the remainder can be written

$$\lambda F + \mu G, \quad \lambda, \mu \in k.$$

Now F and G have a ninth zero in common (by Bezout), and every curve $\lambda F + \mu G = 0$ passes through it.

When the P_i are not in general position, the proof is completed by a case-by-case study (Walker 1950, III 6.2). □

We now write $\ell(P, Q)$ for the line in $\mathbb{P}^2$ through the points P, Q. Let $P, Q, R \in C(k)$, and let

$$S = (P + Q)R, \quad T = P(Q + R).$$

Then $(P + Q) + R = OS$ and $P + (Q + R) = OT$. In order to prove that $(P + Q) + R = P + (Q + R)$, it suffices to show that $S = T$.

Consider the cubic curves:

$$\begin{aligned} C &= 0, \\ \ell(P, Q) \cdot \ell(R, P + Q) \cdot \ell(QR, O) &= 0, \\ \ell(P, QR) \cdot \ell(Q, R \cdot \ell(P, O) &= 0. \end{aligned}$$

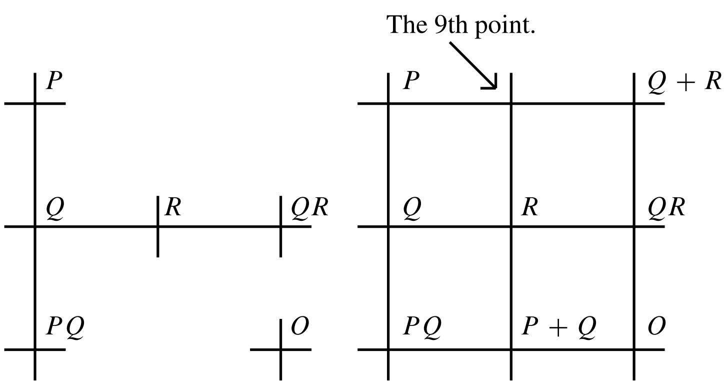
All three pass through the eight points

$$O, P, Q, R, PQ, QR, P + Q, Q + R, \tag{4}$$

and the last two also pass through

$$U \stackrel{\text{def}}{=} \ell(P, Q + R) \cap \ell(P + Q, R).$$

Therefore, if the six lines $\ell(P, Q), \ldots, \ell(P, O)$ are distinct, then the proposition shows that C passes through U, and this implies that $S = U = T$.



To handle the special case when two of the lines coincide, one uses the following stronger form of (3.2):

> Let C, C', C'' be cubic curves in $\mathbb{P}^2$ with C irreducible, and suppose $C \cdot C' = \sum_{i=1}^{9}[P_i]$ where the P_i are nonsingular (not necessarily distinct) points on C; if $C \cdot C'' = \sum_{i=1}^{8}[P_i] + [Q]$, then $Q = P_9$ (Fulton 1969, p. 124).

Here $C \cdot C'$ denotes the divisor $\sum_{P \in C \cap C'} I(P, C \cap C') \cdot [P]$ (see the next section).

EXERCISE 3.3 Find a necessary and sufficient condition for the line $L : Y = cX + d$ to be an inflectional tangent to the affine curve $C : Y^2 = X^3 + aX + b$, i.e., to meet C at a point P with $I(P, L \cap C) = 3$. Hence find a general formula for the elliptic curves C in canonical form having a rational point of order 3.

NOTES In his papers in the 1920s in which he generalized Mordell's finite basis theorem, Weil made systematic use of the commutative group structure on $E(\mathbb{Q})$, and (according to Schappacher 1990) was perhaps the first to do so — earlier mathematicians worked instead with the law of composition $P, Q \mapsto PQ = -(P+Q)$, which is not associative.

4 Regular functions; the Riemann-Roch theorem

Algebraically closed base fields

In this subsection, we assume that k is algebraically closed.

REGULAR FUNCTIONS ON AFFINE CURVES

Let C be the affine plane curve over k defined by an irreducible polynomial $f(X, Y)$. A polynomial $g(X, Y) \in k[X, Y]$ defines a function

$$(a, b) \mapsto g(a, b) \colon C(k) \to k$$

and the functions arising in this way are called the ***regular functions*** on C.

Clearly, any multiple of $f(X, Y)$ in $k[X, Y]$ defines the zero function on $C(k)$, and Hilbert's Nullstellensatz (Fulton 1969, p. 21) implies the converse (recall that (f) is a prime ideal because f is irreducible). Therefore the map sending g to the function $(a, b) \mapsto g(a, b)$ on $C(k)$ defines an isomorphism

$$k[X, Y]/(f(X, Y)) \to \{\text{ring of regular functions on } C\}.$$

Write

$$k[C] \stackrel{\text{def}}{=} k[X, Y]/(f(X, Y)) = k[x, y].$$

Then x and y are the coordinate functions $P \mapsto x(P)$ and $P \mapsto y(P)$ on $C(k)$, and the elements of $k[C]$ are polynomials in x and y. Note that a nonzero regular function on C has only finitely many zeros on C, because a curve $g(X,Y)=0$ intersects C in only finitely many points unless $f(X,Y)$ divides $g(X,Y)$ (by 1.24, for example).

Because (f) is prime, $k[x,y]$ is an integral domain, and we let $k(C)$ be its field of fractions $k(x,y)$. An element $\varphi = g/h$ of $k(x,y)$ defines a function

$$(a,b) \mapsto \frac{g(a,b)}{h(a,b)} : C(k) \smallsetminus \{\text{ zeros of } h\} \to k.$$

We call such a φ a ***rational function*** on C, ***regular on C*** $\smallsetminus \{$ zeros of $h\}$.

EXAMPLE 4.1 (a) Let C be the X-axis, i.e., the affine curve defined by the equation $Y=0$. Then $k[C]=k[X,Y]/(Y)\simeq k[X]$ and $k(C)\simeq k(X)$. The rational functions on C are just the quotients $g(X)/h(X)$, $h\neq 0$, and such a function is regular outside the finite set of zeros of $h(X)$.

(b) Let C be the curve $Y^2=X^3+aX+b$.Then

$$k[C]=k[x,y]=k[X,Y]/(Y^2-X^3-aX-b)=k[x,y].$$

Thus the regular functions on C are polynomials in the coordinate functions x and y, and x and y satisfy the relation

$$y^2=x^3+ax+b.$$

REGULAR FUNCTIONS ON PROJECTIVE CURVES

Let C be the plane projective curve over k defined by an irreducible homogeneous polynomial $F(X,Y,Z)$. If $G(X,Y,Z)$ and $H(X,Y,Z)$ are homogeneous polynomials of the same degree and H is not a multiple of F, then

$$(a:b:c)\mapsto \frac{G(a,b,c)}{H(a,b,c)}$$

is a well-defined function on the complement in $C(k)$ of the (finite) set of zeros of H. This is a rational function on C. More precisely, let

$$k[x,y,z]=k[X,Y,Z]/(F(X,Y,Z))$$

and let $k(x,y,z)$ be the field of fractions of $k[x,y,z]$. Because F is homogeneous, there is a well-defined decomposition

$$k[x,y,z]=\bigoplus_d k[x,y,z]_d$$

where $k[x,y,z]_d$ consists of the elements of $k[x,y,z]$ having a representative in $k[X,Y,Z]$ that is homogeneous of degree d. Define

$$k(C)=k(x,y,z)_0=\{g/h\in k(x,y,z)\mid g,h\in k[x,y,z]_d,\text{ for some } d\}.$$

It is a subfield of $k(x,y,z)$, and its elements are called the ***rational functions*** on C. A rational function defines a(n honest) function on the complement of a finite set in $C(k)$ (the set of poles of the function). Let U be the complement of a finite set in $C(k)$; then a function $\varphi\colon U \to k$ is said to be ***regular*** if there exists a rational function without poles in U and agreeing with φ on U.

REMARK 4.2 Recall that there is a bijection

$$\begin{array}{ccccc} \mathbb{A}^2(k) & \leftrightarrow & U_2(k) & \subset & \mathbb{P}^2(k) \\ (\frac{a}{c}, \frac{b}{c}) & \leftrightarrow & (a:b:c) & & \end{array}$$

To avoid confusion, write $k[X',Y']$ for the polynomial ring associated with $\mathbb{A}^2$ and $k[X,Y,Z]$ for the polynomial ring associated with $\mathbb{P}^2$. A polynomial $g(X',Y')$ defines a function $\mathbb{A}^2(k) \to k$, and the composite

$$U_2(k) \to \mathbb{A}^2(k) \xrightarrow{g} k$$

is

$$(a:b:c) \mapsto g(\tfrac{a}{c}, \tfrac{b}{c}) = \frac{g^*(a,b,c)}{c^{\deg g}}$$

where $g^*(X,Y,Z) \stackrel{\text{def}}{=} g(\frac{X}{Z}, \frac{Y}{Z}) \cdot Z^{\deg g}$ is $g(X,Y)$ made homogeneous by adding the fewest possible Zs. Thus $g(X',Y')$ as a function on $\mathbb{A}^2 \simeq U_2$ agrees with $\frac{g^*(X,Y,Z)}{Z^{\deg g}}$. One see easily that the map

$$\frac{g(X',Y')}{h(X',Y')} \mapsto \frac{g^*(X,Y,Z)}{Z^{\deg g}} \frac{Z^{\deg h}}{h^*(X,Y,Z)} \colon k(X',Y') \to k(X,Y,Z)$$

is an injection, with image the subfield $k(X,Y,Z)_0$ of $k(X,Y,Z)$ of elements that can be expressed as a quotient of homogeneous polynomials of the same degree.

Now let C be an irreducible curve in $\mathbb{P}^2$, and assume that $C \cap U_2 \neq \emptyset$, i.e., that C is not the "line at infinity" $Z = 0$. Then the map

$$\frac{g(x',y')}{h(x',y')} \mapsto \frac{g^*(x,y,z)}{z^{\deg g}} \frac{z^{\deg h}}{h^*(x,y,z)} \colon k(x',y') \to k(x,y,z)_0$$

is a bijection from the field of rational functions on the affine curve $C \cap U_2$ to the field of rational functions on C. Moreover, if $\varphi' \mapsto \varphi$, then $\varphi(a:b:c) = \varphi'(\frac{a}{c}, \frac{b}{c})$ for any point $(a:b:c) \in C(k) \cap U_2$ at which φ is defined.

EXAMPLE 4.3 The rational functions on $\mathbb{P}^1$ are the functions

$$(a:b) \mapsto \frac{G(a,b)}{H(a,b)}$$

where $G(X,Z)$ and $H(X,Z)$ are homogeneous polynomials of the same degree and $H(X,Z)$ is not the zero polynomial.

EXAMPLE 4.4 Let C be a nonsingular projective curve over $\mathbb{C}$. Then $C(\mathbb{C})$ has the structure of a compact Riemann surface, and the meromorphic functions on $C(\mathbb{C})$ in the sense of complex analysis are exactly the rational functions. For example, $\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere, and, written inhomogeneously, the meromorphic functions are the functions $\frac{g(z)}{h(z)}$ with each of $g(z)$ and $h(z)$ polynomials.

In contrast, there are many meromorphic functions on an affine curve that are not rational: for example, e^z is a meromorphic function on $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$ (even holomorphic) that is not rational.

THE RIEMANN-ROCH THEOREM

Let C be the nonsingular projective curve over a field k (still assumed to be algebraically closed) defined by a homogeneous polynomial $F(X, Y, Z)$. As for meromorphic functions on Riemann surfaces, we try to understand the rational functions on a C in terms of their zeros and poles.

The ***group of divisors*** $\mathrm{Div}(C)$ on C is the free abelian group on the set $C(k)$. Thus an element of $\mathrm{Div}(C)$ is a finite sum

$$D = \sum n_P[P], \quad n_P \in \mathbb{Z}, \quad P \in C(k).$$

The ***degree*** of D is $\sum n_P$. There is a partial ordering on $\mathrm{Div}(C)$:

$$\sum n_P[P] \geq \sum m_P[P] \iff n_P \geq m_P \text{ for all } P.$$

In particular, $\sum n_P[P] \geq 0$ if and only if all the n_p are nonnegative.

Let φ be a nonzero rational function on C. Then φ is defined by a quotient $\frac{G(X,Y,Z)}{H(X,Y,Z)}$ of two polynomials of the same degree, say m, such that F doesn't divide H. Because $\varphi \neq 0$, F doesn't divide G either (recall that $k[X, Y, Z]$ is a unique factorization domain). By Bezout's theorem

$$(\deg F) \cdot m = \sum_{\{P \mid F(P)=0=G(P)\}} I(P, C \cap \{G = 0\})$$

$$(\deg F) \cdot m = \sum_{\{P \mid F(P)=0=H(P)\}} I(P, C \cap \{H = 0\}).$$

Define the divisor of φ to be

$$\mathrm{div}(\varphi) = \sum_{\{P \mid G(P)=0=F(P)\}} I(P, C \cap \{G = 0\})[P] - \sum_{\{P \mid H(P)=0=F(P)\}} I(P, C \cap \{H = 0\})[P]$$

The $[P]$ occurring in $\mathrm{div}(\varphi)$ with positive coefficient are called the ***zeros*** of φ, and those occurring with negative coefficient are its ***poles.*** Note the $\mathrm{div}(\varphi)$ has degree zero, and so φ has as many zeros as poles (counting multiplicities). Also, note that only the constant functions has no zeros or poles.

Given a divisor D, we define

$$L(D) = \{\varphi \mid \operatorname{div}(\varphi) + D \geq 0\} \cup \{0\}. \tag{5}$$

For example, if $D = [P]+2[Q]$, then $L(D)$ consists of those rational functions having no poles outside $\{P, Q\}$ and having at worst a single pole at P and a double pole at Q. Each $L(D)$ is a vector space over k, and in fact a finite-dimensional vector space. We denote its dimension by $\ell(D)$.

THEOREM 4.5 (RIEMANN) *There exists an integer g such that for all divisors D,*

$$\ell(D) \geq \deg D + 1 - g, \tag{6}$$

with equality for $\deg D$ *sufficiently positive.*

PROOF. See Fulton 1969, p. 196. □

The integer g determined by the theorem is called the ***genus*** of C.

EXAMPLE 4.6 Let $a_1, \ldots, a_m \in k = \mathbb{A}^1(k) \subset \mathbb{P}^1(k)$, and let $D = \sum r_i[a_i] \in \operatorname{Div}(\mathbb{P}^1)$, $r_i > 0$. The rational functions φ on $\mathbb{A}^1$ with their poles in $\{a_1, \ldots, a_m\}$ and at worst a pole of order r_i at a_i are those of the form

$$\varphi = \frac{f(X)}{(X-a_1)^{r_1}\cdots(X-a_m)^{r_m}}, \quad f(X) \in k[X].$$

The function φ will not have a pole at ∞ if and only if $\deg f \leq \sum r_i = \deg D$. The dimension of $L(D)$ is therefore the dimension of the space of polynomials f of degree $\leq \deg D$, which is $\deg D + 1$. Thus $\mathbb{P}^1$ has genus 0.

REMARK 4.7 (a) Roch's improvement to Riemann's theorem (the *Riemann-Roch theorem*) states that

$$\ell(D) = \deg D + 1 - g + \ell(W - D)$$

for any "canonical" divisor W; moreover, W has degree $2g-2$, and so equality holds in (6) if $\deg D > 2g - 2$. The canonical divisors are exactly the divisors of differentials of K over k. See Fulton 1969, Chap. 8, §§5,6.

(b) The Riemann-Roch theorem holds also for compact Riemann surfaces when $\ell(D)$ is defined to be the dimension of the space of meromorphic functions φ with $\operatorname{div}\varphi + D \geq 0$. For the Riemann surface of a nonsingular projective curve over $\mathbb{C}$, the canonical divisors in the two senses coincide, and so the genuses in the two senses also coincide. On comparing the two theorems, one obtains the statement in the first paragraph of (4.4).

For the Riemann-Roch theorem to be useful, we need to be able to compute the genus of a curve. For a nonsingular projective plane curve, it is given by the formula:

$$g(C) = \frac{(\deg C - 1)(\deg C - 2)}{2} \tag{7}$$

(Fulton 1969, p. 199). For example, a nonsingular projective plane curve of degree 1 or 2 has genus 0, and such a curve of degree 3 has genus 1.

ASIDE 4.8 The reader will have noted that not all integers $g \geq 0$ can occur in the formula (7). There do exist nonsingular projective curves with genus equal to any nonnegative integer — they just don't all occur as *nonsingular* projective *plane* curves.

For every nonsingular projective curve[6] C, there exists a regular map $\varphi\colon C \to C'$ from C onto a plane projective curve C' such that φ is an isomorphism outside a finite set and C' has only ordinary multiple points as singularities (Fulton 1969, p. 177, p. 220). The genus of C is given by the formula

$$g(X) = \frac{(\deg C - 1)(\deg C - 2)}{2} - \sum_{P \in C(k)} \frac{m_P(C)(m_P(C) - 1)}{2}$$

where $m_P(C)$ is the multiplicity of P on C (ibid. p. 199). Conversely, for each plane projective curve C' there exists a regular map $\varphi\colon C \to C'$ such that C is a nonsingular projective curve C and φ is an isomorphism outside the finite set of singular points of C' (ibid. p. 179).

THE GROUP LAW ON A CUBIC CURVE

The divisor of a rational function on C is said to be ***principal.*** Two divisors D and D' are said to be ***linearly equivalent,*** $D \sim D'$, if $D - D'$ is a principal divisor. We have groups

$$\mathrm{Div}(C) \supset \mathrm{Div}^0(C) \supset P(C)$$

where $\mathrm{Div}^0(C)$ is the group of divisors of degree 0 on C and $P(C)$ is the group of principal divisors. Define ***Picard groups:***

$$\mathrm{Pic}(C) = \mathrm{Div}(C)/P(C), \qquad \mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/P(C).$$

ASIDE 4.9 The group $\mathrm{Pic}(C)$ is defined also for affine curves. When C is a nonsingular *affine* curve, the ring $k[C]$ is a Dedekind domain, and $\mathrm{Pic}(C)$ is its ideal class group.

Now consider a nonsingular projective curve of genus 1. In this case, the Riemann-Roch theorem shows that

$$\ell(D) = \deg D \text{ if } \deg D \geq 1.$$

[6] A closed subvariety of $\mathbb{P}^n$ is the zero set of a finite set of homogeneous polynomials in $n+1$ variables. Such a subvariety is irreducible if it can't be written as a union of two proper closed subvarieties. An irreducible closed subvariety C of $\mathbb{P}^n$ is a curve if $C(k)$ is infinite but each proper closed subvariety of C is finite (cf. Fulton 1969, Chap. 6).

PROPOSITION 4.10 *Let C be a nonsingular projective curve of genus 1, and let $O \in C(k)$. The map*

$$P \mapsto [P] - [O] : C(k) \to \operatorname{Pic}^0(C) \tag{8}$$

is bijective.

PROOF. The map is injective because, if $\operatorname{div}(\varphi) = [P] - [O]$, then φ defines an isomorphism of C onto $\mathbb{P}^1$ (apply (4.26a) below to see that it has degree 1, and then apply 4.24b), which is impossible because $\mathbb{P}^1$ has genus 0. For the surjectivity, let D be a divisor of degree 0. Then $D + [O]$ has degree 1, and so there exists a rational function φ, unique up to multiplication by a nonzero constant, such that $\operatorname{div}(\varphi) + D + [O] \geq 0$. The only divisors ≥ 0 of degree 1 are of the form $[P]$. Hence there is a well-defined point P such that $D + [O] \sim [P]$, i.e., such that $D \sim [P] - [O]$. □

The bijection $C(k) \to \operatorname{Pic}^0(C)$ defines the structure of an abelian group on $C(k)$, which is determined by the condition: $P + Q = S$ if and only if $[P] + [Q] \sim [S] + [O]$.

I claim that this is the same structure as defined in the last section. Let $P, Q \in C(k)$, and suppose $P + Q = S$ with the law of composition in §3. Let L_1 be the line through P and Q, and let L_2 be the line through O and S. From the definition of S, we know that L_1 and L_2 have a common point R as their third points of intersection with C. Regard L_1 and L_2 as linear forms in X, Y, Z, and let $\varphi = \frac{L_1}{L_2}$. Then φ has simple zeros at P, Q, R and simple poles at O, S, R, and so

$$\operatorname{div}(\varphi) = [P] + [Q] + [R] - [O] - [S] - [R] = [P] + [Q] - [S] - [O].$$

Hence $[P] + [Q] \sim [S] + [O]$, and $P + Q = S$ according to the group structure defined by (8).

REMARK 4.11 When we choose a different zero $O' \in C(k)$, then the group law we get is just the translate of that given by O.

Perfect base fields

We now allow k to be a perfect field; for example, k could be any field of characteristic zero or any finite field. All curves over k will be assumed to be nonsingular and absolutely irreducible.

AFFINE CURVES

For an affine plane curve C_f over k, we let

$$k[C] = k[X, Y]/(f) = k[x, y],$$

and call it the ***ring of regular functions*** on C. We can no longer identify $k[C]$ with a ring of functions on $C(k)$ because, for example, $C(k)$ may be empty. However, every g in $k[C]$ defines a function $C(K) \to K$ for each field $K \supset k$, and $k[C]$ can be identified with the ring of families of such functions, compatible with inclusions $K \subset L$, defined by polynomials in the coordinate functions x and y. In a fancier terminology, a curve C defines a functor from the category k-fields to sets, and the regular maps are the maps of functors $C \to \mathbb{A}^1$ expressible as polynomials in the coordinate functions. A ***rational function*** on C is an element of the field of fractions $k(C) = k(x, y)$ of $k[C]$.

A ***prime divisor*** on C is a nonzero prime ideal $\mathfrak{p}$ in $k[C]$, and the ***group of divisors*** $\mathrm{Div}(C)$ is the free abelian group on the set of prime divisors. When k is algebraically closed, the Hilbert Nullstellensatz shows that the prime ideals in $k[C]$ are the ideals of the form $(x - a, y - b)$ with $(a, b) \in C(k)$, and so, in this case, the definition agrees with that in the preceding subsection. The degree of a prime divisor $\mathfrak{p}$ is the dimension of $k[C]/\mathfrak{p}$ as a k-vector space, and $\deg(\sum n_\mathfrak{p}\mathfrak{p}) \overset{\text{def}}{=} \sum n_\mathfrak{p} \deg(\mathfrak{p})$.

For any prime divisor $\mathfrak{p}$, the localization of $k[C]$ at $\mathfrak{p}$,

$$k[C]_\mathfrak{p} \overset{\text{def}}{=} \{g/h \in k(C) \mid g, h \in k[C], h \neq 0\},$$

is a ***discrete valuation ring***, i.e., a principal ideal domain with exactly one prime element $t_\mathfrak{p}$ up to associates (cf. Fulton 1969, Chap. 3, Theorem 1, p. 70). For $h \in k(C)^\times$, define $\mathrm{ord}_\mathfrak{p}(h)$ by the rule $h = h_0 t_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(h)}$, $h_0 \in k[C]_\mathfrak{p}^\times$. Then each $h \in k(C)^\times$ defines a (principal) divisor

$$\mathrm{div}(h) = \sum_\mathfrak{p} \mathrm{ord}_\mathfrak{p}(h)\mathfrak{p}.$$

PROJECTIVE CURVES

For a projective plane curve C_F over k,

$$k[x, y, z] \overset{\text{def}}{=} k[X, Y, Z]/(F(X, Y, Z))$$

is an integral domain, and remains so when tensored with any field $K \supset k$ (recall that we are assuming C to be absolutely irreducible). We can define, as before, a subfield $k(x, y, z)_0$ of $k(x, y, z)$ whose elements are the ***rational functions on*** C.

Write C as a union of affine curves

$$C = C_0 \cup C_1 \cup C_2$$

in the usual way. As before, there is a natural identification of $k(C)$ with $k(C_i)$ for each i. Each prime divisor $\mathfrak{p}$ on one of the C_is defines a discrete valuation ring in $k(C)$, and the discrete valuation rings that arise in this way are exactly those with field of fractions $k(C)$ containing $k^{\times}$ in their group of units. We define a ***prime divisor*** on C to be such a discrete valuation ring. We shall use $\mathfrak{p}$ to denote a prime divisor on C, with $\mathcal{O}_{\mathfrak{p}}$ the corresponding discrete valuation ring and $\mathrm{ord}_{\mathfrak{p}}$ the corresponding valuation on $k(C)^{\times}$. The ***group of divisors*** on C is the free abelian group generated by the prime divisors on C. The degree of a prime divisor $\mathfrak{p}$ is the dimension of the residue field of $\mathcal{O}_{\mathfrak{p}}$ as a k-vector space, and $\deg(\sum n_{\mathfrak{p}}\mathfrak{p}) = \sum n_{\mathfrak{p}} \deg(\mathfrak{p})$. Every $h \in k(C)^{\times}$ defines a (principal) divisor

$$\mathrm{div}(h) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(h)\mathfrak{p},$$

which has degree zero.

EXAMPLE 4.12 Consider an elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in k, \quad \Delta \neq 0,$$

over k. Write E_2 for the affine curve

$$Y^2 = X^3 + aX + b$$

and $k[x, y]$ for the ring of regular functions on E_2. A ***divisor*** on E is a finite sum

$$D = \sum n_{\mathfrak{p}}\mathfrak{p}$$

in which $n_{\mathfrak{p}} \in \mathbb{Z}$ and $\mathfrak{p}$ either corresponds to a nonzero prime ideal in $k[x, y]$ or is another symbol $\mathfrak{p}_{\infty}$ (the "prime divisor corresponding to the point at infinity"). The ***degree*** of $\mathfrak{p}$ is the degree of the field extension $[k[x, y]/\mathfrak{p} : k]$ if $\mathfrak{p} \neq \mathfrak{p}_{\infty}$, and is 1 if $\mathfrak{p} = \mathfrak{p}_{\infty}$.

THE RIEMANN-ROCH THEOREM

Let C be a projective plane curve over k (recall that we are assuming curves to be nonsingular and absolutely irreducible). For a divisor D on C, we define $L(D)$ by (5), and we let $\ell(D)$ be its dimension as a k-vector space. We shall need the following weak version of the Riemann-Roch theorem.

THEOREM 4.13 *There exists an integer g such that, for all divisors D,*

$$\ell(D) \geq \deg D + 1 - g,$$

with equality holding for $\deg D > 2g - 2$.

The usual proofs of the Riemann-Roch theorem apply over arbitrary fields. As an alternative, we sketch how to deduce Theorem 4.13 from the same theorem over an algebraically closed field. Fix an algebraic closure $\bar{k}$ and let $\bar{C}$ denote C regarded as a curve over $\bar{k}$. The Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$ acts on $C(\bar{k}) = \bar{C}(\bar{k})$ with finite orbits because each $P \in C(\bar{k})$ has coordinates in some finite extension of k.

Let C_i be one of the standard affine pieces of C, and let $\mathfrak{p}$ be a prime divisor on C_i. To give a k-homomorphism $k[C_i] = k[x, y] \to \bar{k}$ amounts to giving an element of $C_i(\bar{k})$, and the homomorphisms whose kernel contains $\mathfrak{p}$ correspond to the points in a single Γ-orbit in $C(\bar{k})$. In this way, we obtain a bijection from the set of prime divisors on C to the set of Γ-orbits in $C(\bar{k})$. Thus, we have a injective homomorphism

$$D \mapsto \bar{D}\colon \mathrm{Div}(C) \to \mathrm{Div}(\bar{C})$$

whose image consists of the divisors $\sum n_P P$ such that n_P is constant on each Γ-orbit. The map preserves principal divisors, the degrees of divisors, and the dimensions ℓ (because the condition for a function to lie in $L(D)$ is linear). It follows that C and $\bar{C}$ have the same genus, and that the Riemann-Roch theorem for $\bar{C}$ implies it for C.

The group structure on a cubic

For a curve C of genus 1 over k with a point $O \in C(k)$. The Riemann-Roch theorem over k^{al} shows that the map

$$P \mapsto [P] - [O] : C(\bar{k}) \to \mathrm{Pic}^0(\bar{C})$$

is a bijection. Because $O \in C(k)$, this bijection commutes with the action of Γ, and so defines a bijection $C(\bar{k})^{\Gamma} \to \mathrm{Pic}^0(\bar{C})^{\Gamma}$, i.e.,

$$C(k) \to \mathrm{Pic}^0(C) \stackrel{\text{def}}{=} (\mathrm{Pic}^0(\bar{C})^{\Gamma}. \tag{9}$$

Remark 4.14 Except when k is finite, not every class in $\mathrm{Pic}^0(C)$ need be represented by divisor on C (see 1.10, Chap. IV).

Regular maps of curves

Let k be a perfect field.

Affine plane curves

A ***regular map*** $\varphi\colon C_{g_1} \to C_{g_2}$ of affine plane curves is a pair (f_1, f_2) of regular functions on C_{g_1} sending $C_{g_1}(K)$ into $C_{g_2}(K)$ for all fields K containing k, i.e., such that, for all $K \supset k$,

$$P \in C_{g_1}(K) \implies (f_1(P), f_2(P)) \in C_{g_2}(K).$$

Thus a regular map defines a map $C_{g_1}(K) \to C_{g_2}(K)$, functorial in K, and this functorial map determines the pair (f_1, f_2).

LEMMA 4.15 *Let (f_1, f_2) be a pair of regular functions on C_{g_1}. The map*

$$P \mapsto (f_1(P), f_2(P)) \colon C_{g_1}(K) \to \mathbb{A}^2(K)$$

takes values in $C_{g_2}(K)$ for all fields K containing k if and only if $g_2(f_1, f_2) = 0$ (in $k[C_{g_1}]$).

PROOF. If $g_2(f_1, f_2) = 0$, then $g_2(f_1(P), f_2(P)) = 0$ for all $P \in C_{g_1}(K)$ and so $(f_1(P), f_2(P)) \in C_{g_2}(K)$. Conversely, if $(f_1(P), f_2(P)) \in C_{g_2}(k^{\mathrm{al}})$ for all $P \in C_{g_1}(k^{\mathrm{al}})$, then $g_2(f_1, f_2)$ is the zero function on C_{g_1}. □

PROPOSITION 4.16 *Let C_{g_1} and C_{g_2} be absolutely irreducible affine plane curves over k. There are natural one-to-one correspondences between the following objects:*

(a) *regular maps $\varphi \colon C_{g_1} \to C_{g_2}$;*
(b) *pairs (f_1, f_2) of regular functions on C_{g_1} such that $g_2(f_1, f_2) = 0$ (in $k[C_{g_1}]$);*
(c) *functorial maps $\varphi(K) \colon C_{g_1}(K) \to C_{g_2}(K)$ such that $x \circ \varphi(K)$ and $y \circ \varphi(K)$ are regular functions on C_{g_1};*
(d) *homomorphisms of k-algebras $k[C_{g_2}] \to k[C_{g_1}]$.*

PROOF. The lemma shows that the pair (f_1, f_2) defining φ in (a) satisfies the condition in (b). Moreover, it shows that the pair defines functorial map $P \mapsto (f_1(P), f_2(P))$ as in (c). Conversely, the regular functions $f_1 \overset{\text{def}}{=} x \circ \varphi$ and $f_2 = y \circ \varphi$ in (c) satisfy the condition in (b). Finally, write $k[C_{g_2}] = k[x, y]$. A homomorphism $k[C_{g_2}] \to k[C_{g_1}]$ is determined by the images f_1, f_2 of x, y, which can be any regular functions on C_{g_1} such that $g_2(f_1, f_2) = 0$. □

PROJECTIVE PLANE CURVES

Consider polynomials $F_0(X, Y, Z)$, $F_1(X, Y, Z)$, $F_2(X, Y, Z)$ of the same degree. The map

$$(a_0 : a_1 : a_2) \mapsto (F_0(a_0, a_1, a_m) : F_1(a_0, a_1, a_m) : F_2(a_0, a_1, a_m))$$

defines a regular map to $\mathbb{P}^2$ on the subset of $\mathbb{P}^2$ where not all F_i vanish. Its restriction to any curve in of $\mathbb{P}^2$ will also be regular where it is defined. It may be possible to extend the map to a larger set by representing it by different polynomials. Conversely, every regular map to $\mathbb{P}^2$ from an open subset of $\mathbb{P}^2$ arises in this way, at least "locally". Rather than give a precise definition (see, for example, AG, §6), I give an example, and then state the criterion we shall use.

EXAMPLE 4.17 We prove that the circle $X^2 + Y^2 = Z^2$ over $\mathbb{C}$ is isomorphic to $\mathbb{P}^1$. This equation can be rewritten $(X + iY)(X - iY) = Z^2$, and so, after a change of variables, it becomes $C : XZ = Y^2$. Define

$$\varphi\colon \mathbb{P}^1 \to C,\ (a : b) \mapsto (a^2 : ab : b^2).$$

For the inverse, define

$$\psi\colon C \to \mathbb{P}^1 \quad \text{by} \begin{cases} (a : b : c) \mapsto (a : b) & \text{if } a \neq 0 \\ (a : b : c) \mapsto (b : c) & \text{if } b \neq 0 \end{cases}.$$

Note that,

$$a \neq 0 \neq b, \quad ac = b^2 \implies \frac{c}{b} = \frac{b}{a}$$

and so the two maps agree on the set where they are both defined. Both φ and ψ are regular, and they define inverse maps on the sets of points.

Let $L = aX + bY + cZ$ be a nonzero linear form. The map

$$(x\colon y\colon z) \to (x/L(x,y,z), y/L(x,y,z), z/L(x,y,z))$$

is a bijection from the subset of $\mathbb{P}^2$ where $L \neq 0$ onto the plane $L = 1$ in $\mathbb{A}^3$. This last can be identified with $\mathbb{A}^2$ — for example, if $c \neq 0$, the projection $(x, y, z) \mapsto (x, y)\colon \mathbb{A}^3 \to \mathbb{A}^2$ maps the plane $L = 1$ bijectively onto $\mathbb{A}^2$. Therefore, for any curve $C \subset \mathbb{P}^2$ not contained in the plane $L = 0$, $C_L \overset{\text{def}}{=} C \cap \{P \mid L(P) \neq 0\}$ is an affine plane curve. Note that, if $L = Z$, then $C_L = C_2$.

PROPOSITION 4.18 *Let C and C' be nonsingular projective plane curves. Then any regular map $\varphi\colon C_L \to C'_{L'}$ of affine plane curves (L and L' are nonzero linear forms) extends uniquely to a regular map $C \to C'$.*

PROOF. If φ is the constant map, say, $\varphi(C_L) = P$, then the constant map with value P is the unique extension. Otherwise, φ defines an injective homomorphism of k-algebras $k[C'_{L'}] \to k[C_L]$, which extends to a homomorphism of the fields of fractions $k(C'_{L'}) \to k(C_L)$. But $k(C_L) \simeq k(C)$ and $k(C'_{L'}) \simeq k(C')$, and so we have a homomorphism of k-fields $k(C') \to k(C)$. Every such homomorphism arises from a unique homomorphism of curves $C \to C'$ (Fulton 1969, Chap. 7, p. 180), which extends the original morphism. (More geometrically, the Zariski closure of the graph of φ in $C \times C'$ is the graph of the extension of φ.) □

REMARK 4.19 The proposition holds also for singular curves provided that they are absolutely irreducible and all the singularities are contained in C_L and $C'_{L'}$. Moreover, it is true for any nonsingular projective curves C and C' (not necessarily plane) and open affine subsets $U \subset C$ and $U' \subset C'$.

EXAMPLE 4.20 Write

$$E(a,b) : Y^2Z = X^3 + aXZ^2 + bZ^3$$
$$E(a,b)^{\mathrm{aff}} : Y^2 = X^3 + aX + b.$$

Every regular map $\varphi^{\mathrm{aff}}\colon E(a,b)^{\mathrm{aff}} \to E(a',b')^{\mathrm{aff}}$ extends uniquely to a regular map $\varphi\colon E(a,b) \to E(a',b')$. The curve $E(a,b)$ has exactly one point at infinity, namely, $(0:1:0)$, which is the third point of intersection of $E(a,b)$ with any "vertical line" $X = cZ$. If φ^{aff} sends vertical lines (lines $X = c$) to vertical lines, then φ must send the point at infinity on $E(a,b)$ to the point at infinity on $E(a',b')$.

DEFINITION 4.21 Let $\varphi\colon C \to C'$ be a regular map of curves over k.

(a) The map φ is ***constant*** if, for all fields $K \supset k$, the image of $\varphi(K)$ is a single point
(b) The map φ is ***dominating*** if the image of $\varphi(k^{\mathrm{al}})$ omits only finitely many points of $C'(k^{\mathrm{al}})$.
(c) The map φ is ***surjective*** if $\varphi(k^{\mathrm{al}})$ is surjective.

EXAMPLE 4.22 Let $C = \mathbb{A}^1 \smallsetminus \{0\}$.

(a) The regular map $x \mapsto x\colon C \to \mathbb{A}^1$ is dominating.
(b) For $n \in \mathbb{Z}$, the regular map $x \mapsto x^n\colon C \to C$ is constant if $n = 0$ and is otherwise surjective (even though $x \mapsto x^n\colon k^\times \to k^\times$ need not be surjective).

PROPOSITION 4.23 *(a) A regular map of geometrically irreducible curves is either dominating or constant.*

(b) A dominating regular map of projective curves is surjective.

PROOF. For (b), see Fulton 1969, Problem 8-18. □

Let C and C' be geometrically irreducible. A dominating regular map $\varphi\colon C \to C'$ defines (by composition) a homomorphism $k(C') \hookrightarrow k(C)$. The ***degree*** of φ is defined to the degree of $k(C)$ over the image of $k(C')$.

PROPOSITION 4.24 *(a) If the extension $k(C)/\varphi(k(C'))$ is separable (for example, if the degree of φ is prime to the characteristic of k), then $\varphi(k^{\mathrm{al}})\colon C(k^{\mathrm{al}}) \to C'(k^{\mathrm{al}})$ is $n:1$ outside a finite set.*

(b) If the degree of φ is 1, and C and C' are nonsingular projective curves, then φ is an isomorphism.

PROOF. For (a), see Fulton 1969, Problem 8.36. For (b), see ibid., Chap. 7, Theorem 3, p. 179. □

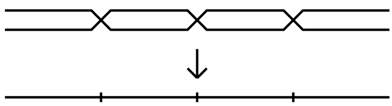
EXAMPLE 4.25 Consider the map $(x, y) \mapsto x \colon E^{\text{aff}}(k) \to \mathbb{A}^1(k)$, where E^{aff} is the curve

$$E^{\text{aff}} : Y^2 = X^3 + aX + b.$$

The map on the rings of regular functions is

$$X \mapsto x \colon k[X] \to k[x, y] \stackrel{\text{def}}{=} k[X, Y]/(Y^2 - X^3 - aX - b).$$

Clearly $k(x, y) = k(x)[\sqrt{x^3 + ax + b}]$, and so the map has degree 2. If $\text{char}(k) \neq 2$, then the field extension is separable, and the map on points is $2 : 1$ except over the roots of $X^3 + aX + b$.



If $\text{char}(k) = 2$, then the field extension is purely inseparable, and the map is $1 : 1$ on points.

REMARK 4.26 (a) A surjective regular map $\varphi \colon C \to C'$ defines, in a natural way, a homomorphism $\varphi^* \colon \text{Div}(C') \to \text{Div}(C)$. This multiplies the degree of a divisor by $\deg(\varphi)$. In other words, when one counts multiplicities, $\varphi^{-1}(P)$ has $\deg(\varphi)$ points for all $P \in C'(k^{\text{al}})$.

(b) We shall need the following criterion: a nonconstant map $\varphi \colon C \to C'$ is separable if, at some point, the map on the tangent spaces is an isomorphism.

5 Defining algebraic curves over subfields

Let $\Omega \supset k$ be fields. A curve C over k is defined by polynomial equations with coefficients in k, and these same equations define a curve C_Ω over Ω. In this section, we examine the functor $C \mapsto C_\Omega$. This functor is faithful (the map on Hom sets is injective) but, in general, it is neither full (the map on Homs is not onto) nor essentially surjective (a curve over Ω need not be isomorphic to a curve of the form C_Ω).

EXAMPLE 5.1 (a) The curves

$$X^2 + Y^2 = 1, \quad X^2 + Y^2 = -1,$$

are not isomorphic over $\mathbb{R}$ (the second has no real points) but become isomorphic over $\mathbb{C}$ by the map $(x, y) \mapsto (ix, iy)$.

(b) An elliptic curve E over Ω with j-invariant $j(E)$ (see Chap. II, 2.1) arises from an elliptic curve over k if and only if $j(E) \in k$.

Let Γ be the group of automorphisms of Ω fixing k (i.e., fixing each element of k). In the remainder of this section, we assume that the only elements of Ω fixed by all elements of Γ are those in k, i.e., we assume that $\Omega^{\Gamma} = k$. For example, Ω could be a Galois extension of k (possibly infinite), or Ω could be $\mathbb{C}$ and k could be any subfield (FT, 8.23). For a curve C over Ω and $\sigma \in \Gamma$, σC denotes the curve over Ω obtained by applying σ to the coefficients of the equations defining C.

Let C be a curve over Ω. A ***descent system on*** C is a family $(\varphi_\sigma)_{\sigma\in\Gamma}$ of isomorphisms $\varphi_\sigma\colon \sigma C \to C$ satisfying the cocycle condition,

$$\varphi_\sigma \circ (\sigma\varphi_\tau) = \varphi_{\sigma\tau} \text{ for all } \sigma, \tau \in \Gamma.$$

The pairs consisting of a curve over Ω and a descent system can be made into a category by defining a morphism $(C, (\varphi_\sigma)_\sigma) \to (C', (\varphi'_\sigma)_\sigma)$ to be a regular map $\alpha\colon C \to C'$ such that $\alpha \circ \varphi_\sigma = \varphi_\sigma \circ \sigma\alpha$ for all $\sigma \in \Gamma$. For a curve C over k, C_Ω has a canonical descent system, because $\sigma(C_\Omega) = C_\Omega$, and so we can take $\varphi_\sigma = \mathrm{id}$.

Let C be a curve over Ω. A ***model*** of C over k is a curve C_0 over k together with an isomorphism $\varphi\colon C \to C_{0\Omega}$. Such a model (C_0, φ) ***splits*** a descent system $(\varphi_\sigma)_{\sigma\in\Gamma}$ on C if $\varphi_\sigma = \varphi^{-1} \circ \sigma\varphi$ for all $\sigma \in \Gamma$. A descent system is ***effective*** if it is split by some model over k.

PROPOSITION 5.2 *The functor sending a curve C over k to C_Ω endowed with its canonical descent system is fully faithful and its essential image consists of the pairs $(C, (\varphi_\sigma))$ with $(\varphi_\sigma)_{\sigma\in\Gamma}$ effective.*

PROOF. See AG, 16.16. □

Let (φ_σ) be a descent system on C. For $P \in C(\Omega)$, define

$${}^{\sigma}P = \varphi_\sigma(\sigma P).$$

Then

$${}^{\sigma\tau}P = \varphi_{\sigma\tau}(\sigma\tau P) = (\varphi_\sigma \circ \sigma\varphi_\tau)(\sigma\tau P) = \varphi_\sigma(\sigma({}^{\tau}P)) = {}^{\sigma}({}^{\tau}P),$$

and so $(\sigma, P) \mapsto {}^{\sigma}P$ is an action of Γ on $C(\Omega)$. Conversely, an action $(\sigma, P) \mapsto {}^{\sigma}P$ of Γ on $C(\Omega)$ arises from a descent system if and only if, for every $\sigma \in \Gamma$, the map $\sigma P \mapsto {}^{\sigma}P\colon (\sigma C)(\Omega) \to C(\Omega)$ is regular; the action is then said to be ***regular***.

A finite set S of points in $C(\Omega)$ is said to ***rigidify*** C if no automorphism of C fixes every point in S except the identity map.

PROPOSITION 5.3 *A descent system (φ_σ) on C is effective if there exists a finite set S of points rigidifying C and a subfield K of Ω, finitely generated over k, such that ${}^{\sigma}P = P$ for every $P \in S$ and every σ fixing K.*

PROOF. See AG, 16.33. □

REMARK 5.4 Let C be a nonsingular projective curve over an algebraically closed field.

If C has genus zero, then it is isomorphic to $\mathbb{P}^1$. The automorphisms of $\mathbb{P}^1$ are the linear fractional transformations, and so any three distinct points rigidify $\mathbb{P}^1$.

If C has genus one, then it has only finitely many automorphisms fixing a given point O on C (see II §2; IV 7.13). For each automorphism $\alpha \neq \mathrm{id}$, choose a P such that $\alpha(P) \neq P$. Then O and the Ps rigidify C.

If C has genus greater than one, then it has only finitely many automorphisms (see, for example, Hartshorne 1977, IV Ex. 5.2, when the field has characteristic zero). Thus, C is rigidified by some finite set.

Recall (FT, §8) that Γ has a natural topology under which the open subgroups of Γ correspond to the subfields of Ω that are finitely generated over k. An action of Γ on $C(\Omega)$ is continuous if and only if the stabilizer of each $P \in C(\Omega)$ is open.

PROPOSITION 5.5 *Assume Ω is algebraically closed. The functor sending a nonsingular projective curve C over k to C_Ω endowed with the natural action of Γ on $C(\Omega)$ is fully faithful, with essential image the nonsingular projective curves endowed with a continuous regular action of Γ.*

PROOF. Because Ω is algebraically closed, to give a descent system on a curve C' over Ω is the same as to give a regular action of Γ on $C'(\Omega)$. Moreover, a regular map of curves with descent systems will preserve the descent systems if and only if it commutes with the actions. According to (5.4), C' is rigidified by a finite set S, and because the action is continuous the condition in (5.3) holds. Thus the statement follows from the preceding propositions. □

NOTES Propositions 5.2 and 5.3 are true for all quasi-projective varieties. It should be noted that they are quite elementary (see AG, §16). In particular, their proofs don't require any of Grothendieck's or Weil's theorems on descent.

Chapter II

Basic Theory of Elliptic Curves

For convenience, we assume that the field k is perfect.

1 Definition of an elliptic curve

DEFINITION 1.1 An ***elliptic curve*** over k can be defined, according to taste, as:

(a) a nonsingular projective plane curve E over k of degree 3 together with a point $O \in E(k)$;
(b) same as (a) except that O is required to be a point of inflection;
(c) a nonsingular projective plane curve over k of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3; \quad (10)$$

(d) a nonsingular projective curve E of genus 1 together with a point $O \in E(k)$.

Let (E, O) be as in (b); we show (Proposition 1.2) that a linear change of variables will carry E into the form (c) and O into the point $(0 : 1 : 0)$. Conversely, let E be as in (c); then $O = (0 : 1 : 0) \in E(k)$ and is a point of inflection (see I 1.20).

Let E be as in (a); then E has genus 1 by formula (7), p. 34. Conversely, let (E, O) be as in (d). We shall see that the Riemann-Roch theorem implies that the k-vector space $L(3O)$ has a basis $1, x, y$ such that the map

$$P \mapsto (x(P) : y(P) : 1) \colon E \smallsetminus \{O\} \to \mathbb{P}^2$$

extends to an isomorphism from E onto a curve as in (c) sending O to $(0 : 1 : 0)$ (p. 47). On combining these statements, we see that a curve E as in (a) can be embedded in $\mathbb{P}^2$ in such a way that O becomes a point of inflection. We also give a direct proof of this (p. 48).

Transforming a cubic equation into standard form

PROPOSITION 1.2 *Let C be a nonsingular cubic projective plane curve over k, and let O be a point of inflection in $C(k)$.*

(a) *After an invertible linear change of variables with coefficients in k, the point O will have coordinates* $(0:1:0)$ *and the tangent line to C at O will be* $L_\infty : Z = 0$.

(b) *If* $(0:1:0) \in C(k)$ *and the tangent line to C at* $(0:1:0)$ *is* $L_\infty : Z = 0$, *then the equation of C has the form (10).*

PROOF. We first prove (a). Let $(a:b:c) \in \mathbb{P}^2(k)$, and assume $b \neq 0$. The regular map

$$(x:y:z) \mapsto (bx - ay : by : bz - cy) \colon \mathbb{P}^2 \to \mathbb{P}^2$$

sends $(a:b:c)$ to $(0:b^2:0) = (0:1:0)$ and is an isomorphism (it has an inverse of a similar form). If $b = 0$, but $c \neq 0$, we first interchange the y and z coordinates. Thus, we may suppose $O = (0:1:0)$.

Let

$$L : aX + bY + cZ = 0, \quad a,b,c \in k, \quad \text{not all of } a,b,c \text{ zero},$$

be the tangent line at $(0:1:0)$. Let $A = (a_{ij})$ be any invertible 3×3 matrix whose first two columns are orthogonal to (a,b,c), and define a change of variables by

$$A \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}.$$

With respect to the variables X', Y', Z', the equation of the line L becomes

$$0 = (a,b,c) \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = (a,b,c)A \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = (0,0,d) \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = dZ'.$$

Moreover, $d \neq 0$, and so we may take the equation of the line to be $Z' = 0$. This completes the proof of (a).

We next prove (b). The general cubic form is $F(X,Y,Z)$:

$$c_1X^3 + c_2X^2Y + c_3X^2Z + c_4XY^2 + c_5XYZ + c_6XZ^2 + \\ c_7Y^3 + c_8Y^2Z + c_9YZ^2 + c_{10}Z^3.$$

Let F be the polynomial defining C. Because C is nonsingular, F is absolutely irreducible (I 1.21).

Because $O = (0:1:0) \in C(k)$, $\boxed{c_7 = 0}$.

Recall that $U_1 = \{(x : y : z) \mid y = 1\}$ and that we identify U_1 with $\mathbb{A}^2$ via $(x : 1 : z) \leftrightarrow (x, z)$. Moreover $C \cap U_1$ is the affine curve defined by $F(X, 1, Z)$:

$$c_1X^3 + c_2X^2 + c_3X^2Z + c_4X + c_5XZ + c_6XZ^2 + c_8Z + c_9Z^2 + c_{10}Z^3.$$

The tangent line at $(0 : 1 : 0) \leftrightarrow (0, 0)$ is

$$c_4X + c_8Z = 0.$$

Because this equals $L_\infty : Z = 0$, $\boxed{c_4 = 0}$. As C is nonsingular, $\boxed{c_8 \neq 0}$.

The intersection number

$$\begin{aligned} I(O, L_\infty \cap C) &= I(Z, F(X, 1, Z)) \\ &= I(Z, c_1X^3 + c_2X^2). \end{aligned}$$

Because O is a point of inflection, $I(O, L_\infty \cap C) \geq 3$, and so $\boxed{c_2 = 0}$.

On combining the boxed statements, we find that our cubic form has become

$$c_1X^3 + c_3X^2Z + c_5XYZ + c_6XZ^2 + c_8Y^2Z + c_9YZ^2 + c_{10}Z^3, \quad c_8 \neq 0.$$

Moreover, $c_1 \neq 0$ because otherwise the polynomial is divisible by Z. After dividing through by c_1 and replacing Z with $-c_1Z/c_8$, we obtain an equation of the form (10). □

Nonsingular projective curves of genus 1 with a rational point.

Let E be a complete nonsingular curve of genus 1 over a field k and let $O \in E(k)$. According to the Riemann-Roch theorem (I 4.13), the rational functions on E having no poles except at O and having at worst a pole of order $m \geq 1$ at O, form a vector space of dimension m over k, i.e., $L(m[O])$ has dimension m for $m \geq 1$. The constant functions lie in $L([O])$, and according to the Riemann-Roch theorem, there are no other. Thus $\{1\}$ is a basis for $L([O])$. Choose x so that $\{1, x\}$ is a basis for $L(2[O])$. Choose y so that $\{1, x, y\}$ is a basis for $L(3[O])$. Then $\{1, x, y, x^2\}$ is a basis for $L(4[O])$ — if it were linearly dependent, x^2 would have to be a linear combination of $1, x, y$, but then it couldn't have a quadruple pole at O. And $\{1, x, y, x^2, xy\}$ is a basis for $L(5[O])$ for a similar reason.

The subset $\{1, x, y, x^2, xy, x^3, y^2\}$ of $L(6[O])$ contains 7 elements, and so it must be linearly dependent: there exist $a_i \in k$ such that

$$a_0y^2 + a_1xy + a_3y = a_0'x^3 + a_2x^2 + a_4x + a_6$$

(as regular functions on $E \setminus \{O\}$). Moreover, a_0 and a_0' must be nonzero, because the set with either x^3 or y^2 omitted is linearly independent, and so, after replacing y with $a_0 y/a_0'$ and x with $a_0 x/a_0'$ and multiplying through by $a_0'^2/a_0^3$, we can suppose both equal 1. The map $P \mapsto (x(P), y(P))$ sends $E \smallsetminus \{O\}$ onto the plane affine curve

$$C : Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6.$$

The function x has a double pole at O and no other pole, and so it has only two zeros. Similarly, $x + c$ has two zeros for any $c \in k$ (counting multiplicities), and so the composite

$$E \smallsetminus \{O\} \to C \to \mathbb{A}^1, \quad P \mapsto (x(P), y(P)) \mapsto x(P)$$

has degree 2 (see I 4.24, 4.26a). Similarly, the composite

$$E \smallsetminus \{O\} \to C \to \mathbb{A}^1, \quad P \mapsto (x(P), y(P)) \mapsto y(P)$$

has degree 3. The degree of $E \smallsetminus \{O\} \to C$ divides both 2 and 3, and therefore is 1. If C were singular, it would have genus 0, which is impossible. Therefore C is nonsingular, and so the map is an isomorphism, and it extends to an isomorphism of E onto

$$\bar{C} : Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3$$

(I 4.18).

Transforming a point to a point of inflection

Let C be a nonsingular projective plane cubic curve over k, and let $O \in C(k)$. Since we wish to transform O to into a point of inflection, we may suppose that it is not already one. Therefore, the tangent line to C at O meets C in a point $P \neq O$. We can make a linear change of variables so that the Y-axis is the tangent line at O and $(0:0:1) = P$. Thus,

$$C : F_1(X,Y)Z^2 + F_2(X,Y)Z + F_3(X,Y) = 0$$

where $F_i(X,Y)$ is homogeneous of degree i. Let $C^{\mathrm{aff}} = C \cap \{Z \neq 0\}$, so that

$$C^{\mathrm{aff}}: F_1(X,Y) + F_2(X,Y) + F_3(X,Y) = 0.$$

Let $O = (0, y)$, $y \neq 0$; then y is a double root of

$$F_1(0,1)y + F_2(0,1)y^2 + F_3(0,1)y^3 = 0,$$

and so

$$F_2(0,1)^2 - 4F_1(0,1)F_3(0,1) = 0. \tag{11}$$

The line $Y = tX$ intersects C^{aff} at the points whose x-coordinates satisfy

$$xF_1(1,t) + x^2 F_2(1,t) + x^3 F_3(1,t) = 0.$$

The solution $x = 0$ gives the origin, and so

$$F_1(1,t) + xF_2(1,t) + x^2F_3(1,t) = 0$$

gives a relation between the functions $P \mapsto x(P)$ and $P \mapsto t(P) = y(P)/x(P)$ on $C^{\mathrm{aff}} \smallsetminus \{(0,0)\}$. This relation can be rewritten as

$$(2F_3(1,t)x + F_2(1,t))^2 = F_2(1,t)^2 - 4F_1(1,t)F_3(1,t).$$

Now

$$\begin{aligned} s &\mapsto 2F_3(1,y/x)x + F_2(1,y/x), \\ t &\mapsto y/x \end{aligned}$$

defines a homomorphism $k[s,t] \to k[x,y][x^{-1}]$ where s,t satisfy

$$s^2 = G(t), \quad G(t) = F_2(1,t)^2 - 4F_1(1,t)F_3(1,t),$$

i.e., a regular map $C^{\mathrm{aff}} \smallsetminus \{O\} \to E$ where E is the affine curve defined by the equation $s^2 = G(t)$ (see I 4.15). The polynomial $G(t)$ has degree 3 because of (11), and $C^{\mathrm{aff}} \to E$ extends to an isomorphism of C onto the projective closure of E sending O to $(0:1:0)$ (Chap. I, §4).

ASIDE 1.3 The ***Hessian*** of a projective plane curve C_F is

$$H(X,Y,Z) = \begin{vmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial X \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Y \partial Z} \\ \frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2} \end{vmatrix}.$$

Assume $\mathrm{char}(k) \neq 2$. A nonsingular point $P = (a:b:c)$ on the curve C_F is a point of inflection if and only if $\det H(a,b,c) = 0$, in which case $I(P, C_F \cap H) = 1$ (Fulton 1969, Chap. 5, p.116). If F has degree d, then H has degree $3(d-2)$, and so a nonsingular cubic has 3 points of inflection in k^{al}. Unfortunately, it might have no point of inflection with coordinates in k. An invertible linear change of variables will not change this (it will only multiply the Hessian by a nonzero constant). The above nonlinear change of variables was found by Nagell (1928–29).

Isogenies

Two elliptic curves are said to be ***isogenous*** if there exists a nonconstant regular map from one to the other. By composing the map with translation, we then get a regular map sending O to O'. Such a map is called an ***isogeny***. Thus a regular map $\varphi\colon E \to E'$ is an isogeny if $\varphi(O) = O'$ and $\varphi(k^{\mathrm{al}})\colon E(k^{\mathrm{al}}) \to E(k^{\mathrm{al}})$ is surjective.

I claim that if $\varphi\colon E \to E'$ is an isogeny, then $\varphi(k^{\mathrm{al}})\colon E(k^{\mathrm{al}}) \to E'(k^{\mathrm{al}})$ is a homomorphism of groups. In proving this, we may replace k with k^{al}. Note that φ defines a homomorphism

$$\varphi_*\colon \mathrm{Div}(E) \to \mathrm{Div}(E'), \quad \sum n_P[P] \mapsto \sum n_P[\varphi(P)],$$

which preserves principal divisors and degrees. Therefore there exists a commutative diagram

$$\begin{array}{ccc} E(k) & \xrightarrow{P\mapsto[P]-[O]} & \mathrm{Pic}^0(E) \\ \Big\downarrow{\scriptstyle\varphi(k)} & & \Big\downarrow{\scriptstyle\varphi_*} \\ E'(k) & \xrightarrow{P\mapsto[P]-[O']} & \mathrm{Pic}^0(E'). \end{array}$$

As the horizontal maps are isomorphisms and all the maps (except possibly $\varphi(k)$) are homomorphisms, $\varphi(k)$ must also be homomorphism.

I also claim that isogeny is an equivalence relation. It is reflexive because the identity map is an isogeny, and it is transitive because a composite of isogenies is an isogeny. Let $\varphi\colon E \to E'$ be an isogeny, and let S be its kernel. Since S is finite, it will be contained in E_n for some n, and the isogeny $n\colon E \to E \simeq E/E_n$ factors through $\varphi\colon E \to E' \simeq E/S$ (cf. Silverman 1986, III 4).

2 The Weierstrass equation for an elliptic curve

Let E be an elliptic curve over k. Any equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \qquad (12)$$

is called a ***Weierstrass equation*** for the elliptic curve.

When k has characteristic $\neq 2, 3$, a change of variables

$$X' = X, \quad Y' = Y + \frac{a_1}{2}X, \quad Z' = Z$$

will eliminate the XYZ term in (12), and a change of variables

$$X' = X + \frac{a_2}{3}, \quad Y' = Y + \frac{a_3}{2}, \quad Z' = Z$$

will then eliminate the X^2 and Y terms. Thus we arrive at the equation:

$$Y^2Z = X^3 + aXZ^2 + bZ^3.$$

THEOREM 2.1 *Let k be a field of characteristic $\neq 2, 3$.*

(a) Every elliptic curve (E, O) is isomorphic to a curve of the form

$$E(a,b) : Y^2Z = X^3 + aXZ^2 + bZ^3, \qquad a, b \in k, \qquad (13)$$

pointed by $(0 : 1 : 0)$. *Conversely, the curve $E(a,b)$ is nonsingular (and so, together with* $(0 : 1 : 0)$ *is an elliptic curve) if and only if* $4a^3 + 27b^2 \neq 0$.

(b) Let $\varphi\colon E(a',b') \to E(a,b)$ be an isomorphism sending $O = (0 : 1 : 0)$ to $O' = (0 : 1 : 0)$; then there exists a $c \in k^\times$ such that $a' = c^4a$, $b' = c^6b$ and φ is the map $(x : y : z) \mapsto (c^2x : c^3y : z)$. Conversely, if $a' = c^4a$, $b' = c^6b$

for some $c \in k^\times$, *then* $(x : y : z) \mapsto (c^2x : c^3y : z)$ *is an isomorphism* $E(a', b') \to E(a, b)$ *sending* O *to* O'.

(c) When (E, O) *is isomorphic to* $(E(a, b), O)$, *we let*

$$j(E) = \frac{1728(4a^3)}{4a^3 + 27b^2}.$$

Then $j(E)$ *depends only on* (E, O), *and two elliptic curves* E *and* E' *become isomorphic over* k^{al} *if and only if* $j(E) = j(E')$.

PROOF. (a) The first statement was proved above. The point $(0 : 1 : 0)$ is always nonsingular on $E(a, b)$, and we showed in (I 1.5) that the affine curve

$$Y^2 = X^3 + aX + b$$

is nonsingular if and only if $4a^3 + 27b^2 \neq 0$.

(b) The regular function $x \circ \varphi$ on $E(a', b')$ has a double pole at O', and so $x \circ \varphi = u_1x' + r$ for some $u_1 \in k^\times$ and $r \in k$ (see the proof of (1.1d)→(1.1c)). Similarly, $y \circ \varphi = u_2y' + sx' + t$ for some $u_2 \in k^\times$ and $s, t \in k$. But $f \mapsto f \circ \varphi$ is a homomorphism $k[x, y] \to k[x', y']$ where x, y and x', y' respectively are coordinate functions on $E(a, b)$ and $E(a', b')$. As x and y satisfy

$$Y^2 = X^3 + aX + b,$$

so also do $x \circ \varphi$ and $y \circ \varphi$, i.e.,

$$(u_2y' + sx' + t)^2 = \left(u_1x' + r\right)^3 + a\left(u_1x' + r\right) + b.$$

But any polynomial satisfied by x', y' is a multiple of

$$Y^2 - \left(X^3 + a'X + b'\right),$$

from which it follows that $u_2^2 = u_1^3$, $r, s, t = 0$, $a' = c^4a$ (where $c = u_2/u_1$), $b' = c^6b$, and φ is as described. The converse is obvious.

(c) If (E, O) is isomorphic to both $(E(a, b), O)$ and $(E(a', b'), O')$, then there exists a $c \in k^\times$ such that $a' = c^4a$ and $b' = c^6b$, and so obviously the two curves give the same j. Conversely, suppose that $j(E) = j(E')$. Note first that

$$a = 0 \iff j(E) = 0 \iff j(E') = 0 \iff a' = 0.$$

As any two curves of the form $Y^2Z = X^3 + bZ^3$ are isomorphic over k^{al}, we may suppose that a and a' are both nonzero. After replacing (a, b) with (c^4a, c^6b) where $c = \sqrt[4]{\frac{a'}{a}}$ we will have that $a = a'$. Now $j(E) = j(E') \Longrightarrow b = \pm b'$. A minus sign can be removed by a change of variables with $c = \sqrt{-1}$. □

REMARK 2.2 Two elliptic curves can have the same j-invariant and yet not be isomorphic over k. For example, if c is not a square in k, then

$$Y^2Z = X^3 + ac^2XZ^2 + bc^3Z^3$$

has the same j invariant as $E(a,b)$, but it is not isomorphic to it.

REMARK 2.3 For every $j \in k$, there exists an elliptic curve E over k with $j(E) = j$, for example,

$$\begin{aligned}
&Y^2Z = X^3 + Z^3, \quad j = 0,\\
&Y^2Z = X^3 + XZ^2, \quad j = 1728,\\
&Y^2Z = X^3 - \frac{27}{4}\frac{j}{j-1728}XZ^2 - \frac{27}{4}\frac{j}{j-1728}Z^3, \quad j \neq 0, 1728.
\end{aligned}$$

We next give the formulas for the addition and doubling of points on the curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a,b \in k \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

To derive the formulas, first find the x-coordinate of the point sought by using that the sum of the roots of a polynomial $f(X)$ is $-$(coefficient of $X^{\deg f - 1}$).

ADDITION FORMULA

Let $P = (x,y)$ be the sum of $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. If $P_2 = -P_1$, then $P = O$, and if $P_1 = P_2$, we can apply the duplication formula below. Otherwise, $x_1 \neq x_2$, and (x,y) is determined by the following formulas:

$$x(x_1 - x_2)^2 = x_1x_2^2 + x_1^2x_2 - 2y_1y_2 + a(x_1 + x_2) + 2b$$

and

$$y(x_1 - x_2)^3 = W_2y_2 - W_1y_1$$

where

$$\begin{aligned}
W_1 &= 3x_1x_2^2 + x_2^3 + a(x_1 + 3x_2) + 4b\\
W_2 &= 3x_1^2x_2 + x_1^3 + a(3x_1 + x_2) + 4b.
\end{aligned}$$

DUPLICATION FORMULA

Let $P = (x,y)$ and $2P = (x_2, y_2)$. If $y = 0$, then $2P = 0$. Otherwise $y \neq 0$, and (x_2, y_2) is determined by the following formulas:

$$\begin{aligned}
x_2 &= \frac{(3x^2 + a)^2 - 8xy^2}{4y^2} = \frac{x^4 - 2ax^2 - 8bx + a^2}{4(x^3 + ax + b)}\\
y_2 &= \frac{x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - a^3 - 8b^2}{(2y)^3}.
\end{aligned}$$

GENERAL BASE FIELD

We state the analogue of Theorem 2.1 for a general field k. Associated with an equation (12), there are the following quantities:

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 & c_4 &= b_2^2 - 24b_4 \\ b_4 &= a_1a_3 + 2a_4 & c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \\ b_6 &= a_3^2 + 4a_6 & \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\ b_8 &= b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 & j &= c_4^3/\Delta \end{aligned}$$

THEOREM 2.4 *Let k be an arbitrary field.*

(a) Every elliptic curve (E, O) is isomorphic to a curve of the form

$$E(a_1, \ldots, a_4, a_6) : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

pointed by $(0:1:0)$. Conversely, the curve $E(a_1, \ldots)$ is nonsingular (and so, together with $(0:1:0)$ an elliptic curve) if and only if $\Delta \neq 0$.

(b) Let $\varphi\colon E(a_1', \ldots) \to E(a_1, \ldots)$ be an isomorphism sending O to O'; then there exist $u \in k^\times$ and $r, s, t \in k$ such that

$$\begin{aligned} ua_1' &= a_1 + 2s \\ u^2a_2' &= a_2 - sa_1 + 3r - s^2 \\ u^3a_3' &= a_3 + ra_1 + 2t \\ u^4a_4' &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\ u^6a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1 \end{aligned}$$

and φ is the map sending $(x:y:z)$ to $(u^2x + rz : u^3y + su^2x + tz : z)$. Conversely, if there exist $u \in k^\times$ and $r, s, t \in k$ satisfying these equations, then

$$(x:y:z) \mapsto (u^2x + rz : u^3y + su^2x + tz : z)$$

is an isomorphism $E(a_1', \ldots) \to E(a_1, \ldots)$ sending O to O'.

(c) When (E, O) is isomorphic to $(E(a_1, \ldots), O)$, we let

$$j(E) = c_4^3/\Delta.$$

Then $j(E)$ depends only on (E, O), and two elliptic curves (E, O) and (E', O') become isomorphic over k^{al} if and only if $j(E) = j(E')$.

PROOF. The proof is the same as that of Theorem 2.1, only a little more complicated. □

REMARK 2.5 In realizing a curve of genus 1 as a nonsingular plane cubic, it was crucial that the curve have a point in $E(k)$. Without this assumption, it may only be possible to realize the curve as a (singular) plane curve of (possibly much) higher degree.

NOTES Cassels (1991, p. 34) notes that while “older geometrical techniques (adjoint curves etc.) had shown that every elliptic curve is birationally equivalent to a cubic, Nagell (1928–29) was the first to show that it can be reduced to canonical form.” In arbitrary characteristic, Tate wrote out the formulas in Theorem 2.4 in a letter to Cassels (Tate 1975), which has been copied (and, on occasion, miscopied) by all later authors. The name “Weierstrass equation” for (12) is a little misleading since Weierstrass wrote his elliptic curves as

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3, \tag{14}$$

but, as Tate (1974, §2) writes: “We call [12] a Weierstrass equation because in characteristics $\neq 2, 3$, we can replace x and y by

$$\wp = x + \frac{a_1^2 + 4a_2}{12}, \quad \wp' = 2y + a_1x + a_3$$

and [12] becomes [14].”

3 Reduction of an elliptic curve modulo p

Consider an elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in \mathbb{Q}, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

After a change of variables $X \mapsto X/c^2$, $Y \mapsto Y/c^3$, $Z \mapsto Z$, we may suppose that the coefficients a, b lie in $\mathbb{Z}$, and so we may look at them modulo p to obtain a curve $\bar{E}$ over the field $\mathbb{F}_p \overset{\text{def}}{=} \mathbb{Z}/p\mathbb{Z}$. In this section, we examine the curves obtained in this way.

Algebraic groups of dimension 1

Let k be an arbitrary perfect field. The following is a complete list of irreducible algebraic curves over k having group structures defined by regular maps.

ELLIPTIC CURVES

These are the only irreducible projective curves having a group structure defined by polynomial maps.

THE ADDITIVE GROUP

The affine line $\mathbb{A}^1$ is a group under addition,

$$\mathbb{A}^1(k) = k, \quad (x, y) \mapsto x + y : k \times k \to k.$$

We sometimes write $\mathbb{G}_a$ for $\mathbb{A}^1$ endowed with this group structure.

THE MULTIPLICATIVE GROUP

The affine line with the origin removed is a group under multiplication,

$$\mathbb{A}^1(k) \smallsetminus \{0\} = k^\times, \quad (x, y) \mapsto xy : k^\times \times k^\times \to k^\times.$$

We sometimes write $\mathbb{G}_m$ for $\mathbb{A}^1 \smallsetminus \{0\}$ endowed with this group structure. Note that the map $x \mapsto (x, x^{-1})$ identifies $\mathbb{G}_m$ with the affine plane curve $XY = 1$.

TWISTED MULTIPLICATIVE GROUPS

Let a be a nonsquare in $k^\times$, and let $L = k[\sqrt{a}]$. There is an algebraic group $\mathbb{G}_m[a]$ over k such that

$$\mathbb{G}_m[a](k) = \{\gamma \in L^\times \mid \mathrm{Nm}_{L/k}\, \gamma = 1\}.$$

Let $\alpha = \sqrt{a}$, so that $\{1, \alpha\}$ is a basis for L as a k-vector space. Then

$$(x + \alpha y)(x' + \alpha y') = xx' + ayy' + \alpha(xy' + x'y)$$

and

$$\mathrm{Nm}(x + \alpha y) = (x + \alpha y)(x - \alpha y) = x^2 - ay^2.$$

We define $\mathbb{G}_m[a]$ to be the affine plane curve $X^2 - aY^2 = 1$ with the group structure

$$(x, y) \cdot (x', y') = (xx' + ayy', xy' + x'y).$$

For example, when $k = \mathbb{R}$ and $a = -1$, we get the circle group $X^2 + Y^2 = 1$.

An invertible change of variables transforms $\mathbb{G}_m[a]$ into $\mathbb{G}_m[ac^2]$, $c \in k^\times$, and so, up to such a change, $\mathbb{G}_m[a]$ depends only on the field $k[\sqrt{a}]$. The equations defining $\mathbb{G}_m[a]$ still define an algebraic group when a is a square in k, say $a = \alpha^2$, but then $X^2 - aY^2 = (X + \alpha Y)(X - \alpha Y)$, and so the change of variables $X' = X + \alpha Y$, $Y' = X - \alpha Y$ transforms the group into $\mathbb{G}_m$. In particular, this shows that $\mathbb{G}_m[a]$ becomes isomorphic to $\mathbb{G}_m$ over $k[\sqrt{a}]$, and so it can be thought of as a "twist" of $\mathbb{G}_m$.

EXAMPLE 3.1 Let $k = \mathbb{F}_q$, the field with q-elements. Then $\mathbb{G}_a(k)$ has q-elements, $\mathbb{G}_m(k)$ has $q - 1$ elements, and $\mathbb{G}_m[a](k)$ has $q + 1$ elements for any nonsquare a in k. Only the last is not obvious. As $\mathbb{F}_q[\sqrt{a}]$ is the field $\mathbb{F}_{q^2}$, there is an exact sequence

$$0 \to \mathbb{G}_m[a](\mathbb{F}_q) \to \mathbb{F}_{q^2}^\times \xrightarrow{\mathrm{Nm}} \mathbb{F}_q^\times \to 0.$$

The norm map is surjective because every quadratic form in at least three variables over a finite field has a nontrivial zero (e.g., Serre 1964, Chap. I, §2), and so the order of $\mathbb{G}_m[a](\mathbb{F}_q)$ is $(q^2 - 1)/(q - 1) = q + 1$.

We make a few remarks concerning the proofs of the above statements. We have seen that if a nonsingular projective curve has genus 1, then it has a group structure, but why is the converse true? The simplest explanation when $k = \mathbb{C}$ comes from topology. The Lefschetz fixed point theorem (e.g., Greenberg 1967, 30.9) says that, if M is a compact oriented manifold, then for any continuous map $\alpha: M \to M$,

$$(\Delta \cdot \Gamma_\alpha) = \sum (-1)^i \mathrm{Trace}(\alpha | H^i(M, \mathbb{Q})).$$

Here Δ is the diagonal in $M \times M$ and Γ_α is the graph of α, so that $(\Delta \cdot \Gamma_\alpha)$ is the number of "fixed points of α counting multiplicities". Let $L(\alpha)$ be the integer on the right, and assume M has a group structure. For any nonzero a in M, the translation map $\tau_a = (x \mapsto x + a)$ is without fixed points, and so

$$L(\tau_a) = (\Delta \cdot \Gamma_\alpha) = 0.$$

But the map $a \mapsto L(\tau_a): M \to \mathbb{Z}$ is continuous, and hence constant on each connected component. On letting a tend to zero, we find that $L(\tau_0) = 0$. But τ_0 is the identity map, and so

$$L(\tau_0) = \sum (-1)^i \operatorname{Tr}(\mathrm{id} \,|H^i(M, \mathbb{Q})) = \sum (-1)^i \dim_{\mathbb{Q}} H^i(M, \mathbb{Q}).$$

Thus, if the manifold M has a group structure, then its Euler-Poincaré characteristic $\sum (-1)^i \dim_{\mathbb{Q}} H^i(M, \mathbb{Q})$ must be zero. The Euler-Poincaré characteristic of a complex compact Riemann surface of genus g is $1 - 2g + 1 = 2 - 2g$, and so $g = 1$ if the Riemann surface has a continuous group structure.

A similar argument works over any field. One proves directly that for the diagonal Δ in $C \times C$,

$$(\Delta \cdot \Gamma_{\tau_a}) = 0 \text{ if } a \neq 0,$$
$$(\Delta \cdot \Delta) = 2 - 2g,$$

and then "by continuity" that $(\Delta \cdot \Delta) = (\Delta \cdot \Gamma_{\tau_a})$.

The proof that $\mathbb{G}_a$ and $\mathbb{G}_m$ are the only affine algebraic groups of dimension one over an algebraically closed field can be found in most books on algebraic groups[1] (e.g., Borel 1991, 10.9, who notes that the first published proof appears to be in a lecture of Grothendieck). The extension to nonalgebraically closed fields is an easy exercise in Galois cohomology (see IV, §1, below).

Singular cubic curves

Let E be a singular plane projective curve over a perfect field k of characteristic $\neq 2$. As we observed on p. 19, it will have exactly one singular point S, and

[1]Embed the group G as the complement of a finite set S of points in a nonsingular projective curve C. Then C must have infinitely many automorphisms stabilizing the set S (see I 4.19), from which one can deduce that $C = \mathbb{P}^1$ and that S consists of one or two points.

S will have coordinates in k. Assume $E(k)$ contains a point $O \neq S$. Then the same definition as in the nonsingular case turns $E^{\mathrm{ns}}(k) \stackrel{\mathrm{def}}{=} E(k) \smallsetminus \{S\}$ into a group with zero O. Namely, consider the line through two nonsingular points P and Q. According to Bezout's theorem (I 1.18) and (I 1.13), it will intersect the curve in exactly one additional point PQ, which can't be singular. Define $P+Q$ to be the third point of intersection of the line through PQ and O with the cubic. We examine this in the two possible cases.

Cubic curves with a cusp

The projective plane curve

$$E : Y^2 Z = X^3$$

has a cusp at $S = (0:0:1)$ because the affine curve $Y^2 = X^3$ has a cusp at $(0,0)$ (see I 1.12). Note that S is the only point on the projective curve with Y-coordinate zero, and so $E(k) \smallsetminus \{S\}$ is equal to the set of points on the affine curve $E \cap \{Y \neq 0\}$, i.e., on the curve

$$E_1 : Z = X^3.$$

The line $Z = \alpha X + \beta$ intersects E_1 at the points $P_1 = (x_1, z_1)$, $P_2 = (x_2, z_2)$, $P_3 = (x_3, z_3)$ with x_1, x_2, x_3 roots of

$$X^3 - \alpha X - \beta.$$

Because the coefficient of X^2 in this polynomial is zero, the sum $x_1 + x_2 + x_3$ of its roots is zero. Therefore the map

$$P \mapsto x(P) : E_1(k) \to k$$

has the property that

$$P_1 + P_2 + P_3 = 0 \implies x(P_1) + x(P_2) + x(P_3) = 0.$$

Since $O = (0,0)$, the map $P \mapsto -P$ is $(x,z) \mapsto (-x,-z)$, and so $P \mapsto x(P)$ also has the property that

$$x(-P) = -P.$$

These two properties imply that $P \mapsto x(P) : E_1(k) \to k$ is a homomorphism. In fact, it is an isomorphism of algebraic groups. Therefore, the map $P \mapsto \frac{x(P)}{y(P)} : E \smallsetminus \{S\} \to \mathbb{G}_a$ is an isomorphism of algebraic groups.

Cubic curves with a node

The curve

$$Y^2 Z = X^3 + cX^2 Z, \quad c \neq 0,$$

has a node at $(0:0:1)$ because the affine curve

$$Y^2 = X^3 + cX^2, \quad c \neq 0,$$

has a node at $(0,0)$ (see I 1.6). The tangent lines at $(0,0)$ are given by the equation

$$Y^2 - cX^2 = 0.$$

If c is a square in k, this factors as

$$(Y - \sqrt{c}X)(Y + \sqrt{c}X) = 0$$

and we get two tangent lines. In this case the tangent lines are said to be ***defined over*** (or ***rational over***) over k. When endowed with its group structure, $E^{\mathrm{ns}} \overset{\mathrm{def}}{=} E \smallsetminus \{\text{ singular point}\}$ becomes isomorphic to $\mathbb{G}_m$.

If c is not a square, so the tangent lines are not rational over k, then $E^{\mathrm{ns}} \approx \mathbb{G}_m[c]$. See Cassels 1991, Chapter 9.

Criterion

We now derive a criterion for deciding which of the above cases the curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in k, \quad \Delta = 4a^3 + 27b^2 = 0$$

falls into. For simplicity, we assume $\mathrm{char}(k) \neq 2, 3$. Since the point $(0:1:0)$ is always nonsingular, we only need to study the affine curve

$$Y^2 = X^3 + aX + b.$$

We try to find a t such that equation is

$$\begin{aligned} Y^2 &= (X-t)^2(X+2t) \\ &= X^3 - 3t^2X + 2t^3. \end{aligned}$$

For this, we need to choose t so that

$$t^2 = -\frac{a}{3}, \quad t^3 = \frac{b}{2}.$$

Hence $t = \frac{b/2}{-a/3} = -\frac{3}{2}\frac{b}{a}$. Using that $\Delta = 0$, one checks that this works.

Now, we can rewrite the equation as

$$Y^2 = 3t(X-t)^2 + (X-t)^3.$$

This has a singularity at $(t,0)$, which is a cusp if $3t = 0$, a node with rational tangents if $3t$ is a square in $k^\times$, and a node with nonrational tangents if $3t$ is a nonsquare in $k^\times$. Note that

$$-2ab = -2(-3t^2)(2t^3) = (2t^2)^2(3t),$$

and so $3t$ is zero or nonzero, a square or a nonsquare, according as $-2ab$ is.

Reduction of an elliptic curve

Consider an elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in \mathbb{Q}, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

We make a change a variables $X \mapsto X/c^2$, $Y \mapsto Y/c^3$ with c chosen so that the new a, b are integers and $|\Delta|$ is minimal — the equation is then said to be ***minimal.*** The equation

$$\bar{E} : Y^2Z = X^3 + \bar{a}XZ^2 + \bar{b}Z^3$$

with $\bar{a}$ and $\bar{b}$ the images of a and b in $\mathbb{F}_p$ is called the ***reduction of*** E ***modulo*** p. There are three cases to consider (and two subcases).

(a) ***Good reduction.*** If $p \neq 2$ and p does not divide Δ, then $\bar{E}$ is an elliptic curve over $\mathbb{F}_p$. For a point $P = (x : y : z)$ on E, we can choose a representative (x, y, z) for P with $x, y, z \in \mathbb{Z}$ and having no common factor, and then $\bar{P} \stackrel{\text{def}}{=} (\bar{x} : \bar{y} : \bar{z})$ is a well-defined point on $\bar{E}$. Since $(0 : 1 : 0)$ reduces to $(0 : 1 : 0)$ and lines reduce to lines, the map $E(\mathbb{Q}) \to \bar{E}(\mathbb{F}_p)$ is a homomorphism.

(b) ***Cuspidal, or additive, reduction.*** This is the case in which the reduced curve $\bar{E}$ has a cusp, and so $\bar{E}^{\text{ns}} \approx \mathbb{G}_a$. For $p \neq 2, 3$, it occurs exactly when $p|4a^3 + 27b^2$ and $p| - 2ab$.

(c) ***Nodal, or multiplicative, reduction.*** This is the case in which the reduced curve $\bar{E}$ has a node. For $p \neq 2, 3$, it occurs exactly when $p|4a^3 + 27b^2$ and p does not divide $-2ab$. The tangents at the node are rational over $\mathbb{F}_p$ if and only if $-2ab$ becomes a square in $\mathbb{F}_p$, in which case $\bar{E}^{\text{ns}} \approx \mathbb{G}_m$ and E is said to have ***split multiplicative reduction***. On the other hand, if $-2ab$ is not a square modulo p, then $\bar{E}^{\text{ns}} \approx \mathbb{G}_m[\sqrt{-2ab}]$ and E is said to have ***nonsplit multiplicative reduction.***

The following table summarizes the above results on the reduction of elliptic curves ($p \neq 2, 3$, N is the number of nonsingular points on $\bar{E}$ with coordinates in $\mathbb{F}_p$).

Type	tangents	$\Delta \mod p$	$-2ab \mod p$	$\bar{E}^{\text{ns}}$	N
good		$\neq 0$		$\bar{E}$	IV, §9
cusp		0	0	$\mathbb{G}_a$	p
node	rational	0	$\square$	$\mathbb{G}_m$	$p - 1$
node	not rational	0	$\neq \square$	$\mathbb{G}_m[\sqrt{-2ab}]$	$p + 1$

Semistable reduction

If E has good or nodal reduction, then the minimal equation remains minimal after replacing the ground field (here $\mathbb{Q}$) by a larger field. This is not so for cuspidal reduction. Consider, for example, the curve

$$E: Y^2Z = X^3 + pXZ^2 + pZ^3.$$

After passing to an extension field in which p becomes a sixth power, say, $\pi^6 = p$, we can make a change of variables so that the equation becomes

$$E: Y^2Z = X^3 + \pi^2XZ^2 + Z^3.$$

This reduces modulo π to

$$Y^2Z = X^3 + Z^3,$$

which is nonsingular. In fact, for any curve E with cuspidal reduction at p, there will exist a finite extension of the ground field such that E will have either good or nodal reduction at the primes over p (Silverman 1986, VII 5.4).

In summary: good and nodal reduction are not changed by a field extension (in fact, the minimal equation remains minimal) but cuspidal reduction always becomes good or nodal reduction in an appropriate finite extension (and the minimal equation changes). For this reason, a curve is said to have ***semistable reduction*** at p if it has good or nodal reduction there.

Reduction modulo 2 and 3

When considering reduction at 2 or 3, one needs to consider the full equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \qquad (15)$$

because it may be possible to find an equation of this form that is "more minimal" for 2 or 3 than any of the form

$$Y^2Z = X^3 + aXZ^2 + bZ^3. \qquad (16)$$

For example,

$$Y^2 + Y = X^3 - X^2$$

defines a nonsingular curve over $\mathbb{F}_2$, whereas all equations of the form (16) define singular curves over $\mathbb{F}_2$ (see I 1.5).

Again, an equation (15) for E is said to be ***minimal*** if the $a_i \in \mathbb{Z}$ and $|\Delta|$ is minimal. Clearly, a minimal equation exists, and one shows that it is unique up to a change of variables

$$\begin{aligned} x &\mapsto u^2x + r \\ y &\mapsto u^3y + su^2x + t \end{aligned}$$

with u invertible in $\mathbb{Z}$ (so $u = \pm 1$).

EXERCISE 3.2 Show that the curve

$$E: Y^2 + Y = X^3 - X^2 - 10X - 20$$

has good reduction at all primes except 11.

Other fields

Throughout this section, we can replace $\mathbb{Q}$ and $\mathbb{Z}$ with $\mathbb{Q}_p$ and $\mathbb{Z}_p$, or, in fact, with any p-adic field (i.e., finite extension of $\mathbb{Q}_p$) and its ring of integers. Note that Hensel's lemma (I 2.12) implies that the image of $E(\mathbb{Q}_p) \to E(\mathbb{F}_p)$ includes every nonsingular point. Also, we can replace $\mathbb{Q}$ and $\mathbb{Z}$ with a number field K and its ring of integers, with the caution that, if the ring of integers in K is not a principal ideal domain, then it may not be possible to find an equation for the elliptic curve that is minimal for all primes simultaneously.

EXERCISE 3.3 (a) Find examples of elliptic curves E over $\mathbb{Q}$ such that

i) E_p has a cusp S which lifts to a point in $E(\mathbb{Q}_p)$;
ii) E_p has a node S which lifts to a point in $E(\mathbb{Q}_p)$;
iii) E_p has a node S which does not lift to a point in $E(\mathbb{Q}_p)$.

Here E_p is the reduction of the curve modulo a prime $p \neq 2,3$. The equation you give for E should be a minimal equation of the standard form $Y^2Z = X^3 + aXZ^2 + bZ^3$.

(b) For the example you gave in (a)(i), decide whether it acquires good or nodal reduction in a finite extension of $\mathbb{Q}$.

4 Elliptic curves over $\mathbb{Q}_p$

Consider a curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a,b \in \mathbb{Q}_p, \quad 4a^3 + 27b^2 \neq 0.$$

After a change of variables $X \mapsto X/c^2$, $Y \mapsto Y/c^3$, $Z \mapsto Z$, we may suppose that $a,b \in \mathbb{Z}_p$; we may also suppose that $\mathrm{ord}_p(\Delta)$ is minimal, but that is not necessary for the results of this section. As in the last section, we obtain from E a curve $\bar{E}$ over $\mathbb{F}_p$ and a reduction map

$$P \mapsto \bar{P} \colon E(\mathbb{Q}_p) \to \bar{E}(\mathbb{F}_p).$$

We shall define a filtration

$$E(\mathbb{Q}_p) \supset E^0(\mathbb{Q}_p) \supset E^1(\mathbb{Q}_p) \supset \cdots \supset E^n(\mathbb{Q}_p) \supset \cdots$$

and identify the quotients. First, define

$$E^0(\mathbb{Q}_p) = \{P \mid \bar{P} \text{ is nonsingular}\}.$$

It is a subgroup because $(0:1:0)$ is always nonsingular and a line through two nonsingular points on a cubic (or tangent to a nonsingular point) will meet the cubic again at a nonsingular point.

Write $\bar{E}^{\mathrm{ns}}$ for $\bar{E}$ with the singular point (if any) removed. The reduction map

$$P \mapsto \bar{P}\colon E^0(\mathbb{Q}_p) \to \bar{E}^{\,\mathrm{ns}}(\mathbb{F}_p)$$

is a homomorphism, and we define $E^1(\mathbb{Q}_p)$ be its kernel. Thus $E^1(\mathbb{Q}_p)$ consists of the points P that can be represented as $(x : y : z)$ with x and z divisible by p but y not divisible by p. In particular, $P \in E^1(\mathbb{Q}_p) \implies y(P) \neq 0$. Define

$$E^n(\mathbb{Q}_p) = \left\{ P \in E^1(\mathbb{Q}_p) \;\middle|\; \frac{x(P)}{y(P)} \in p^n\mathbb{Z}_p \right\}.$$

THEOREM 4.1 *The filtration*

$$E(\mathbb{Q}_p) \supset E^0(\mathbb{Q}_p) \supset E^1(\mathbb{Q}_p) \supset \cdots \supset E^n(\mathbb{Q}_p) \supset \cdots$$

has the following properties:

(a) *the quotient $E(\mathbb{Q}_p)/E^0(\mathbb{Q}_p)$ is finite;*
(b) *the map $P \mapsto \bar{P}$ defines an isomorphism $E^0(\mathbb{Q}_p)/E^1(\mathbb{Q}_p) \to \bar{E}^{\mathrm{ns}}(\mathbb{F}_p)$;*
(c) *for $n \geq 1$, $E^n(\mathbb{Q}_p)$ is a subgroup of $E(\mathbb{Q}_p)$, and the map $P \mapsto p^{-n}\frac{x(P)}{y(P)} \bmod p$ is an isomorphism of groups $E^n(\mathbb{Q}_p)/E^{n+1}(\mathbb{Q}_p) \to \mathbb{F}_p$;*
(d) *the filtration is exhaustive, i.e., $\bigcap_n E^n(\mathbb{Q}_p) = \{0\}$.*

PROOF. (a) We prove that $E(\mathbb{Q}_p)$ has a natural topology with respect to which it is compact and $E^0(\mathbb{Q}_p)$ is an open subgroup. Since $E(\mathbb{Q}_p)$ is a union of the cosets of $E^0(\mathbb{Q}_p)$, it follows that there can only be finitely many of them.

Endow $\mathbb{Q}_p \times \mathbb{Q}_p \times \mathbb{Q}_p$ with the product topology, $\mathbb{Q}_p^3 \smallsetminus \{(0,0,0)\}$ with the subspace topology, and $\mathbb{P}^2(\mathbb{Q}_p)$ with the quotient topology via

$$\mathbb{Q}_p^3 \smallsetminus \{(0,0,0)\} \to \mathbb{P}^2(\mathbb{Q}_p).$$

Then $\mathbb{P}^2(\mathbb{Q}_p)$ is the union of the images of the sets $\mathbb{Z}_p^\times \times \mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p^\times \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p^\times$, each of which is compact and open. Therefore $\mathbb{P}^2(\mathbb{Q}_p)$ is compact. Its subset $E(\mathbb{Q}_p)$ is closed, because it is the zero set of a polynomial, and so it also is compact. Relative to this topology on $\mathbb{P}^2(\mathbb{Q}_p)$, two points that are close will have the same reduction modulo p. Therefore $E^0(\mathbb{Q}_p)$ is the intersection of $E(\mathbb{Q}_p)$ with an open subset of $\mathbb{P}^2(\mathbb{Q}_p)$.

(b) Hensel's lemma (I 2.12) implies that the reduction map $E^0(\mathbb{Q}_p) \to \bar{E}^{\mathrm{ns}}(\mathbb{F}_p)$ is surjective, and we defined $E^1(\mathbb{Q}_p)$ to be its kernel.

(c) We assume inductively that $E^n(\mathbb{Q}_p)$ is a subgroup of $E(\mathbb{Q}_p)$. If $P = (x : y : 1)$ lies in $E^1(\mathbb{Q}_p)$, then $y \notin \mathbb{Z}_p$. Set $x = p^{-m}x_0$ and $y = p^{-m'}y_0$ with x_0 and y_0 units in $\mathbb{Z}_p$ and $m' \geq 1$. Then

$$p^{-2m'}y_0^2 = p^{-3m}x_0^3 + ap^{-m}x_0 + b.$$

On taking ord_p of the two sides, we find that $2m' = 3m$. Since m' and m are integers, this implies that there is an integer $n \geq 1$ such $m = 2n$ and $m' = 3n$; in fact, $n = m' - m$.

The above discussion shows that if $P = (x : y : z) \in E^n(\mathbb{Q}_p) \smallsetminus E^{n+1}(\mathbb{Q}_p)$, $n \geq 1$, then

$$\begin{cases} \mathrm{ord}_p(x) &= \mathrm{ord}_p(z) - 2n \\ \mathrm{ord}_p(y) &= \mathrm{ord}_p(z) - 3n. \end{cases}$$

Hence P can be expressed $P = (p^n x_0 : y_0 : p^{3n} z_0)$ with $\mathrm{ord}_p(y_0) = 0$ and $x_0, z_0 \in \mathbb{Z}_p$; in fact, this is true for all $P \in E^n(\mathbb{Q}_p)$. Since P lies on E,

$$p^{3n} y_0^2 z_0 = p^{3n} x_0^3 + a p^{7n} x_0 z_0^2 + b p^{9n} z_0^3,$$

and so $P_0 \stackrel{\text{def}}{=} (\bar{x}_0 : \bar{y}_0 : \bar{z}_0)$ lies on the curve

$$E_0 : Y^2 Z = X^3.$$

As $\bar{y}_0 \neq 0$, P_0 is not the singular point of E_0. From the description of the group laws in terms of chords and tangents, we see that the map

$$P \mapsto P_0 : E^n(\mathbb{Q}_p) \to E_0(\mathbb{F}_p)$$

is a homomorphism. Its kernel is $E^{n+1}(\mathbb{Q}_p)$, which is therefore a subgroup, and it follows from Hensel's lemma that its image is the set of nonsingular points of $E_0(\mathbb{F}_p)$. We know from the preceding section that $Q \mapsto \frac{x(Q)}{y(Q)}$ is an isomorphism $E_0^{\mathrm{ns}}(\mathbb{F}_p) \to \mathbb{F}_p$. The composite $P \mapsto P_0 \mapsto \frac{x(P_0)}{y(P_0)}$ is $P \mapsto \frac{p^{-n} x(P)}{y(P)} \mod p$.

(d) If $P \in \bigcap_n E^n(\mathbb{Q}_p)$, then $x(P) = 0$, $y(P) \neq 0$. This implies that either $z(P) = 0$ or $y(P)^2 = bz(P)^3$, but the second equality contradicts $P \in E^1(\mathbb{Q}_p)$. Hence $z(P) = 0$ and $P = (0 : 1 : 0)$. □

COROLLARY 4.2 *For every integer m not divisible by p, the map*

$$P \mapsto mP : E^1(\mathbb{Q}_p) \to E^1(\mathbb{Q}_p)$$

is a bijection.

PROOF. Let $P \in E^1(\mathbb{Q}_p)$ be such that $mP = 0$. If $P \neq 0$, then $P \in E^n(\mathbb{Q}_p) \smallsetminus E^{n+1}(\mathbb{Q}_p)$ for some n (by 4.1d), but $E^n(\mathbb{Q}_p)/E^{n+1}(\mathbb{Q}_p) \simeq \mathbb{Z}/p\mathbb{Z}$ (by 4.1c). The image of P in $\mathbb{Z}/p\mathbb{Z}$ is nonzero, and so m times it is also nonzero, which contradicts the fact that $mP = 0$. Therefore the map is injective.

Let $P \in E^1(\mathbb{Q}_p)$. Because $E^1(\mathbb{Q}_p)/E^2(\mathbb{Q}_p) \simeq \mathbb{Z}/p\mathbb{Z}$ and p doesn't divide m, multiplication by m is an isomorphism on $E^1(\mathbb{Q}_p)/E^2(\mathbb{Q}_p)$. Therefore there exists a $Q_1 \in E^1(\mathbb{Q}_p)$ such that

$$P = mQ_1 \mod E^2(\mathbb{Q}_p).$$

Similarly, there exists a $Q_2 \in E^2(\mathbb{Q}_p)$ such that

$$(P - mQ_1) = mQ_2 \mod E^3(\mathbb{Q}_p)$$

Continuing in this fashion, we obtain a sequence $Q_1, Q_2, \ldots$ of points in $E(\mathbb{Q}_p)$ such that

$$Q_i \in E^i(\mathbb{Q}_p), \quad P - m\sum_{i=1}^{n} Q_i \in E^{n+1}(\mathbb{Q}_p).$$

The first condition implies that the series $\sum Q_i$ converges to a point in $E(\mathbb{Q}_p)$ (recall that $E(\mathbb{Q}_p)$ is compact), and the second condition implies that its limit Q has the property that $P = mQ$. □

REMARK 4.3 It is possible to say much more about the structure of $E(\mathbb{Q}_p)$. A ***one-parameter commutative formal group*** over a (commutative) ring R is a power series $F(X,Y) = \sum_{i,j\geq 0} a_{i,j} X^i Y^j \in R[[X,Y]]$ satisfying the following conditions:

(a) $F(X,Y) = X + Y +$ terms of degree ≥ 2;
(b) $F(X, F(Y,Z)) = F(F(X,Y),Z)$;
(c) $F(X,Y) = F(Y,X)$.

These conditions imply that $F(X,0) = X$ and $F(0,Y) = Y$ and that there exists a unique power series $i(T) = -T + \sum_{n\geq 2} a_n T^n \in R[[T]]$ such that $F(T, i(T)) = 0$.

If F is such a formal group over $\mathbb{Z}_p$, then the series $F(a,b)$ converges for $a, b \in p\mathbb{Z}_p$, and so F makes $p\mathbb{Z}_p$ into a group. One can show (Silverman 1986, Chap. IV) that an elliptic curve E over $\mathbb{Q}_p$ defines a formal group F over $\mathbb{Z}_p$, and that there are power series $x(T)$ and $y(T)$ such that $t \mapsto (x(t) : y(t) : 1)$ is an isomorphism of $p\mathbb{Z}_p$ (endowed with the group structure provided by F) onto $E^1(\mathbb{Q}_p)$. This is useful because it allows us to derive results about elliptic curves from results about formal groups, which are simpler.

5 Torsion points

Throughout this section, E will be the elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in \mathbb{Z}, \quad \Delta = 4a^3 + 27b^2 \neq 0,$$

except that in second half of the section, we allow $a, b \in \mathbb{Z}_p$. Let $E(\mathbb{Q})_{\mathrm{tors}}$ be the torsion subgroup of $E(\mathbb{Q})$, i.e., the subgroup of $E(\mathbb{Q})$ of points of finite order.

THEOREM 5.1 (LUTZ-NAGELL) *If $P = (x : y : 1) \in E(\mathbb{Q})_{\mathrm{tors}}$, then $x, y \in \mathbb{Z}$ and either $y = 0$ or $y|\Delta$.*

REMARK 5.2 (a) The theorem provides an algorithm for finding all the torsion points on E: for each $y = 0$ or $y|\Delta$, find the integers x that are roots of $X^3 + aX + b - y^2$ — they will divide $b - y^2$ — and then check to see whether $(x : y : 1)$ is a torsion point. This will be faster if the equation of E has been chosen to be minimal.

(b) The converse of the theorem is not true: a point $P = (x : y : 1) \in E(\mathbb{Q})$ can satisfy the conditions in the theorem without being a torsion point.

(c) The theorem can often be used to prove that a point $P \in E(\mathbb{Q})$ is of infinite order: compute multiples nP of P until you arrive at one whose coordinates are not integers, or better, just compute the x-coordinates of $2P$, $4P$, $8P$, using the duplication formula p. 52.

The theorem will follow from the next two results: the first says that if P and $2P$ have integer coordinates (when we set $z = 1$), then either $y = 0$ or $y|\Delta$; the second implies that torsion points all have integer coordinates.

PROPOSITION 5.3 *Let $P = (x_1 : y_1 : 1) \in E(\mathbb{Q})$. If P and $2P$ have integer coordinates (when we set $z = 1$), then either $y_1 = 0$ or $y_1|\Delta$.*

PROOF. Assume $y_1 \neq 0$, and set $2P = (x_2 : y_2 : 1)$. Then $2P$ is the second point of intersection of the tangent line at P with the affine curve $Y^2 = X^3 + aX + b$. Let $f(X) = X^3 + aX + b$. The tangent line at P is $Y = \alpha X + \beta$, where $\alpha = \left(\frac{dY}{dX}\right)_P = \frac{f'(x_1)}{2y_1}$, and so the X-coordinates of its points of intersection with the curve satisfy

$$0 = (\alpha X + \beta)^2 - (X^3 + aX + b) = -X^3 + \alpha^2 X^2 + \cdots .$$

But we know that these X-coordinates are x_1, x_1, x_2, and so

$$x_1 + x_1 + x_2 = \alpha^2.$$

By assumption, x_1, x_2 are integers, and so α^2 and $\alpha = f'(x_1)/2y_1$ are integers. Thus $y_1|f'(x_1)$, and directly from the equation $y_1^2 = f(x_1)$ we see that $y_1|f(x_1)$. Hence y_1 divides both $f(x_1)$ and $f'(x_1)$. The theory of resultants (I, §1) shows that[2]

$$\Delta = r(X)f(X) + s(X)f'(X), \quad \text{some } r(X), s(X) \in \mathbb{Z}[X],$$

and so this implies that $y_1|\Delta$. □

PROPOSITION 5.4 *The group $E^1(\mathbb{Q}_p)$ is torsion-free.*

[2] In fact,

$$\Delta = \left(-27(X^3 + aX - b)\right) \cdot f(X) + (3X^2 + 4a)(3X^2 + a) \cdot f'(X).$$

Before proving the proposition, we derive some consequences.

COROLLARY 5.5 *If* $P = (x : y : 1) \in E(\mathbb{Q}_p)_{\mathrm{tors}}$, *then* $x, y \in \mathbb{Z}_p$.

PROOF. Recall that $\bar{P}$ is obtained from P by choosing primitive coordinates $(x : y : z)$ for P (i.e., coordinates such that $x, y, z \in \mathbb{Z}_p$ but not all of $x, y, z \in p\mathbb{Z}_p$), and setting $\bar{P} = (\bar{x} : \bar{y} : \bar{z})$, and that $E^1(\mathbb{Q}_p) = \{P \in E(\mathbb{Q}_p) \mid \bar{P} = (0 : 1 : 0)\}$. If $P = (x : y : 1)$ with x or y not in $\mathbb{Z}_p$, then any primitive coordinates $(x' : y' : z')$ for P will have $z' \in p\mathbb{Z}_p$. Hence $z(\bar{P}) = 0$, which implies $\bar{P} = (0 : 1 : 0)$, and so $P \in E^1(\mathbb{Q}_p)$. Thus (contrapositively) if $P = (x : y : 1) \notin E^1(\mathbb{Q}_p)$, then $x, y \in \mathbb{Z}_p$.

The proposition shows that if P is a nonzero torsion point, then P does not lie in $E^1(\mathbb{Q}_p)$. □

COROLLARY 5.6 *If* $P = (x : y : 1) \in E(\mathbb{Q})_{\mathrm{tors}}$, *then* $x, y \in \mathbb{Z}$.

PROOF. This follows from the previous corollary, because if a rational number r is not an integer, then $\mathrm{ord}_p(r) < 0$ for some p, and so $r \notin \mathbb{Z}_p$. □

COROLLARY 5.7 *If E has good reduction at p (i.e., $p \neq 2$ and p does not divide Δ), then the reduction map*

$$E(\mathbb{Q})_{\mathrm{tors}} \to \bar{E}(\mathbb{F}_p)$$

is injective.

PROOF. Because E has good reduction, $E^0(\mathbb{Q}_p) = E(\mathbb{Q}_p)$. The reduction map $E(\mathbb{Q}_p) \to \bar{E}(\mathbb{Q}_p)$ has kernel $E^1(\mathbb{Q}_p)$, which intersects $E(\mathbb{Q})_{\mathrm{tors}}$ in $\{O\}$. □

REMARK 5.8 This puts a very serious restriction on the size of $E(\mathbb{Q})_{\mathrm{tors}}$. For example, if E has good reduction at 5, then, according to the congruence Riemann hypothesis (see IV, §9), $\bar{E}$ will have at most $5 + 1 + 2\sqrt{5}$ points with coordinates in $\mathbb{F}_5$, and so E will have at most 10 torsion points with coordinates in $\mathbb{Q}$.

We now begin the proof of Proposition 5.4. After Proposition 4.2, it remains to show that $E^1(\mathbb{Q})$ contains no point $P \neq 0$ such that $pP = 0$. For this, we have to analyse the filtration more carefully.

For $P \in E^1(\mathbb{Q}_p)$, we have $y(P) \neq 0$, which suggests that we look at the affine curve $E \cap \{(x : y : z\} \mid y \neq 0\}$:

$$E_1 : Z = X^3 + aXZ^2 + bZ^3. \tag{17}$$

A point $P = (x : y : z)$ on E has coordinates $x'(P) \stackrel{\mathrm{def}}{=} \frac{x(P)}{y(P)}$, $z'(P) \stackrel{\mathrm{def}}{=} \frac{z(P)}{y(P)}$ on E_1. For example, $O = (0 : 1 : 0)$ becomes the origin on E_1, and so

$P \mapsto -P$ becomes reflection in the origin $(x', z') \mapsto (-x', -z')$. As before, $P + Q + R = 0$ if and only if P, Q, R lie on a line.

In terms of our new picture,

$$E^n(\mathbb{Q}_p) = \{P \in E^1(\mathbb{Q}_p) \mid x'(P) \in p^n\mathbb{Z}_p\}.$$

Thus the $E^n(\mathbb{Q}_p)$'s form a fundamental system of neighbourhoods of the origin in $E_1(\mathbb{Q}_p)$. The key lemma is following:

LEMMA 5.9 *Let $P_1, P_2, P_3 \in E(\mathbb{Q}_p)$ be such that $P_1 + P_2 + P_3 = O$. If $P_1, P_2 \in E^n(\mathbb{Q}_p)$, then $P_3 \in E^n(\mathbb{Q}_p)$, and*

$$x'(P_1) + x'(P_2) + x'(P_3) \in p^{5n}\mathbb{Z}_p.$$

PROOF. We saw in §3 that if $P = (x : y : 1) \in E^n(\mathbb{Q}_p) \smallsetminus E^{n+1}(\mathbb{Q}_p)$, then $\mathrm{ord}_p(x) = -2n$, $\mathrm{ord}_p(y) = -3n$. In terms of homogeneous coordinates $P = (x : y : z)$, this means that

$$P \in E^n(\mathbb{Q}_p) \smallsetminus E^{n+1}(\mathbb{Q}_p) \Rightarrow \begin{cases} \mathrm{ord}_p \frac{x(P)}{z(P)} &= -2n \\ \mathrm{ord}_p \frac{y(P)}{z(P)} &= -3n \end{cases}$$
$$\Rightarrow \begin{cases} \mathrm{ord}_p \frac{x(P)}{y(P)} &= n \\ \mathrm{ord}_p \frac{z(P)}{y(P)} &= 3n. \end{cases}$$

Thus

$$P \in E^n(\mathbb{Q}_p) \implies x'(P) \in p^n\mathbb{Z}_p, \quad z'(P) \in p^{3n}\mathbb{Z}_p.$$

Let $x_i' = x'(P_i)$ and $z_i' = z'(P_i)$ for $i = 1, 2, 3$. The line through P_1, P_2 (assumed distinct) is $Z = \alpha X + \beta$ where

$$\begin{aligned} \alpha &= \frac{z_2' - z_1'}{x_2' - x_1'} \\ &\overset{(17)}{=} \frac{(x_2'^3 - x_1'^3) + a(x_2' - x_1')(z_2'^2 - z_1'^2) + b(z_2'^3 - z_1'^3)}{x_2' - x_1'} \\ &= \cdots \\ &= \frac{x_2'^2 + x_1'x_2' + x_1'^2 + az_2'^2}{1 - ax_1'(z_2' + z_1') - b(z_2'^2 + z_1'z_2 + z_1'^2)}. \end{aligned}$$

The bottom line is a unit in $\mathbb{Z}_p$, and so $\alpha \in p^{2n}\mathbb{Z}_p$. Moreover

$$\beta = z_1' - \alpha x_1' \in p^{3n}\mathbb{Z}_p.$$

On substituting $\alpha X + \beta$ for Z in the equation for E_1, we obtain the equation

$$\alpha X + \beta = X^3 + aX(\alpha X + \beta)^2 + b(\alpha X + \beta)^3.$$

We know that the solutions of this equation are x_1', x_2', x_3', and so

$$x_1' + x_2' + x_3' = \frac{2a\alpha\beta + 3b\alpha^2\beta}{1 + a\alpha^2 + b\alpha^3} \in p^{5n}\mathbb{Z}_p.$$

The proof when $P_1 = P_2$ is similar. □

We now complete the proof of Proposition 5.4. For $P \in E^n(\mathbb{Q}_p)$, let $\bar{x}(P) = x'(P) \mod p^{5n}\mathbb{Z}_p$. The lemma shows that the map

$$P \mapsto \bar{x}(P) : E^n(\mathbb{Q}_p) \to p^n\mathbb{Z}_p/p^{5n}\mathbb{Z}_p$$

has the property

$$P_1 + P_2 + P_3 = 0 \implies \bar{x}(P_1) + \bar{x}(P_2) + \bar{x}(P_3) = 0.$$

As $\bar{x}(-P) = -\bar{x}(P)$, $P \mapsto \bar{x}(P)$ is a homomorphism of abelian groups. Suppose that $P \in E^1(\mathbb{Q}_p)$ has order p. As P is nonzero, it lies in $E^n(\mathbb{Q}_p) \smallsetminus E^{n+1}(\mathbb{Q}_p)$ for some n. Then $\bar{x}(P) \in p^n\mathbb{Z}_p \smallsetminus p^{n+1}\mathbb{Z}_p \mod p^{5n}\mathbb{Z}_p$, and so

$$\bar{x}(pP) = p\bar{x}(P) \in p^{n+1}\mathbb{Z}_p \smallsetminus p^{n+2}\mathbb{Z}_p \mod p^{5n}\mathbb{Z}_p.$$

This contradicts the fact that $pP = 0$.

REMARK 5.10 When $\mathbb{Q}$ is replaced by a number field K, the above argument may fail to show that torsion elements of $E(K)$ have coordinates that are algebraic integers (when z is taken to be 1). Let π be a prime element in K_v. The same argument as above shows that there is an isomorphism

$$E^n(K_v)/E^{5n}(K_v) \to \pi^n\mathcal{O}_v/\pi^{5n}\mathcal{O}_v.$$

However, if p is a high power of π (i.e., the extension $K/\mathbb{Q}$ is highly ramified v) and n is small, this no longer excludes the possibility that $E^n(K_v)$ may contain an element of order p.

REMARK 5.11 It was conjectured by Beppo Levi[3] at the International Congress of Mathematicians in 1906, and proved by Mazur in 1975 (Mazur 1977, III 5.1), that $E(\mathbb{Q})_{\mathrm{tors}}$ is isomorphic to one of the following groups:

$$\begin{array}{lll} \mathbb{Z}/m\mathbb{Z} & \text{for} & m = 1, 2, \ldots, 10, 12; \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} & \text{for} & m = 2, 4, 6, 8. \end{array}$$

This can be interpreted as a statement about the curves considered in Chapter V (see p189). The 15 curves in the exercise below exhibit all possible torsion subgroups (in order). By contrast, $E(\mathbb{Q}^{\mathrm{al}})_{\mathrm{tors}} \approx \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$. The fact that $E(\mathbb{Q})_{\mathrm{tors}}$ is so much smaller than $E(\mathbb{Q}^{\mathrm{al}})_{\mathrm{tors}}$ indicates that the image of the Galois group in the automorphism group of $E(\mathbb{Q}^{\mathrm{al}})_{\mathrm{tors}}$ is large.

[3]The conjecture was forgotten, and then re-conjectured by Ogg in 1975 (Schappacher and Schoof 1996).

EXERCISE 5.12 For four of the following elliptic curves (including at least one of the last four), compute the torsion subgroups of $E(\mathbb{Q})$.

$$\begin{aligned}
Y^2 &= X^3+2\\
Y^2 &= X^3+X\\
Y^2 &= X^3+4\\
Y^2 &= X^3+4X\\
Y^2+Y &= X^3-X^2\\
Y^2 &= X^3+1\\
Y^2-XY+2Y &= X^3+2X^2\\
Y^2+7XY-6Y &= X^3-6X^2\\
Y^2+3XY+6Y &= X^3+6X^2\\
Y^2-7XY-36Y &= X^3-18X^2\\
Y^2+43XY-210Y &= X^3-210X^2\\
Y^2 &= X^3-X\\
Y^2 &= X^3+5X^2+4X\\
Y^2+5XY-6Y &= X^3-3X^2\\
Y^2 &= X^3+337X^2+20736X
\end{aligned}$$

6 Néron models

Recall that an elliptic curve E over $\mathbb{Q}_p$ has a Weierstrass equation

$$Y^2Z+a_1XYZ+a_3YZ^2 = X^3+a_2X^2Z+a_4XZ^2+a_6Z^3, \quad a_i \in \mathbb{Q}_p,$$

which is uniquely determined up to a change of variables of the form

$$\begin{aligned}
X &= u^2X'+r\\
Y &= u^3Y'+su^2X'+t
\end{aligned}$$

with $u,r,s,t \in \mathbb{Q}_p$ and $u \neq 0$. Under such a change, the discriminant Δ transforms according to $\Delta = u^{12}\Delta'$. We make a change of variables so that the $a_i \in \mathbb{Z}_p$ and $\mathrm{ord}_p(\Delta)$ is as small as possible. The new "minimal" equation is uniquely determined up to a change of variables of the above form with $u,r,s,t \in \mathbb{Z}_p$ and $u \in \mathbb{Z}_p^\times$. We can think of this minimal Weierstrass equation as defining a curve over $\mathbb{Z}_p$, which will be the best "model" of E over $\mathbb{Z}_p$ among plane projective curves. We call it the ***Weierstrass minimal model*** of E.

Néron showed that if we allow our models to be curves over $\mathbb{Z}_p$ that are not necessarily embeddable in $\mathbb{P}^2$, then we may obtain a model that is better in some respects than any plane model. I'll attempt to explain what these Néron models are in this section. Unfortunately, this is a difficult topic, which requires the theory of schemes for a satisfactory explanation[4] and so I'll have to be very superficial. For a detailed account, see Silverman 1994, Chap. IV.

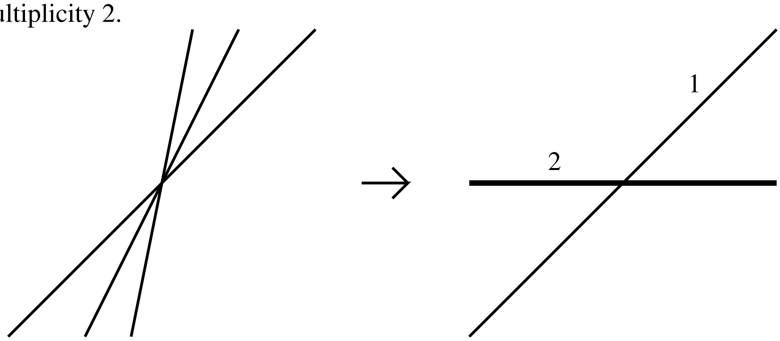
In order to be able to state Néron's results, and the earlier results of Kodaira, we need to expand our notion of a curve to allow "multiple components". For an affine plane curve, this means simply that we allow curves to be defined by polynomials f with repeated factors. For example, the equation

$$(Y-X)(Y-pX)(Y-p^2X)=0$$

defines a curve in the sense of Chapter I. It consists of three components, namely, three lines through the origin. Modulo p, the equation becomes

$$(Y-X)Y^2=0,$$

which is the union of two lines $Y-X=0$ and $Y=0$, the second of which has multiplicity 2.



For more general curves, the idea is the same.

The work of Kodaira

Before considering Néron models, we look at an analogous situation, which was its precursor. Consider an equation

$$Y^2Z=X^3+a(T)XZ^2+b(T)Z^3,\quad \Delta(T)\stackrel{\text{def}}{=}4a(T)^3+27b(T)^2\neq 0$$

with $a(T),b(T)\in\mathbb{C}[T]$. We can view this in three different ways:

(a) as defining an elliptic curve E over the field $\mathbb{C}(T)$;

[4]Néron himself didn't use schemes. For a long period, the only rigorous foundations for algebraic geometry were provided by Weil 1962, which didn't allow mixed characteristic. Consequently, those working in mixed characteristic were forced to devise their own extension of Weil's foundations, which makes their work difficult to understand by the modern reader. This, of course, all changed in the early 1960s with Grothendieck's schemes, but some authors, Néron included, continued with the old way.

(b) as defining a surface S in $\mathbb{P}^2(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C})$ whose points are the pairs $((x\colon y\colon z), t)$ satisfying the equation;
(c) as defining a family of (possibly degenerate) elliptic curves $E(T)$ parametrized by T.

By (c) we mean the following: for each $t_0 \in \mathbb{C}$ we have a curve

$$E(t_0) : Y^2Z = X^3 + a(t_0)XZ^2 + b(t_0)Z^3, \quad a(t_0), b(t_0) \in \mathbb{C},$$

with discriminant $\Delta(t_0)$. This is nonsingular, and hence an elliptic curve, if and only if t_0 is not a root of the polynomial $\Delta(T)$; otherwise, it will have a singularity, and we view it as a degenerate elliptic curve. Note that the projection map $\mathbb{P}^2(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C}) \to \mathbb{A}^1(\mathbb{C})$ induces a map $S \to \mathbb{A}^1(\mathbb{C})$ whose fibres are the curves $E(t)$. We can view S as a "model" of E over $\mathbb{C}[T]$ (or over $\mathbb{A}^1(\mathbb{C})$). We should choose the equation of E so that $\Delta(T)$ has the smallest possible degree so that there are as few singular fibres as possible.

For convenience, we now drop the Z, and consider the curve

$$S : Y^2 = X^3 + a(T)X + b(T), \quad a(T), b(T) \in \mathbb{C}[T],$$

— strictly, we should work with the family of projective curves.

Let $P = (x, y, t) \in S(\mathbb{C})$, and let $f(X, Y, T) = X^3 + a(T)X + b(T) - Y^2$. Then P is singular on the curve $E(t)$ if and only if it satisfies the following equations,

$$\frac{\partial f}{\partial Y} = -2Y = 0$$
$$\frac{\partial f}{\partial X} = 3X^2 + a(T) = 0.$$

It is singular on the surface S if in addition it satisfies the equation,

$$\frac{\partial f}{\partial T} = \frac{da}{dT}X + \frac{db}{dT} = 0.$$

Thus, P might be singular in its fibre $E(t)$ without being singular on S.

EXAMPLE 6.1 (a) Consider the equation

$$Y^2 = X^3 - T, \quad \Delta(T) = 27T^2.$$

The origin is singular (in fact, it is a cusp) when regarded as a point on the curve $E(0) : Y^2 = X^3$, but not when regarded as a point on the surface $S : Y^2 = X^3 - T$. In fact, the tangent plane to S at the origin is the (X, Y)-plane, $T = 0$.

(b) Consider the equation

$$Y^2 = X^3 - T^2, \quad \Delta(T) = 27T^4.$$

In this case, the origin is singular when regarded as a point on $E(0)$ *and* when regarded as a point on S.

(c) Consider the equation

$$\begin{aligned} Y^2 &= (X-1+T)(X-1-T)(X+2) \\ &= X^3-(3+T^2)X+2-2T^2. \end{aligned}$$

The discriminant is

$$\Delta(T) = -324T^2 + 72T^4 - 4T^6.$$

The curve $E(0)$ is

$$Y^2 = X^3 - 3X + 2 = (X-1)^2(X+2),$$

which has a node at $(1,0)$. When we replace $X-1$ in the original equation with X in order to translate $(1,0,0)$ to the origin, the equation becomes

$$\begin{aligned} Y^2 &= (X+T)(X-T)(X+3) \\ &= (X^2-T^2)(X+3) \\ &= X^3+3X^2-T^2X-3T^2. \end{aligned}$$

This surface has a singularity at the origin because its equation has no linear term.

Kodaira (1960) showed that, by blowing up points, and blowing down curves, etc., it is possible to obtain from the surface

$$S : Y^2Z = X^3 + a(T)XZ^2 + b(T)Z^3, \quad a(T), b(T) \in \mathbb{C}[T], \quad \Delta[T] \neq 0$$

a new surface S' endowed with a regular map $S' \to \mathbb{A}^1$ having the following properties:

(a) S' is nonsingular;
(b) S' regarded as a curve over $\mathbb{C}(T)$ is equal to S regarded as a curve over $\mathbb{C}(T)$ (for the experts, the maps $S \to \mathbb{A}^1$ and $S' \to \mathbb{A}^1$ have the same generic fibres);
(c) the fibres $E'(t_0)$ of S' over $\mathbb{A}^1(\mathbb{C})$ are projective curves; moreover $E'(t_0) = E(t_0)$ if the points of $E(t_0)$ are nonsingular when regarded as points on S (for example, if $E(t_0)$ itself is nonsingular);
(d) S' is minimal with the above properties: if $S'' \to \mathbb{A}^1$ is a second map with properties (a,b,c), then any regular map $S' \to S''$ over $\mathbb{A}^1$ giving an isomorphism on the generic fibres is an isomorphism.

The map $S' \to \mathbb{A}^1$ is uniquely determined by these properties up to a unique isomorphism. Kodaira classified the possible fibres of $S' \to \mathbb{A}^1$.

"Blowing up" a nonsingular point P in a surface S leaves the surface unchanged except that it replaces the point P with the projective space of lines through the origin in the tangent plane to S at P. For a curve C in S through P, the inverse image of $C \smallsetminus P$ is a curve in the blown up variety whose closure meets the projective space at the point corresponding to the tangent line of C at P. Even when $S \subset \mathbb{P}^m$, the blown-up surface doesn't have a natural embedding into a projective space.

EXAMPLE 6.2 To illustrate the phenomenon of "blowing up", consider the map

$$\sigma: k^2 \to k^2, \quad (x, y) \mapsto (x, xy).$$

Its image omits only the points on the Y-axis where $Y \neq 0$. A point in the image is the image of a unique point in k^2 except for $(0,0)$, which has been "blown up" to the whole of the Y-axis. In other words, the map is one-to-one, except that the Y-axis has been "blown down" to a point. The line $L : Y = \alpha X$ has inverse image $XY = \alpha X$, which is the union of the Y-axis and the line $Y = \alpha$; the closure of the inverse image of $L \smallsetminus \{(0,0)\}$ is the line $Y = \alpha$. The singular curve

$$C : Y^2 = X^3 + \alpha X^2$$

has as inverse image the curve

$$Y^2X^2 = X^3 + aX^2,$$

which is the union of the curve $X^2 = 0$ (the Y-axis with "multiplicity" 2) and the nonsingular curve $Y^2 = X + \alpha$. Note that the latter meets the Y-axis at the points $(0, \pm\sqrt{\alpha})$, i.e., at the points corresponding to the slopes of the tangents of C at $(0,0)$.

In this example, $(0,0)$ in $\mathbb{A}^2(k)$ is blown up to an affine line. In a true blowing-up, it would be replaced by a projective line, and the description of the map would be more complicated. (See Fulton 1969, Chap. VII, for blow-ups of points in $\mathbb{P}^2$).

The work of Néron

Néron proved an analogue of Kodaira's result for elliptic curves over $\mathbb{Q}_p$.[5] To explain his result, we need to talk about schemes. For the nonexperts, a ***scheme*** $\mathcal{E}$ over $\mathbb{Z}_p$ is simply the object defined by a collection of polynomial equations with coefficients in $\mathbb{Z}_p$. The object defined by the same equations regarded as having coefficients in $\mathbb{Q}_p$ is a variety E over $\mathbb{Q}_p$ called the ***generic fibre*** of

[5]More accurately, he showed that Kodaira's results for curves over the discrete valuation ring $\mathbb{C}[[T]]$ applied to curves over any complete valuation ring with perfect residue field.

$\mathcal{E}/\mathbb{Z}_p$, and the object defined by the equations with the coefficients reduced modulo p is a variety $\bar{E}$ over $\mathbb{F}_p$ called the ***special fibre*** of $\mathcal{E}/\mathbb{Z}_p$. For example, if $\mathcal{E}$ is the scheme defined by the equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3, \quad a_i \in \mathbb{Z}_p,$$

then E is the elliptic curve over $\mathbb{Q}_p$ defined by the same equation, and $\bar{E}$ is the elliptic curve over $\mathbb{F}_p$ by the equation

$$Y^2Z + \bar{a}_1XYZ + \bar{a}_3YZ^2 = X^3 + \bar{a}_2X^2Z + \bar{a}_4XZ^2 + \bar{a}_6Z^3$$

where $\bar{a}_i$ is the image of a_i in $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

Given an elliptic curve $E/\mathbb{Q}_p$, Néron constructs a scheme $\mathcal{E}$ over $\mathbb{Z}_p$ having the following properties:

(a) $\mathcal{E}$ is regular; this means that all the local rings associated with $\mathcal{E}$ are regular local rings (i.e., their maximal ideals can be generated by 2 elements; for a surface over an algebraically closed field, "regular" is equivalent to "nonsingular");
(b) the generic fibre of $\mathcal{E}$ is the original curve E;
(c) $\mathcal{E}$ is proper over $\mathbb{Z}_p$; this simply means that $\bar{E}$ is a projective curve;
(d) $\mathcal{E}$ is minimal with the above properties: if $\mathcal{E}'$ is a second scheme over $\mathbb{Z}_p$ having the properties (a,b,c), then any regular map $\mathcal{E} \to \mathcal{E}'$ over $\mathbb{Z}_p$ giving an isomorphism on the generic fibres is an isomorphism.

Moreover, Néron classified the possible special fibres over $\mathbb{F}_p^{\mathrm{al}}$ and obtained essentially the same list as Kodaira.

The curve $\mathcal{E}$ over $\mathbb{Z}_p$ is called the ***complete Néron (minimal) model***. It has some defects: we need not have $\mathcal{E}(\mathbb{Z}_p) \simeq E(\mathbb{Q}_p)$; it doesn't have a group structure; its special fibre $\bar{E}$ may be singular. All three defects are eliminated by simply removing all singular points and multiple curves in the special fibre. One then obtains the ***smooth Néron (minimal) model***, which however has the defect that it not complete.

Given an elliptic curve E over $\mathbb{Q}_p$ we now have three models over $\mathbb{Z}_p$:

(a) $\mathcal{E}^w$, the Weierstrass minimal model of E;
(b) $\mathcal{E}$, the complete Néron model of E;
(c) $\mathcal{E}'$, the smooth Néron model of E.

They are related as follows: to get $\mathcal{E}'$ from $\mathcal{E}$, remove all multiple curves and singular points; to get the Weierstrass model with the singular point in the closed fibre removed from $\mathcal{E}'$, remove all connected components of the special fibre except that containing O.

ASIDE 6.3 For an abelian variety of dimension > 1, only the smooth Néron model exists.

The different types

We describe three of the possible ten (or eleven, depending how one counts) different types of models using both Kodaira's numbering ($I_0, I_n, II, \ldots$) and Néron's numbering ($a, b_n, c1, \ldots$). We describe the special fibre over $\mathbb{F}_p^{\mathrm{al}}$ rather than $\mathbb{F}_p$. For example, in the case of nodal reduction (type (I_n, b_n)), the identity component[6] of special fibre of the smooth Néron model will be a twisted $\mathbb{G}_m$ over $\mathbb{F}_p$ unless the tangents are rational, and some of the components of the special fibre might not be rational over $\mathbb{F}_p$.

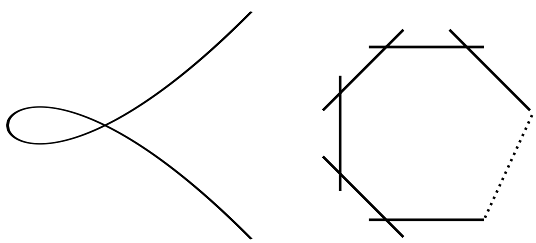
(I_0, a) In this case E has good reduction and all three models are the same.

(I_n, b_n), $n > 1$ In this case E has nodal reduction; let $n = \mathrm{ord}_p(\Delta)$; the special fibres for the three models are:

(a) a cubic curve with a node;

(b) n curves, each of genus 0, each intersecting exactly two other of the curves;

(c) an algebraic group G such that the identity component G° of G is $\mathbb{G}_m$, and such that G/G^0 is a cyclic group of order n.



Same, but with the points of intersection removed.

(a) (b) (c)

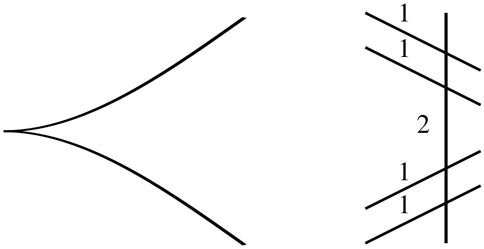
(I_0^*, c_4) In this case E has cuspidal reduction and $\mathrm{ord}_p(\Delta) = 6$ ($p \neq 2$); the special fibres for the three models are:

(a) a cubic curve with a cusp;

(b) four disjoint curves of genus 0 and multiplicity 1, together with one curve of genus 0 and multiplicity 2 crossing each of the four curves transversally;

(c) an algebraic group G whose identity component is $\mathbb{G}_a$ and such that G/G° is a group of order 4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

[6]The identity component of an algebraic group is the (unique) irreducible component containing the zero (identity element) of the algebraic group.



(a) (b)

Same, but with the points of intersection and the multiple curve removed.
(c)

When the minimal equation is used, the mysterious quotient $E(\mathbb{Q}_p)/E^0(\mathbb{Q}_p)$ of the last section is equal to $G(\mathbb{F}_p)/G^\circ(\mathbb{F}_p)$ where G is the special fibre of the smooth Néron model and G° is the identity component of G. In the above three examples, it is (a) the trivial group; (b) a group of a cyclic group of order n (and equal to a cyclic group of order n if E has split nodal reduction); (c) a subgroup of $(\mathbb{Z}/2\mathbb{Z})^2$.

Summary of the minimal models

	Weierstrass	complete Néron	smooth Néron
Plane curve	Yes	Not always	Not always
Regular?	Not always	Yes	Yes
$\bar{E}$ projective?	Yes	Yes	Not always
$\bar{E}$ nonsingular?	Not always	Not always	Yes
$\bar{E}$ a group?	Not always	Not always	Yes

7 Algorithms for elliptic curves

Tate (1975) gives an algorithm for computing the minimum Weierstrass equation, discriminant, conductor, j-invariant, the fibres of its Néron model, etc. of an elliptic curve over $\mathbb{Q}$. These are painful for humans to use, but easy for computers. Fortunately, they have been implemented in computer programs, for example, in the program Pari,[7] which is specifically designed for calculations in algebraic number theory (including elliptic curves). In the following, I explain how to use Pari as a supercalculator. You can also program it, but for that you will have to read the manual. Some terms introduced below will only be defined later.

[7] Apparently, initially the authors planned to write Pari in Pascal; hence the name **Pa**scal A**ri**thmétique.

Recall that the general Weierstrass equation of an elliptic curve E over a field k is

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

One attaches to the curve the following quantities:

$$\begin{aligned} b_2 &= a_1^2 + 4a_2 & c_4 &= b_2^2 - 24b_4 \\ b_4 &= a_1a_3 + 2a_4 & c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 \\ b_6 &= a_3^2 + 4a_6 & \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\ b_8 &= b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 & j &= c_4^3/\Delta. \end{aligned}$$

The curve is nonsingular if and only if $\Delta \neq 0$. The differential $\omega = \frac{dx}{2y+a_1x+a_3}$ is invariant under translation. A Weierstrass equation for an elliptic curve E is unique up to a coordinate transformation of the form

$$x = u^2x' + r \qquad y = u^3y' + su^2x' + t \;\;, \quad u, r, s, t \in k, \quad u \neq 0.$$

The quantities Δ, j, ω transform according to the rules:

$$u^{12}\Delta' = \Delta, \quad j' = j, \quad \omega' = u\omega.$$

Two curves become isomorphic over the algebraic closure of k if and only if they have the same j-invariant. When k has characteristic $\neq 2, 3$, the terms involving a_1, a_3, a_2 can be eliminated from the Weierstrass equation, and the above equations become those of (2.1).

A ***minimum*** Weierstrass equation for an elliptic curve E over $\mathbb{Q}$ is an equation of the above form with the $a_i \in \mathbb{Z}$ and Δ minimal. It is unique up to a coordinate transformation of the above form with $r, s, t, u \in \mathbb{Z}$ and $u \in \mathbb{Z}^\times = \{\pm 1\}$.

To start Pari, type (or click on, or ...)
gp (**g**rande calculatrice **p**ari).
An elliptic curve is specified by giving a vector
`v=[a1,a2,a3,a4,a6]`.

`e=ellinit(v,1)` Defines `e` to be the elliptic curve $Y^2 + a_1XYZ + \cdots$ and computes the 13-component vector

$$[a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6, b_8, c_4, c_6, \Delta, j]$$

`e=ellinit(v)` Defines `e` to be the elliptic curve $Y^2+a_1XYZ+\cdots$ and computes the above 13-component vector plus some other information useful for other computations.

`elladd(e,z1,z2)` Computes the sum of the points `z1=[x1,y2]` and `z2=[x2,y2]` on `e`.

`elltors(e)` Computes `[t,v1,v2]` where `t` is the order of $E(\mathbb{Q})_{\mathrm{tors}}$, `v1` gives the structure of $E(\mathbb{Q})_{\mathrm{tors}}$ as a product of cyclic groups, and `v2` gives the generators of the cyclic groups.

`ellglobalred(e)` Computes the vector `[N,v,c]` where `N` is the conductor of the curve and `v=[u,r,s,t]` is the coordinate transformation giving the Weierstrass minimum model with $a_1 = 0$ or 1, $a_2 = 0, 1, -1$, and $a_3 = 0, 1$. Such a model is unique.

`e'=ellchangecurve(e,v)` Changes `e` to `e'`, where `e'` is the 13+-component vector corresponding to the curve obtained by the change of coordinates `v=[u,r,s,t]`.

`ellocalred(e',p)` Computes the type of the reduction at p using Kodaira's notation. It produces `[f,n,...]` where `f` is the exponent of p in the conductor of e, $n = 1$ means good reduction (type I_0), $n = 2, 3, 4$ means reduction of type II,III,IV, $n = 4 + \nu$ means type I_ν, and -1, -2 etc. mean I* II* etc..

`ellap(e,p)` Computes $a_p \overset{\text{def}}{=} p + 1 - \#E(\mathbb{F}_p)$. Requires e to be minimal at p.

`ellan(e,n)` Computes the first n a_ks. Requires e to be minimal at p.

`ellgenerators(e)` Computes a basis for $E(\mathbb{Q})/\mathrm{tors}$ (requires John Cremona's elliptic curve data to be available.

To quit, type `quit` (or `\q`).

Example.

`gp`

`gp> v=[0,-4,0,0,16]` Sets `v` equal to the vector $(0, -4, \ldots)$.

`%1=[0,-4,0,0,16]`

`gp> ellinit(v,1)` Computes the vector $(a_1, \ldots, j)$.

`%2=[0,-4,0,0,16,-16,0,64,-256,256,-9728,-45056,-4096/11]` For example, $\Delta = -45056$ and $j = -4096/11$.

`gp> e=ellinit(v)` Computes the vector $(a_1, \ldots, j, \ldots)$ and sets `e` equal to the elliptic curve $Y^2 = X^3 - 4X^2 + 16$.

`%3=[0,-4,0,0,16,...]`

`gp> elltors(e)` Computes the torsion subgroup (it's cyclic of order 5).

`%4=[5,[5],[0,4]]`

`gp> ellglobalred(e)` Computes the minimum conductor and the change of coordinates required to give the minimal equation.

`%5=[11, [2,0,0,4],1]`

`gp> ellchangecurve(e,[2,0,0,4])`

`%6=[0,-1,1,0,0,-4,0,1,-1,16,-152,-11,-4096/11,...]`

Computes the minimal Weierstrass equation for E, $Y^2 + Y = X^3 - X^2$, which now has discriminant -11 but (of course) the same j-invariant.

`gp> elllocalred(%6,2)`

`%7 = [0,1,[1,0,0,0],1]` So E now has good reduction at 2.

`gp> elllocalred(%6,11)`

`%8 = [1,5,[1,0,0,0],1]` So E has bad reduction at 11, with conductor 11^1 (hence the singularity is a node), and the Kodaira type of the special fibre of the Néron model is I_1.

`gp> ellap(%6,13)` Computes a_{13}.
`%9=4` So $\#E(\mathbb{F}_{13}) = 13 + 1 - 4 = 10$.
`gp> ellgenerators(e)`
`%10=[]` So $E(\mathbb{Q})$ is finite.
`gp> ellinit([6,-3,9,-16,-14])`
`%11=[6,-3,9,-16,-14,24,22,25,29,48,-216,37,110592/37,...]`
`gp> ellgenerators(%11)`
`%12=[[-2,2]]` So $E(\mathbb{Q})/\mathrm{tors}$ is generated free of rank one generated by $(-2, 2)$.
Pari is freely available from http://pari.math.u-bordeaux.fr/. Most of the elliptic curve algorithms incorporated into Pari are explained in the books Cohen 1993, 2000, and Cremona 1992.

Chapter III

Elliptic Curves over the Complex Numbers

In this chapter, we discuss the theory of elliptic curves over $\mathbb{C}$.

1 Lattices and bases

A ***lattice*** in $\mathbb{C}$ is the subgroup generated by two complex numbers that are linearly independent over $\mathbb{R}$. Thus

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad \text{some } \omega_1, \omega_2 \in \mathbb{C},$$

and since neither ω_1 nor ω_2 is a real multiple of the other, we can order them so that $\Im(\omega_1/\omega_2) > 0$. If $\{\omega_1', \omega_2'\}$ is a second pair of elements of Λ, then

$$\omega_1' = a\omega_1 + b\omega_2, \quad \omega_2' = c\omega_1 + d\omega_2, \quad \text{some } a, b, c, d \in \mathbb{Z},$$

i.e.,

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

with A a 2×2 matrix with integer coefficients. The pair (ω_1', ω_2') will be a $\mathbb{Z}$-basis for Λ if and only if A is invertible and so has determinant ± 1. Let $z = \omega_1/\omega_2$ and $z' = \omega_1'/\omega_2'$; then

$$\Im(z') = \Im\left(\frac{az+b}{cz+d}\right) = \frac{\Im(adz + bc\bar{z})}{|cz+d|^2} = \frac{(ad-bc)\Im(z)}{|cz+d|^2}$$

and so $\Im(\omega_1'/\omega_2') > 0$ if and only if $\det A > 0$. Therefore, the group $\mathrm{SL}_2(\mathbb{Z})$ of matrices with integer coefficients and determinant 1 acts transitively on the set of bases (ω_1, ω_2) for Λ with $\Im(\omega_1/\omega_2) > 0$. We have proved the following statement:

PROPOSITION 1.1 *Let M be the set of pairs of complex numbers (ω_1, ω_2) such that $\Im(\omega_1/\omega_2) > 0$, and let $\mathcal{L}$ be the set of lattices in $\mathbb{C}$. Then the map $(\omega_1, \omega_2) \mapsto \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ induces a bijection*

$$\mathrm{SL}_2(\mathbb{Z})\backslash M \to \mathcal{L}.$$

Here $\mathrm{SL}_2(\mathbb{Z})\backslash M$ means the set of orbits in M for the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a\omega_1 + b\omega_2 \\ c\omega_1 + d\omega_2 \end{pmatrix}.$$

Let $\mathbb{H}$ be the complex upper half-plane:

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

Let $z \in \mathbb{C}^\times$ act on M by the rule $z(\omega_1, \omega_2) = (z\omega_1, z\omega_2)$ and on $\mathcal{L}$ by the rule $z\Lambda = \{z\lambda \mid \lambda \in \Lambda\}$. The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ induces a bijection $M/\mathbb{C}^\times \to \mathbb{H}$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on M corresponds to the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

on $\mathbb{H}$. We have bijections

$$\begin{array}{ccccc} \mathcal{L}/\mathbb{C}^\times & \xleftrightarrow{1:1} & \mathrm{SL}_2(\mathbb{Z})\backslash M/\mathbb{C}^\times & \xleftrightarrow{1:1} & \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}. \\ (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)\cdot\mathbb{C}^\times & \longleftrightarrow & \mathrm{SL}_2(\mathbb{Z})\cdot(\omega_1, \omega_2)\cdot\mathbb{C}^\times & \longleftrightarrow & \mathrm{SL}_2(\mathbb{Z})\cdot\dfrac{\omega_1}{\omega_2} \end{array} \tag{18}$$

For a lattice Λ with basis $\{\omega_1, \omega_2\}$, the interior of any parallelogram with vertices z_0, $z_0+\omega_1$, $z_0+\omega_2$, $z_0+\omega_1+\omega_2$ will be called a ***fundamental domain*** or ***period parallelogram*** D for Λ. We usually choose z_0 so that D to contain 0.

2 Doubly periodic functions

Let Λ be a lattice in $\mathbb{C}$. To give a function on $\mathbb{C}/\Lambda$ amounts to giving a function on $\mathbb{C}$ such that

$$f(z+\omega) = f(z) \qquad \text{(as functions on } \mathbb{C}) \tag{19}$$

for all $\omega \in \Lambda$. If $\{\omega_1, \omega_2\}$ is a basis for Λ, then this condition is equivalent to

$$\begin{cases} f(z+\omega_1) = f(z) \\ f(z+\omega_2) = f(z) \end{cases}.$$

For this reason, functions satisfying (19) are said to be ***doubly periodic*** for Λ. In this section, we study the doubly periodic meromorphic functions for a lattice Λ, and in the next section we interpret these functions as meromorphic functions on the quotient Riemann surface $\mathbb{C}/\Lambda$. Throughout "doubly periodic" will mean "doubly periodic and meromorphic".

PROPOSITION 2.1 *Let $f(z)$ be a doubly periodic function for Λ, not identically zero, and let D be a fundamental domain for Λ such that f has no zeros or poles on the boundary of D. Then*

(a) $\sum_{P\in D} \mathrm{Res}_P(f) = 0$;
(b) $\sum_{P\in D} \mathrm{ord}_P(f) = 0$;
(c) $\sum_{P\in D} \mathrm{ord}_P(f)\cdot P \equiv 0 \mod \Lambda$.

The first sum is over the points in D where f has a pole, and the other sums are over the points where it has a zero or pole (and $\mathrm{ord}_P(f)$ is the order of the zero or the negative of the order of the pole). Each sum is finite.

PROOF. According to the residue theorem (Cartan 1963, III 5.2),

$$\int_\Gamma f(z)dz = 2\pi i\left(\sum_{P\in D} \mathrm{Res}_P(f)\right),$$

where Γ is the boundary of D. Because f is periodic, the integrals of it over opposite sides of D cancel, and so the integral is zero. This gives (a). For (b) one applies the residue theorem to f'/f, noting that this is again doubly periodic and that $\mathrm{Res}_P(f'/f) = \mathrm{ord}_P(f)$. For (c) one applies the residue theorem to $z\cdot f'(z)/f(z)$. This is no longer doubly periodic, but the integral of it around Γ lies in Λ. □

COROLLARY 2.2 *A nonconstant doubly periodic function has at least two poles (or one double pole).*

PROOF. A holomorphic doubly periodic function is bounded on the closure of any fundamental domain (by compactness), and hence on the entire plane (by periodicity). It is therefore constant by Liouville's theorem (Cartan 1963, III 1.2). It is impossible for a doubly periodic function to have a single simple pole in a period parallelogram because, by (a) of the proposition, the residue at the pole would have to be zero there, and so the pole couldn't be simple. □

The Weierstrass $\wp$ function

Let Λ be a lattice in $\mathbb{C}$. The Riemann-Roch theorem applied to the quotient $\mathbb{C}/\Lambda$ proves the existence of nonconstant doubly periodic meromorphic functions for Λ, but here we shall construct them explicitly for Λ. When G is a *finite* group acting on a set S, it is easy to construct functions invariant under the action of G: take f to be any function $f\colon S\to\mathbb{C}$, and define

$$F(s) = \sum_{g\in G} f(gs);$$

then $F(g's) = \sum_{g\in G} f(g'gs) = F(s)$ because, as g runs over G, so does $g'g$; thus F is invariant, and (obviously) all invariant functions are of this form. When G is not finite, one has to verify that the series converges — in fact, in order to be able to change the order of summation, one needs (at least) absolute convergence.

Let D be an open subset of $\mathbb{C}$, and let $f_0, f_1, \ldots$ be a sequence of holomorphic functions on D. Recall (Cartan 1963, I 2) that the series $\sum_n f_n$ is said to ***converge normally*** on a subset A of D if the series of positive terms $\sum_n \|f_n\|$ converges, where $\|f_n\| = \sup_{z\in A} |f_n(z)|$. The series $\sum_n f_n$ is then both uniformly convergent and absolutely convergent on A. When $f_0, f_1, \ldots$ is a sequence of *meromorphic* functions, the series is said to ***converges normally*** on A if, after a finite number of terms f_n have been removed, it becomes a normally convergent series of holomorphic functions. If a series $\sum_n f_n$ of meromorphic functions is normally convergent on compact subsets of D, then the sum f of the series is a meromorphic function on D; moreover, the series of derivatives converges normally on compact subsets of D, and its sum is the derivative of f (ibid. V 2).

Now let $\varphi(z)$ be a meromorphic function $\mathbb{C}$ and write

$$\Phi(z) = \sum_{\omega\in\Lambda} \varphi(z+\omega).$$

Assume that as $|z| \to \infty$, $\varphi(z) \to 0$ so fast that the series for $\Phi(z)$ is normally convergent on compact subsets. Then $\Phi(z)$ is doubly periodic with respect to Λ, because replacing z by $z+\omega_0$ for some $\omega_0 \in \Lambda$ merely rearranges the terms in the sum.

To prove the normal convergence for the functions we are interested in, we shall need the following result.

LEMMA 2.3 *For any lattice Λ in $\mathbb{C}$, the series $\sum_{\omega\in\Lambda,\omega\neq 0} 1/|\omega|^3$ converges.*

PROOF. Let ω_1, ω_2 be a basis for Λ, and, for each integer $n \geq 1$, consider the parallelogram

$$P(n) = \{a_1\omega_1 + a_2\omega_2 \mid a_1, a_2 \in \mathbb{R},\ \max(|a_1|, |a_2|) = n\}.$$

There are $8n$ points of Λ on $P(n)$, and the distance between 0 and any of them is at least kn, where k is the shortest distance from 0 to a point of $P(1) \cap \Lambda$. Therefore, the contribution of the points on $P(n)$ to the sum is bounded by $8n/k^3n^3$, and so

$$\sum_{\omega\in\Lambda,\omega\neq 0} \frac{1}{|\omega|^3} \leq \frac{8}{k^3}\sum_n \frac{1}{n^2} < \infty.$$

□

We know from Corollary 2.2 that the simplest possible nonconstant doubly periodic function is one with a double pole at each point of Λ and no other poles. Suppose $f(z)$ is such a function. Then $f(z)-f(-z)$ is a doubly periodic function with no poles except perhaps simple ones at the points of Λ. Hence it must be constant, and since it is an odd function it must vanish. Thus $f(z)$ is even, and we can make it unique by imposing the normalization condition

$$f(z) = z^{-2} + 0 + z^2 g(z)$$

with $g(z)$ holomorphic near $z = 0$. There is such a function, namely, the Weierstrass function $\wp(z)$, but we can't define it directly from $1/z^2$ by the method at the start of this subsection because $\sum_{\omega\in\Lambda} 1/(z+\omega)^2$ is not normally convergent. Instead, we define

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda,\ \omega\neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$
$$\wp'(z) = \sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^{-3}}.$$

PROPOSITION 2.4 *The two series above converge normally on compact subsets of $\mathbb{C}$, and their sums $\wp$ and $\wp'$ are doubly periodic meromorphic functions on $\mathbb{C}$ with $\wp' = \frac{d\wp}{dz}$.*

PROOF. Note that $\wp'(z) = \sum_{\omega\in\Lambda} \varphi(z)$ with $\varphi(z) = \frac{-2}{z^3} = \frac{d}{dz}(\frac{1}{z^2})$, and that $\sum_{\omega\in\Lambda} \varphi(z+\omega)$ converges normally on any compact disk $|z| \leq r$ by comparison with $\sum \frac{1}{|\omega|^3}$. Thus, $\wp'(z)$ is a doubly periodic meromorphic function on $\mathbb{C}$ by the above remarks.

For $|z| \leq r$, and for all but the finitely many ω with $|\omega| \leq 2r$, we have that

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{-z^2+2\omega z}{\omega^2(z-\omega)^2}\right| = \frac{\left|z\left(2-\frac{z}{\omega}\right)\right|}{|\omega^3|\left|1-\frac{z}{\omega}\right|^2} \leq \frac{r\frac{5}{2}}{|\omega^3|\cdot\frac{1}{4}} = \frac{10r}{|\omega|^3}$$

and so $\wp(z)$ also converges normally on the compact disk $|z| \leq r$. Because its derivative is doubly periodic, so also is $\wp(z)$. □

Eisenstein series

Let Λ be a lattice in $\mathbb{C}$, and consider the sum

$$\sum_{\omega\in\Lambda,\ \omega\neq 0} \frac{1}{\omega^n}.$$

The map $\omega \mapsto -\omega\colon \Lambda \to \Lambda$ has order 2, and its only fixed point is 0. Therefore $\Lambda \smallsetminus \{0\}$ is a disjoint union of its orbits, and it follows that the sum is zero if n is odd. We write

$$G_{2k}(\Lambda) = \sum_{\omega\in\Lambda,\ \omega\neq 0} \frac{1}{\omega^{2k}},$$

and we let $G_{2k}(z) = G_{2k}(\mathbb{Z}z+\mathbb{Z})$, $z \in \mathbb{H}$. Note that $G_{2k}(c\Lambda) = c^{-2k}G_{2k}(\Lambda)$ for $c \in \mathbb{C}^{\times}$, and so $G_{2k}(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-2k}G_{2k}(\mathbb{Z}\frac{\omega_1}{\omega_2} + \mathbb{Z})$.

PROPOSITION 2.5 *For all integers $k \geq 2$, $G_{2k}(z)$ converges to a holomorphic function on $\mathbb{H}$.*

PROOF. Let

$$D = \{z \in \mathbb{C} \mid |z| \geq 1, \quad |\Re(z)| \leq 1/2\}.$$

For $z \in D$,

$$|mz+n|^2 = m^2 z\bar{z} + 2mn\Re(z) + n^2 \geq m^2 - mn + n^2 = |m\rho - n|^2$$

where $\rho = e^{2\pi i/3}$. Therefore, Lemma 2.3 shows that, for $k \geq 2$, $G_{2k}(z)$ converges normally on D. For any $A \in \mathrm{SL}_2(\mathbb{Z})$, $G_{2k}(A^{-1}z)$ also converges normally on D, which shows that $G_{2k}(z)$ converges normally on AD. In Chapter V, Proposition 1.3, we shall see that the sets AD cover $\mathbb{H}$, and so this shows that $G_{2k}(z)$ is holomorphic on the whole of $\mathbb{H}$. □

The functions $G_{2k}(\Lambda)$ and $G_{2k}(z)$ are called ***Eisenstein series.***

The field of doubly periodic functions

Let Λ be a lattice in $\mathbb{C}$. The meromorphic functions on $\mathbb{C}$ form a field $M(\mathbb{C})$ (Cartan 1963, I 4.3), and the doubly periodic functions form a subfield of $M(\mathbb{C})$, which the next two propositions determine.

PROPOSITION 2.6 *There is the following relation between $\wp$ and $\wp'$:*

$$\wp'(z)^2 = 4\wp(z)^3 - g_4\wp(z) - g_6$$

where $g_4 = 60G_4(\Lambda)$ and $g_6 = 140G_6(\Lambda)$.

PROOF. We compute the Laurent expansion of $\wp(z)$ near 0. Recall that for $|t| < 1$,

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots.$$

On differentiating this, we find that

$$\frac{1}{(1-t)^2} = \sum_{n\geq 1} nt^{n-1} = \sum_{n\geq 0}(n+1)t^n.$$

Hence, for $|z| < |\omega|$,

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{\left(1-\frac{z}{\omega}\right)^2} - 1\right) = \sum_{n\geq 1}(n+1)\frac{z^n}{\omega^{n+2}}.$$

On putting this into the definition of $\wp(z)$ and changing the order of summation, we find that for $|z| < |\omega|$,

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n\geq 1}\sum_{\omega\neq 0}(n+1)\frac{z^n}{\omega^{n+2}} \\ &= \frac{1}{z^2} + \sum_{k\geq 1}(2k+1)G_{2k+2}(\Lambda)z^{2k} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots.\end{aligned}$$

Therefore,

$$\wp'(z) = -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + \cdots.$$

These expressions contain enough terms to show that the Laurent expansion of

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4(\Lambda)\wp(z) + 140G_6(\Lambda)$$

has no nonzero term in z^n with $n \leq 0$. Therefore this function is holomorphic at 0 and takes the value 0 there. Since it is doubly periodic and has no poles in a suitable fundamental domain containing 0, we see that it is constant, and in fact zero. □

PROPOSITION 2.7 *The field of doubly periodic meromorphic functions is the subfield $\mathbb{C}(\wp, \wp')$ of $M(\mathbb{C})$ generated by $\wp$ and $\wp'$, i.e., every doubly periodic meromorphic function can be expressed as a rational function of $\wp$ and $\wp'$.*

PROOF. We begin by showing that every even doubly periodic function f lies in $\mathbb{C}(\wp)$.

Observe that, because $f(z) = f(-z)$, the kth derivative of f,

$$f^{(k)}(z) = (-1)^k f^{(k)}(-z).$$

Therefore, if f has a zero of order m at z_0, then it has a zero of order m at $-z_0$. On applying this remark to $1/f$, we obtain the same statement with "zero" replaced by "pole".

Similarly, because $f^{(2k+1)}(z_0) = -f^{(2k+1)}(-z_0)$, if $z_0 \equiv -z_0 \bmod \Lambda$, then the order of zero (or pole) of f at z_0 is even.

Choose a set of representatives mod Λ for the zeros and poles of f not in Λ and number them $z_1, \ldots, z_m, -z_1, \ldots, -z_m, z_{m+1}, \ldots, z_n$ so that (modulo Λ)

$$\begin{aligned} z_i &\not\equiv -z_i, && 1 \le i \le m \\ z_i &\equiv -z_i \not\equiv 0, && m < i \le n. \end{aligned}$$

Let m_i be the order of f at z_i; according to the second observation, m_i is even for $i > m$.

Now $\wp(z) - \wp(z_i)$ is also an even doubly periodic function. Since it has exactly two poles in a fundamental domain, it must have exactly two zeros there. When $i \le m$, it has simple zeros at $\pm z_i$; when $i > m$, it has a double zero at z_i (by the second observation). Define

$$g(z) = \prod_{i=1}^{m} (\wp(z) - \wp(z_i))^{m_i} \cdot \prod_{i=m+1}^{n} (\wp(z) - \wp(z_i))^{m_i/2}.$$

Then $f(z)$ and $g(z)$ have exactly the same zeros and poles at points z not on Λ. We deduce from (2.1b) that they also have the same order at $z = 0$, and so f/g, being holomorphic and doubly periodic, is constant: $f = cg \in \mathbb{C}(\wp)$.

Now consider an arbitrary doubly periodic function f. Such an f decomposes into the sum of an even and of an odd doubly periodic function:

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}.$$

We know that the even doubly periodic functions lie in $\mathbb{C}(\wp)$, and clearly the odd doubly periodic functions lie in $\wp' \cdot \mathbb{C}(\wp)$. □

Let $\mathbb{C}[x, y] = \mathbb{C}[X, Y]/(Y^2 - 4X^3 + g_4 X + g_6)$, and let $\mathbb{C}[\wp, \wp']$ be the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{C}$ generated by $\wp$ and $\wp'$. Proposition 2.6 shows that $(X, Y) \mapsto (\wp(z), \wp'(z))$ defines a homomorphism

$$\mathbb{C}[x, y] \to \mathbb{C}[\wp, \wp'],$$

which I claim to be an isomorphism. For this, we have to show that the only polynomials $g(X, Y) \in \mathbb{C}[X, Y]$ such that $g(\wp, \wp') = 0$ are those divisible by $f(X, Y) \stackrel{\text{def}}{=} Y^2 - X^3 + g_4 X + g_6$. The theory of resultants (see Chap. I, §1) shows that for any polynomial $g(X, Y)$, there exist polynomials $a(X, Y)$ and $b(X, Y)$ such that

$$a(X, Y) f(X, Y) + b(X, Y) g(X, Y) = R(X) \in \mathbb{C}[X]$$

with $\deg_Y(b) < \deg_Y(f)$. If $g(\wp, \wp') = 0$, then $R(\wp) = 0$, but it is easy to see that $\wp$ is transcendental over $\mathbb{C}$ (for example, it has infinitely many poles). Therefore $R = 0$, and so $f(X, Y)$ divides $b(X, Y) g(X, Y)$. Any polynomial with the form of $f(X, Y)$ is irreducible, and so $f(X, Y)$ divides either $b(X, Y)$ or $g(X, Y)$. Because of the degrees, it can't divide b, and so it must divide g.

The isomorphism $\mathbb{C}[x,y] \to \mathbb{C}[\wp,\wp']$ induces an isomorphism of the fields of fractions

$$\mathbb{C}(x,y) \to \mathbb{C}(\wp,\wp').$$

Thus, $\mathbb{C}(x,y)$ is canonically isomorphic to the field of all doubly periodic meromorphic functions for Λ.

3 Elliptic curves as Riemann surfaces

The notion of a Riemann surface

Let X be a connected Hausdorff topological space admitting a countable base for its open sets. A ***coordinate neighbourhood*** for X is a pair (U,z) with U an open subset of X and z a homeomorphism of U onto an open subset of the complex plane $\mathbb{C}$. Two coordinate neighbourhoods (U_1,z_1) and (U_2,z_2) are ***compatible*** if the function

$$z_1 \circ z_2^{-1} \colon z_2(U_1 \cap U_2) \to z_1(U_1 \cap U_2)$$

and its inverse are holomorphic. A family of coordinate neighbourhoods $(U_i,z_i)_{i\in I}$ is a ***coordinate covering*** if $X = \bigcup_i U_i$ and (U_i,z_i) is compatible with (U_j,z_j) for all pairs $(i,j) \in I \times I$. Two coordinate coverings are said to be ***equivalent*** if the their union is also a coordinate covering. This defines an equivalence relation on the set of coordinate coverings, and an equivalence class is called a ***complex structure*** on X. A Hausdorff topological space X together with a complex structure is a ***Riemann surface.***

Let X be a Riemann surface. A function $f\colon U \to \mathbb{C}$ on an open subset U of X is ***holomorphic*** if it satisfies the following condition for one (hence every) coordinate covering $(U_i,z_i)_{i\in I}$ in the equivalence class defining the complex structure: $f \circ z_i^{-1} \colon z_i(U \cap U_i) \to \mathbb{C}$ is holomorphic for all $i \in I$.

A map $f\colon X \to X'$ from one Riemann surface to a second is ***holomorphic*** if $g \circ f$ is holomorphic whenever g is a holomorphic function on an open subset of X'. For this, it suffices to check that for every point P in X, there are coordinate neighbourhoods (U,z) of P and (U',z') of $f(P)$ such that $z' \circ f \circ z^{-1} \colon z(U) \to z'(U')$ is holomorphic. An ***isomorphism*** of Riemann surfaces is a bijective holomorphic map whose inverse is also holomorphic.

Recall that a ***meromorphic function*** on an open subset U of $\mathbb{C}$ is a holomorphic function f on $U \smallsetminus \Xi$ for some discrete subset $\Xi \subset U$ having at worst a pole at each point of Ξ, i.e., such that for each $a \in \Xi$, there exists an m for which $(z-a)^m f(z)$ is holomorphic in a neighbourhood of a. A ***meromorphic function*** on an open subset of a Riemann surface is defined similarly. Equivalently it is a holomorphic map from the Riemann surface to the Riemann sphere (Cartan 1963, VI 4.5).

EXAMPLE 3.1 Any open subset of $\mathbb{C}$ is a Riemann surface with a single coordinate neighbourhood — U itself with the identity function z.

EXAMPLE 3.2 Consider the unit sphere

$$S_2 : X^2 + Y^2 + Z^2 = 1$$

in $\mathbb{R}^3$, and let P be the north pole $(0, 0, 1)$. Stereographic projection from P is the map

$$(x, y, z) \mapsto \frac{x + iy}{1 - z} : S_2 \smallsetminus P \to \mathbb{C}.$$

Take this to be a coordinate neighbourhood for S_2. Stereographic projection from the south pole gives a second coordinate neighbourhood. These two coordinate neighbourhoods define a complex structure on S_2, and S_2 together with the complex structure is called the ***Riemann sphere***. The map $(x, y, z) \mapsto (x \colon y \colon z)$ identifies S_2 with $\mathbb{P}^1(\mathbb{C})$.

Quotients of $\mathbb{C}$ by lattices

Let Λ be a lattice in $\mathbb{C}$. Topologically the quotient $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{R}^2/\mathbb{Z}^2$, which is a one-holed torus (the surface of a donut). Write $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ for the quotient map. Then $\mathbb{C}/\Lambda$ can be given a complex structure for which a function $\varphi \colon U \to \mathbb{C}$ on an open subset U of $\mathbb{C}/\Lambda$ is holomorphic (resp. meromorphic) if and only if the composite $\varphi \circ \pi \colon \pi^{-1}(U) \to \mathbb{C}$ is holomorphic (resp. meromorphic) in the usual sense. It is the unique complex structure for which π is a local isomorphism of Riemann surfaces.

We shall see that, although any two quotients $\mathbb{C}/\Lambda$, $\mathbb{C}/\Lambda'$ are homeomorphic, they will be isomorphic as Riemann surfaces only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^\times$.

The holomorphic maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$

Let Λ and Λ' be lattices in $\mathbb{C}$. The map $\pi \colon \mathbb{C} \to \mathbb{C}/\Lambda$ realizes $\mathbb{C}$ as the universal covering space of $\mathbb{C}/\Lambda$. Since the same is true of $\pi' \colon \mathbb{C} \to \mathbb{C}/\Lambda'$, a continuous map $\varphi \colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ such that $\varphi(0) = 0$ will lift uniquely to a continuous map $\tilde{\varphi} \colon \mathbb{C} \to \mathbb{C}$ such that $\tilde{\varphi}(0) = 0$:

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\tilde{\varphi}} & \mathbb{C} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\
\mathbb{C}/\Lambda & \xrightarrow{\varphi} & \mathbb{C}/\Lambda'
\end{array}$$

(see, for example, Greenberg 1967, 5.1, 6.4). Because π and π' are local isomorphisms of Riemann surfaces, the map φ will be holomorphic if and only if $\tilde{\varphi}$ is holomorphic.

PROPOSITION 3.3 *Let Λ and Λ' be lattices in $\mathbb{C}$. A complex number α such that $\alpha\Lambda \subset \Lambda'$ defines a holomorphic map*

$$[z] \mapsto [\alpha z] \colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$$

sending 0 to 0, and every holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ sending 0 to 0 is of this form (for a unique α).

PROOF. It is obvious from the above remarks that α defines a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$. Conversely, let $\varphi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ be a holomorphic map such that $\varphi(0) = 0$, and let $\tilde{\varphi}$ be its unique lifting to a holomorphic map $\mathbb{C} \to \mathbb{C}$ sending 0 to 0. For any $\omega \in \Lambda$, the map $z \mapsto \tilde{\varphi}(z+\omega) - \tilde{\varphi}(z)$ is continuous and takes values in $\Lambda' \subset \mathbb{C}$; because $\mathbb{C}$ is connected and Λ' is discrete, it must be constant, and so its derivative is zero:

$$\tilde{\varphi}'(z+\omega) = \tilde{\varphi}'(z).$$

Therefore $\tilde{\varphi}'(z)$ is doubly periodic. As it is holomorphic, it must be constant, say $\tilde{\varphi}'(z) = \alpha$ for all z. On integrating, we find that $\tilde{\varphi}(z) = \alpha z + \beta$, and $\beta = \tilde{\varphi}(0) = 0$. □

COROLLARY 3.4 *The Riemann surfaces $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic if and only if $\Lambda' = \alpha\Lambda$ for some $\alpha \in \mathbb{C}^\times$.*

PROOF. This is obvious from the proposition. □

COROLLARY 3.5 *Every holomorphic map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ sending 0 to 0 is a homomorphism of groups.*

PROOF. Clearly $[z] \mapsto [\alpha z]$ is a homomorphism of groups. □

The proposition shows that

$$\mathrm{Hom}(\mathbb{C}/\Lambda, \mathbb{C}/\Lambda') \simeq \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda'\},$$

and the corollary shows that there is a one-to-one correspondence

$$\{\mathbb{C}/\Lambda\}/{\approx} \xleftrightarrow{1:1} \mathcal{L}/\mathbb{C}^\times.$$

The elliptic curve $E(\Lambda)$

Let Λ be a lattice in $\mathbb{C}$.

LEMMA 3.6 *The polynomial $f(X) = 4X^3 - g_4(\Lambda)X - g_6(\Lambda)$ has distinct roots.*

PROOF. The function $\wp'(z)$ is odd, so $\wp'(\omega_1/2) = -\wp'(-\omega_1/2)$, and doubly periodic, so $\wp'(\omega_1/2) = \wp'(-\omega_1/2)$. Thus, $\wp'(\omega_1/2) = 0$ and Proposition 2.6 shows that $\wp(\omega_1/2)$ is a root of $f(X)$. The same argument shows that $\wp(\omega_2/2)$ and $\wp((\omega_1+\omega_2)/2)$ are also roots of $f(X)$. It remains to prove that these three numbers are distinct.

The function $\wp(z) - \wp(\omega_1/2)$ has a zero at $\omega_1/2$, which must be a double zero because its derivative is also 0 there. Since $\wp(z) - \wp(\omega_1/2)$ has only one (double) pole in a fundamental domain D containing 0, Proposition 2.1 shows that $\omega_1/2$ is the only zero of $\wp(z) - \wp(\omega_1/2)$ in D, i.e., that $\wp(z)$ takes the value $\wp(\omega_1/2)$ only at $z = \omega_1/2$ within D. In particular, $\wp(\omega_1/2)$ is not equal to $\wp(\omega_2/2)$ or $\wp((\omega_1+\omega_2)/2)$. Similarly, $\wp(\omega_2/2)$ is not equal to $\wp((\omega_1+\omega_2)/2)$. □

From the lemma, we see that

$$E(\Lambda) : Y^2Z = 4X^3 - g_4(\Lambda)XZ^2 - g_6(\Lambda)Z^3$$

is an elliptic curve. Recall that $c^4 g_4(c\Lambda) = g_4(\Lambda)$ and $c^6 g_6(c\Lambda) = g_6(\Lambda)$ for any $c \in \mathbb{C}^\times$, and so $c\Lambda$ defines essentially the same elliptic curve as Λ.

For any elliptic curve

$$E : Y^2Z = X^3 + a_4XZ^2 + a_6Z^3,$$

the closed subspace $E(\mathbb{C})$ of $\mathbb{P}^2(\mathbb{C})$ has a natural complex structure: for example, in a neighbourhood of a point $P \in E(\mathbb{C})$ such that $y(P) \neq 0 \neq z(P)$, the function x/z provides a local coordinate.

PROPOSITION 3.7 *The map*

$$\begin{cases} z \mapsto (\wp(z) : \wp'(z) : 1), & z \neq 0 \\ 0 \mapsto (0 : 1 : 0) \end{cases}$$

is an isomorphism of Riemann surfaces $\mathbb{C}/\Lambda \to E(\Lambda)(\mathbb{C})$.

PROOF. It is certainly a well-defined map. The function $\wp(z)\colon \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$ is $2:1$ in a fundamental domain containing 0, except at the points $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_1+\omega_2}{2}$, where it is one-to-one. Therefore, $\wp$ realizes $\mathbb{C}/\Lambda$ as a covering of degree 2 of the Riemann sphere, and it is a local isomorphism except at the four listed points. Similarly, x/z realizes $E(\Lambda)(\mathbb{C})$ as a covering of degree 2 of the Riemann sphere, and it is a local isomorphism except at $(0:1:0)$ and the three points where $y = 0$. It follows that $\mathbb{C}/\Lambda \to E(\Lambda)(\mathbb{C})$ is an isomorphism outside the two sets of four points. A similar argument shows that it is a local isomorphism at the remaining four points. □

THE ADDITION FORMULA

Consider $\wp(z+z')$. For a fixed z', it is a doubly periodic function of z, and therefore it is a rational function of $\wp$ and $\wp'$. The next result exhibits the rational function.

PROPOSITION 3.8 *The following formula holds:*

$$\wp(z+z') = \frac{1}{4}\left(\frac{\wp'(z)-\wp'(z')}{\wp(z)-\wp(z')}\right)^2 - \wp(z) - \wp(z').$$

PROOF. Let $f(z)$ denote the difference of the left and the right sides. Its only possible poles (in a fundamental domain for Λ) are at 0 or $\pm z'$, and by examining the Laurent expansion of $f(z)$ near these points one sees that it has no pole at 0 or $-z'$, and at worst a simple pole at z'. Since it is doubly periodic, it must be constant, and since $f(0)=0$, it must be identically zero. □

COROLLARY 3.9 *The map* $z \mapsto (\wp(z) : \wp'(z) : 1) : \mathbb{C}/\Lambda \to E(\Lambda)$ *is a homomorphism of groups.*

PROOF. The above formula agrees with the formula for the x-coordinate of the sum of two points on $E(\Lambda)$.[1] □

Classification of elliptic curves over $\mathbb{C}$

THEOREM 3.10 *Every elliptic curve E over $\mathbb{C}$ is isomorphic to $E(\Lambda)$ for some lattice Λ.*

PROOF. Recall (II 2.1c) that, over an algebraically closed field, the elliptic curves are classified (up to isomorphism) by their j-invariants. For any lattice Λ in $\mathbb{C}$, the curve

$$E(\Lambda) : Y^2 Z = 4X^3 - g_4(\Lambda) X Z^2 - g_4(\Lambda) Z^3$$

has discriminant $\Delta(\Lambda) = g_4(\Lambda)^3 - 27 g_6(\Lambda)^2$ and j-invariant

$$j(\Lambda) = \frac{1728 g_4(\Lambda)^3}{g_4(\Lambda)^3 - 27 g_6(\Lambda)^2}.$$

[1]Let $Y = mX + c$ be the line through the points $P = (x, y)$ and $P' = (x', y')$ on the curve $Y^2 = 4X^3 - g_4 X - g_6$. Then the x, x', and $x(P+P')$ are the roots of the polynomial

$$(mX+c)^2 - 4X^3 + g_4 X + g_6,$$

and so

$$x(P+P') + x + x' = \frac{m^2}{4} = \frac{1}{4}\left(\frac{y-y'}{x-x'}\right)^2.$$

For $c \in \mathbb{C}^{\times}$, $g_4(c\Lambda) = c^{-4}g_4(\Lambda)$ and $g_6(c\Lambda) = c^{-6}g_6(\Lambda)$, and so the isomorphism class of $E(\Lambda)$ depends only on Λ up to scaling. Define

$$j(\tau) = j(\mathbb{Z}\tau + \mathbb{Z}).$$

Then, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$j(\frac{a\tau + b}{c\tau + d}) = j(\tau).$$

In Chapter V (Proposition 2.2) we shall show that $j\colon \mathbb{H} \to \mathbb{C}$ is surjective, which completes the proof. □

SUMMARY

For any subfield k of $\mathbb{C}$, we have the diagram:

$$\begin{tikzcd}
\{\text{Elliptic curves}/\mathbb{C}\}/\approx \arrow[r, leftrightarrow, "1:1"] & \mathcal{L}/\mathbb{C}^{\times} \arrow[r, leftrightarrow, "1:1"] & \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \arrow[r, "j"] & \mathbb{C} \\
\{\text{Elliptic curves}/k\}/\approx \arrow[u] \arrow[rrr, "j"] & & & k \arrow[u]
\end{tikzcd} \tag{20}$$

The upper j is an isomorphism (V 2.2) and the lower j is surjective (II 2.3). The left hand vertical map and the bottom map are injective if k is algebraically closed.

REMARK 3.11 The above picture can be made a little more precise. Consider the isomorphism

$$z \mapsto (\wp(z) : \wp'(z) : 1)\colon \mathbb{C}/\Lambda \to E(\mathbb{C}).$$

Since $x = \wp(z)$ and $y = \wp'(z)$,

$$\frac{dx}{y} = \frac{\wp'(z)dz}{\wp'(z)} = dz.$$

Thus the differential dz on $\mathbb{C}$ corresponds to the differential $\frac{dx}{y}$ on $E(\mathbb{C})$. Conversely, from a holomorphic differential ω on $E(\mathbb{C})$ we can obtain an realization of E as a quotient $\mathbb{C}/\Lambda$ as follows. For $P \in E(\mathbb{C})$, consider $\varphi(P) = \int_O^P \omega \in \mathbb{C}$. This is a not well defined because it depends on the choice of a path from O to P. However, if we choose a $\mathbb{Z}$-basis (γ_1, γ_2) for $H_1(E(\mathbb{C}), \mathbb{Z})$, and set $\omega_1 = \int_{\gamma_1} \omega$, $\omega_2 = \int_{\gamma_2} \omega$, then $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$, and $P \mapsto \varphi(P)$ is an isomorphism $E(\mathbb{C}) \to \mathbb{C}/\Lambda$. In this way, we obtain a natural one-to-one correspondence between $\mathcal{L}$ and the set of isomorphism classes of pairs (E, ω)

consisting of an elliptic curve E over $\mathbb{C}$ and a holomorphic differential ω on E. Thus:

$$\begin{array}{ccc} \{(E,\omega)\}/\approx & \xleftrightarrow{1:1} & \mathcal{L} \\ \downarrow & & \downarrow \\ \{E\}/\approx & \xleftrightarrow{1:1} & \mathcal{L}/\mathbb{C}^{\times}. \end{array}$$

Torsion points

Let E be an elliptic curve over $\mathbb{C}$. From an isomorphism $E(\mathbb{C}) \approx \mathbb{C}/\Lambda$, we obtain an isomorphism

$$E(\mathbb{C})_n \approx \tfrac{1}{n}\Lambda/\Lambda = \{\tfrac{a}{n}\omega_1 + \tfrac{b}{n}\omega_2 \mid a,b \in \mathbb{Z}\}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2.$$

This is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2. Because of this description over $\mathbb{C}$, torsion points on elliptic curves are often called ***division points.***

THEOREM 3.12 *For any elliptic curve E over an algebraically closed field k of characteristic zero, $E(k)_n$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2.*

PROOF. There exists an algebraically closed subfield k_0 of finite transcendence degree over $\mathbb{Q}$ such that E arises from a curve E_0 over k_0 (e.g., take k_0 to be the algebraic closure in k of the subfield generated over $\mathbb{Q}$ by the coefficients of the equation defining E). Now k_0 can be embedded into $\mathbb{C}$, and so we can apply the next proposition (twice). □

PROPOSITION 3.13 *Let E be an elliptic curve over an algebraically closed field k, and let Ω be an algebraically closed field containing k. Then the map $E(k) \to E(\Omega)$ induces an isomorphism on the torsion subgroups.*

PROOF. The first lemma below shows that the elements of $E(k)_n$ are the common solutions of

$$\begin{aligned} Y^2 &= X^3 + aX + b \\ \psi_n(X,Y) &= 0 \end{aligned}$$

for a certain polynomial ψ_n not divisible by $Y^2 - X^3 - aX - b$ and so we can apply the second lemma below. □

LEMMA 3.14 *Let E be the curve $Y^2Z = X^3 + aXZ^2 + bZ^3$. Let*

$$\begin{aligned} &\psi_1 = 1, \quad \psi_2 = 2Y, \\ &\psi_3 = 3X^4 + 6aX^2 + 12bX - a^2 \\ &\psi_4 = 4Y(X^6 + 5aX^4 + 20bX^3 - 5a^2X^2 - 4abX - 8b^2 - a^3) \end{aligned}$$

and, inductively,

$$Y\psi_{2n} = \psi_n\left(\psi_{n-1}^2\psi_{n+2} - \psi_{n+1}^2\psi_{n-2}\right)$$
$$\psi_{2n+1} = \psi_n^3\psi_{n+2} - \psi_n^3\psi_{n-1}.$$

Then, for any point $P = (x : y : 1)$ of E,

$$nP = (X\psi_n^4 - \psi_{n-1}\psi_n^2\psi_{n+1} : \tfrac{1}{2}\psi_{2n} : \psi_n^4).$$

PROOF. Use induction (cf. Cassels 1966, 7.2 and Cassels 1991, p. 133.) □

LEMMA 3.15 *Let $k \subset \Omega$ be algebraically closed fields. If $F(X,Y), G(X,Y) \in k[X,Y]$ have no common factor, then any common solution to the equations*

$$\begin{cases} F(X,Y) &= 0 \\ G(X,Y) &= 0 \end{cases}$$

with coordinates in Ω in fact has coordinates in k.

PROOF. From the theory of resultants (see Chap. I, §1), we know that there exist polynomials $a(X,Y)$, $b(X,Y)$, and $R(X)$ with coefficients in k such that

$$a(X,Y)F(X,Y) + b(X,Y)G(X,Y) = R(X)$$

and $R(x_0) = 0$ if and only if $F(x_0,Y)$ and $G(x_0,Y)$ have a common zero. In other words, the roots of R are the x-coordinates of the common zeros of $F(X,Y)$ and $G(X,Y)$. Since $R(X)$ is a polynomial in one variable, its roots all lie in k. Moreover, for a given $x_0 \in k$, all the common roots of $F(x_0,Y)$ and $G(x_0,Y)$ lie in k. □

REMARK 3.16 Theorem 3.12 holds also when k has characteristic $p \neq 0$ provided p does not divide n. For a power of p, either an elliptic curve has no points of order p (the "supersingular" case), or else has p^m points of order dividing p^m for every $m \geq 1$ (the "ordinary" case).

Endomorphisms

Let K be a number field. Each $\alpha \in K$ satisfies an equation,

$$\alpha^m + a_1\alpha^{m-1} + \cdots + a_m = 0, \quad a_i \in \mathbb{Q}. \tag{21}$$

If it satisfies such an equation with the $a_i \in \mathbb{Z}$, then α is said to be an ***(algebraic) integer*** of K. The algebraic integers form a subring $\mathcal{O}_K$ of K, which is a free

$\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$.[2] For example, if $K = \mathbb{Q}[\sqrt{d}]$ with $d \in \mathbb{Z}$ and square-free, then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}1 + \mathbb{Z}\sqrt{d} & d \not\equiv 1 \mod 4 \\ \mathbb{Z}1 + \mathbb{Z}\frac{1+\sqrt{d}}{2} & d \equiv 1 \mod 4. \end{cases}$$

PROPOSITION 3.17 *Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\tau = \omega_1/\omega_2 \in \mathbb{H}$. The ring of endomorphisms of the Riemann surface $\mathbb{C}/\Lambda$ is $\mathbb{Z}$ unless $[\mathbb{Q}[\tau] : \mathbb{Q}] = 2$, in which case it is a subring of $\mathbb{Q}[\tau]$ of rank 2 as a $\mathbb{Z}$-module.*

PROOF. Suppose that there exists an $\alpha \in \mathbb{C}$, $\alpha \notin \mathbb{Z}$, such that $\alpha\Lambda \subset \Lambda$. Then

$$\begin{aligned} \alpha\omega_1 &= a\omega_1 + b\omega_2 \\ \alpha\omega_2 &= c\omega_1 + d\omega_2, \end{aligned}$$

with $a, b, c, d \in \mathbb{Z}$. On dividing through by ω_2 we obtain the equations

$$\begin{aligned} \alpha\tau &= a\tau + b \\ \alpha &= c\tau + d. \end{aligned}$$

Note that $c \neq 0$.

On eliminating α from the two equations, we find that

$$c\tau^2 + (d - a)\tau + b = 0.$$

Therefore $\mathbb{Q}[\tau]$ is of degree 2 over $\mathbb{Q}$.

On eliminating τ from the two equations, we find that

$$\alpha^2 - (a + d)\alpha - bc = 0.$$

Therefore α is integral over $\mathbb{Z}$, and hence is contained in the ring of integers of $\mathbb{Q}[\tau]$. □

REMARK 3.18 The endomorphisms of the elliptic curve $E(\Lambda)$ coincide with the endomorphisms of the Riemann surface $\mathbb{C}/\Lambda$. Thus the proposition holds also for $\mathrm{End}(E(\Lambda))$. We exhibit some endomorphisms of elliptic curves not in $\mathbb{Z}$.

(a) Consider

$$E : Y^2Z = X^3 + aXZ^2,$$

and let $i = \sqrt{-1} = 1^{\frac{1}{4}}$. Then $(x : y : z) \mapsto (-x : iy : z)$ is an endomorphism of E of order 4, and so $\mathrm{End}(E) = \mathbb{Z}[i]$. Note that E has j-invariant 1728.

[2] More generally, let R be an integral domain with field of fractions F, and let K be a finite extension of F. The set of elements α of K satisfying an equation of the form (21) with $a_i \in R$ is a ring, called the integral closure of R in K. Its field of fractions in K.

(b) Consider

$$E : Y^2Z = X^3 + bZ^3,$$

and let $\rho = e^{2\pi i/3} = 1^{\frac{1}{3}}$. Then $(x : y : z) \mapsto (\rho x : y : z)$ is an endomorphism of E of order 3 of E. In this case, E has j-invariant is 0.

ASIDE 3.19 A complex number α is said to be ***algebraic*** if it is algebraic over $\mathbb{Q}$, and is otherwise said to be ***transcendental***. There is a general philosophy that a transcendental meromorphic function f should take transcendental values at the algebraic points in $\mathbb{C}$, except at some "special" points, where it has interesting "special values". We illustrate this for two functions.

(a) Define $e(z) = e^{2\pi iz}$. If z is algebraic but not rational, then $e(z)$ is transcendental[3]. On the other hand, if $z \in \mathbb{Q}$, then $e(z)$ is algebraic — in fact, it is a root of 1, and $\mathbb{Q}[e(z)]$ is a finite extension of $\mathbb{Q}$ with abelian Galois group. The famous Kronecker-Weber theorem states that, conversely, every abelian extension of $\mathbb{Q}$ is contained in $\mathbb{Q}[e(\frac{1}{m})]$ for some m.

(b) Let $\tau \in \mathbb{H}$ be algebraic. If τ generates a quadratic extension of $\mathbb{Q}$, then $j(\tau)$ is algebraic, and otherwise $j(\tau)$ is transcendental (the second statement was proved by Siegel in 1949).

In fact, when $[\mathbb{Q}[\tau]\colon\mathbb{Q}] = 2$, one can say much more. Assume that $\mathbb{Z}[\tau]$ is the ring of integers in $K \overset{\text{def}}{=} \mathbb{Q}[\tau]$. Then $j(\tau)$ is an algebraic integer, and

$$[\mathbb{Q}[j(\tau)] : \mathbb{Q}] = [K[j(\tau)] : K] = h_K$$

where h_K is the class number of K. Moreover, $K[j(\tau)]$ is the Hilbert class field of K (the largest abelian extension of K unramified at all primes of K including the infinite primes).

The characteristic polynomial of an endomorphism

When Λ is a free module over some ring R and $\alpha\colon \Lambda \to \Lambda$ is R-linear, $\det(\alpha)$ denotes the determinant of the matrix of α relative to some basis for Λ — it is independent of the choice of the basis.

LEMMA 3.20 *Let Λ be a free $\mathbb{Z}$-module of finite rank, and let $\alpha\colon \Lambda \to \Lambda$ be a $\mathbb{Z}$-linear map with nonzero determinant.*

(a) *The cokernel of α is finite, with order equal to $|\det(\alpha)|$.*

[3] More generally, if α and β are algebraic, $\alpha \neq 0, 1$, and β is irrational, then α^β is transcendental — Hilbert stated this as the seventh of his famous problems (International Congress 1900), and Gelfond and Schneider proved it in 1934. It implies the statement for e because $e(z) = (e^{\pi i})^{2z}$. About 1920 Hilbert gave a lecture in which he said that he was very hopeful that the Riemann hypothesis would be proved within his lifetime, that perhaps the youngest members of his audience would live to see Fermat's Last Theorem proved, but that no one in the audience would see the transcendence of $2^{\sqrt{2}}$ established (Tijdeman 1976, p243). He was close with Fermat's Last Theorem.

(b) *The kernel of the map*

$$\tilde{\alpha}: (\Lambda \otimes \mathbb{Q})/\Lambda \to (\Lambda \otimes \mathbb{Q})/\Lambda$$

defined by α *is finite with order equal to* $|\det(\alpha)|$.

PROOF. (a) Let $e_1, \ldots, e_m$ be a basis for Λ. A standard result on commutative groups shows that there exists a basis $e'_1, \ldots, e'_m$ for Λ such that $\varphi(e_i) = n_i e'_i$ for some $n_i \in \mathbb{Z}$, $n_i > 0$, and so the cokernel of α is finite, with order equal to $n_1 \cdots n_m$. The matrix of α with respect to the bases $e_1, \ldots e_m$ and $e'_1, \ldots e'_m$ is $\mathrm{diag}(n_1, \ldots, n_m)$. As the transition matrix from one basis to the second has determinant ± 1, we see that

$$\det(\alpha) = \pm\det(\mathrm{diag}(n_1, \ldots, n_m)) = n_1 \cdots n_m.$$

(b) Consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda \otimes \mathbb{Q} & \longrightarrow & (\Lambda \otimes \mathbb{Q})/\Lambda & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha\otimes 1} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda \otimes \mathbb{Q} & \longrightarrow & (\Lambda \otimes \mathbb{Q})/\Lambda & \longrightarrow & 0.
\end{array}$$

Because $\det(\alpha) \neq 0$, the middle vertical map is an isomorphism. Therefore the snake lemma gives an isomorphism

$$\mathrm{Ker}(\tilde{\alpha}) \to \mathrm{Coker}(\alpha),$$

and so (b) follows from (a). □

We apply this to an elliptic curve E over $\mathbb{C}$. Then $E(\mathbb{C}) = \mathbb{C}/\Lambda$ for some lattice Λ, and $E(\mathbb{C})_{\mathrm{tors}} = \mathbb{Q}\Lambda/\Lambda$ where

$$\mathbb{Q}\Lambda = \{r\lambda \in \mathbb{C} \mid r \in \mathbb{Q}, \lambda \in \Lambda\} = \{z \in \mathbb{C} \mid mz \in \Lambda \text{ some } m \in \mathbb{Z}\} \simeq \mathbb{Q}\otimes_{\mathbb{Z}}\Lambda.$$

Let α be a nonzero endomorphism of E, and let n be the order of its kernel on $E(\mathbb{C})$ (which equals its kernel on $E(\mathbb{C})_{\mathrm{tors}}$). Because $\alpha(\mathbb{C})$ is a homomorphism of groups, it is $n : 1$, and so α has degree n (see I 4.24). Now the lemma implies the following statement.

PROPOSITION 3.21 *The degree of a nonzero endomorphism* α *of an elliptic curve* E *with* $E(\mathbb{C}) = \mathbb{C}/\Lambda$ *is the determinant of* α *acting on* Λ.

We wish to restate this more algebraically. Let E be an elliptic curve over an algebraically closed field k, and let ℓ be a prime not equal to the characteristic of k. Then $E(k)_{\ell^n} \approx (\mathbb{Z}/\ell^n\mathbb{Z})^2$. The ***Tate module*** $T_\ell E$ of E is defined to be

$$T_\ell E = \varprojlim E(k)_{\ell^n}.$$

Thus, it is a free $\mathbb{Z}_\ell$-module of rank 2 such that $T_\ell E/\ell^n T_\ell E \simeq E(k)_{\ell^n}$ for all n. For example, if $k = \mathbb{C}$ and $E(\mathbb{C}) = \mathbb{C}/\Lambda$, then

$$E(\mathbb{C})_{\ell^n} = \tfrac{1}{\ell^n}\Lambda/\Lambda = \Lambda/\ell^n\Lambda = \Lambda \otimes (\mathbb{Z}/\ell^n\mathbb{Z}),$$

and so

$$T_\ell E = \Lambda \otimes \mathbb{Z}_\ell.$$

PROPOSITION 3.22 *Let E and ℓ be as above. For any nonzero endomorphism α of E,*

$$\det(\alpha|T_\ell E) = \deg\alpha.$$

PROOF. When $k = \mathbb{C}$, then the statement follows from the above discussion. For k of characteristic zero, it follows from the case $k = \mathbb{C}$. For a proof for an arbitrary k in a more general setting, see Milne 1986a, 12.9. □

COROLLARY 3.23 *Let $X^2 + cX + d$ be the characteristic polynomial of α acting on $T_\ell E$. Then*

(a) *c, d lie in $\mathbb{Z}$ and are independent of ℓ (in fact, $d = \deg(\alpha)$),*
(b) *$c^2 - 4d \geq 0$, and*
(c) *$\alpha^2 + c\alpha + d = 0$ (as an endomorphism of E).*

PROOF. By definition $X^2 + cX + d = \det(X - \alpha|T_\ell E)$. Therefore,

$$\begin{aligned} d &= \det(-\alpha|T_\ell E) = \deg(\alpha) \in \mathbb{Z} \\ 1 + c + d &= \det(1_E - \alpha|T_\ell E) = \deg(1_E - \alpha) \in \mathbb{Z}, \end{aligned}$$

which proves (a). As neither $\deg(\alpha)$ or $\deg(1_E - \alpha)$ depends on ℓ, nor do c or d. By Cayley's theorem, $\alpha^2 - c\alpha + d$ acts as the zero map on $T_\ell E$. Therefore it is zero on all points of ℓ-power order in $E(k^{\mathrm{al}})$, which implies that it is zero on E. □

EXERCISE 3.24 (a) Prove that, for all z_1, z_2,

$$\begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_1+z_2) & -\wp'(z_1+z_2) & 1 \end{vmatrix} = 0.$$

(b) Compute sufficiently many initial terms for the Laurent expansions of $\wp'(z)$, $\wp'(z)^2$, etc., to verify the equation in Proposition 2.6.

Chapter IV

The Arithmetic of Elliptic Curves

The fundamental theorem proved in this chapter is the finite basis, or Mordell-Weil, theorem.

THEOREM (FINITE BASIS) *For any elliptic curve E over a number field K, $E(K)$ is finitely generated.*

The theorem was proved by Mordell (1922) when $K = \mathbb{Q}$, and for all number fields by Weil in his thesis (1928). Weil in fact proved a much more general result, namely, he showed that for any nonsingular projective curve C over a number field K, the group $\mathrm{Pic}^0(C)$ is finitely generated. For an elliptic curve, $\mathrm{Pic}^0(C) = C(K)$ (see I 4.10), and, for a general curve $\mathrm{Pic}^0(C) = J(K)$ where J is the jacobian variety of C. Thus, once the jacobian variety of a curve over a number field had been defined, Weil's result proved the finite basis theorem for it.

The first step in proving the theorem is to prove a weaker result:

THEOREM (WEAK FINITE BASIS) *For any elliptic curve E over a number field K and any integer n, $E(K)/nE(K)$ is finite.*

Clearly, for an abelian group M,

$$M \text{ finitely generated} \implies M/nM \text{ finite for all } n > 1,$$

but the converse is false. For example, $(\mathbb{Q}, +)$ has the property that $\mathbb{Q} = n\mathbb{Q}$, but the elements of any finitely generated subgroup of $\mathbb{Q}$ have bounded denominators, and so $\mathbb{Q}$ is not finitely generated.

We assume now that $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite, and sketch how one deduces that $E(\mathbb{Q})$ is finitely generated. Recall that the height of a point $P \in \mathbb{P}^2(\mathbb{Q})$ is $H(P) = \max(|a|, |b|, |c|)$ where $P = (a : b : c)$ and a, b, c have been chosen to be integers with no common factor. We use H to define a canonical height function h on $E(\mathbb{Q})$. Let $P_1, \ldots, P_s \in E(\mathbb{Q})$ be a set of representatives for the elements of $E(\mathbb{Q})/2E(\mathbb{Q})$. Then any $Q \in E(\mathbb{Q})$ can be written

$$Q = P_i + 2Q'$$

for some i and for some $Q' \in E(\mathbb{Q})$. We shall show that h has the property that then $h(Q') < h(Q)$ provided $h(Q)$ is greater than some fixed constant C. If $h(Q') > C$, we can repeat the argument with Q', etc., to obtain

$$Q = P_i + 2Q' = P_i + 2(P_{i'} + 2Q'') = \cdots .$$

Let $Q_1, \ldots, Q_t$ be the set of points in $E(\mathbb{Q})$ with height $\leq C$. Then the above equation exhibits Q as a linear combination of P_is and a Q_j, and so the P_is and Q_js generate $E(\mathbb{Q})$.

NOTES The argument in the last paragraph is called "proof by descent". Fermat is generally credited with originating this method in his proof of Fermat's Last Theorem for the exponent 4 (which *was* short enough to fit in the margin). However, in some sense it goes back to the early Greek mathematicians. Consider the proof that $Y^2 = 2X^2$ has no solution in integers. Define the height of a pair (m, n) of integers to be $\max(|m|, |n|)$. The usual argument shows that if (m, n) is one solution to the equation, then there exists another of smaller height, which leads to a contradiction.

1 Group cohomology

In proving the weak finite basis theorem, and also later in the study of the Tate-Shafarevich group, we shall use a little of the theory of the cohomology of groups.

Cohomology of finite groups

Let G be a finite group, and let M be an abelian group. An ***action*** of G on M is a map $G \times M \to M$ such that

(a) $\sigma(m + m') = \sigma m + \sigma m'$ for all $\sigma \in G, m, m' \in M$;
(b) $(\sigma\tau)(m) = \sigma(\tau m)$ for all $\sigma, \tau \in G, m \in M$;
(c) $1_G m = m$ for all $m \in M$.

Thus, to give an action of G on M is the same as to give a homomorphism $G \to \mathrm{Aut}(M)$. A ***G-module*** is an abelian group together with an action of G.

EXAMPLE 1.1 Let L be a finite Galois extension of a field K with Galois group G, and let E be an elliptic curve over K. Then L, $L^\times$, and $E(L)$ all have obvious G-actions.

For a G-module M, we define

$$H^0(G, M) = M^G = \{m \in M \mid \sigma m = m, \text{all } \sigma \in G\}.$$

In the above examples,

$$H^0(G, L) = K, \quad H^0(G, L^\times) = K^\times \text{, and } H^0(G, E(L)) = E(K).$$

A ***crossed homomorphism*** is a map $f: G \to M$ such that

$$f(\sigma\tau) = f(\sigma) + \sigma f(\tau), \quad \text{all } \sigma, \tau \in G.$$

Note that the condition implies that $f(1) = f(1 \cdot 1) = f(1) + f(1)$, and so $f(1) = 0$. For any $m \in M$, we obtain a crossed homomorphism by putting

$$f(\sigma) = \sigma m - m, \qquad \text{all } \sigma \in G.$$

Such crossed homomorphisms are said to be ***principal***. The sum and difference of two crossed homomorphisms is again a crossed homomorphism, and the sum and difference of two principal crossed homomorphisms is again principal. Thus we can define

$$H^1(G, M) = \frac{\{\text{crossed homomorphisms}\}}{\{\text{principal crossed homomorphisms}\}}$$

(quotient abelian group). There are also cohomology groups $H^n(G, M)$ for $n > 1$, but we won't need them.

EXAMPLE 1.2 When G acts trivially on M, i.e., $\sigma m = m$ for all $\sigma \in G$ and $m \in M$, a crossed homomorphism is simply a homomorphism, and every principal crossed homomorphism is zero. Hence $H^1(G, M) = \mathrm{Hom}(G, M)$.

PROPOSITION 1.3 *Let L be a finite Galois extension of K with Galois group G; then $H^1(G, L^\times) = 0$, i.e., every crossed homomorphism $G \to L^\times$ is principal.*

PROOF. Let f be a crossed homomorphism $G \to L^\times$. In multiplicative notation, this means that

$$f(\sigma\tau) = f(\sigma) \cdot \sigma(f(\tau)), \quad \sigma, \tau \in G,$$

and we have to find a $\gamma \in L^\times$ such that $f(\sigma) = \sigma\gamma/\gamma$ for all $\sigma \in G$. Because the $f(\tau)$ are nonzero, Dedekind's theorem on the independence of characters (FT, 5.14) shows that

$$\sum\nolimits_{\tau \in G} f(\tau)\tau : L \to L$$

is not the zero map, i.e., that there exists an $\alpha \in L$ such that

$$\beta \overset{\text{def}}{=} \sum_{\tau \in G} f(\tau)\tau\alpha \neq 0.$$

But then, for $\sigma \in G$,

$$\begin{aligned}
\sigma\beta &= \sum_{\tau \in G} \sigma(f(\tau)) \cdot \sigma\tau(\alpha) \\
&= \sum_{\tau \in G} f(\sigma)^{-1} \cdot f(\sigma\tau) \cdot \sigma\tau(\alpha) \\
&= f(\sigma)^{-1} \sum_{\tau \in G} f(\sigma\tau) \cdot \sigma\tau(\alpha) \\
&= f(\sigma)^{-1}\beta,
\end{aligned}$$

and so $f(\sigma) = \beta/\sigma\beta = \sigma(\beta^{-1})/\beta^{-1}$. □

COROLLARY 1.4 *A point $P = (x_0 : \cdots : x_n) \in \mathbb{P}^n(L)$ is fixed by G if and only if it is represented by an $n+1$-tuple in K.*

PROOF. Suppose that $\sigma P = P$ for all $\sigma \in G$. Then

$$\sigma(x_0, \ldots, x_n) = c(\sigma)(x_0, \ldots, x_n)$$

for some $c(\sigma) \in L^\times$. One checks that $\sigma \mapsto c(\sigma)$ is a crossed homomorphism, and so $c(\sigma) = c/\sigma c$ for some $c \in L^\times$. Hence

$$\sigma(cx_0, \ldots, cx_n) = (cx_0, \ldots, cx_n),$$

and so the cx_i lie in K. □

PROPOSITION 1.5 *For any exact sequence of G-modules*

$$0 \to M \to N \to P \to 0,$$

there is a canonical exact sequence

$$\begin{aligned}
0 \to H^0(G, M) \to H^0(G, N) \to H^0(G, P) \xrightarrow{\delta} \\
H^1(G, M) \to H^1(G, N) \to H^1(G, P)
\end{aligned}$$

PROOF. The map δ is defined as follows: let $p \in P^G$; there exists an $n \in N$ mapping to p, and $\sigma n - n \in M$ for all $\sigma \in G$; the map $\sigma \mapsto \sigma n - n : G \to M$ is a crossed homomorphism, whose class we define to be $\delta(p)$. Another n' mapping to p gives rise to a crossed homomorphism differing from the first by the principal crossed homomorphism $\sigma \mapsto \sigma(n' - n) - (n' - n)$, and so $\delta(p)$ is well-defined. The rest of the proof is routine (and should be written out by any reader unfamiliar with group cohomology). □

Let H be a subgroup of G. The restriction map $f \mapsto f|H$ defines a homomorphism Res: $H^1(G, M) \to H^1(H, M)$.

PROPOSITION 1.6 *If G has order m, then $mH^1(G, M) = 0$.*

PROOF. In general, if H is a subgroup of finite index m in G, there is a corestriction map Cor: $H^1(H, M) \to H^1(G, M)$ such that $\mathrm{Cor} \circ \mathrm{Res} = m$. Apply this with $H = 1$. □

REMARK 1.7 Let H be a normal subgroup of a group G, and let M be a G-module. Then M^H is a G/H-module, and a crossed homomorphism $f\colon G/H \to M^H$ defines a crossed homomorphism $G \to M$ by composition:

$$\begin{tikzcd} G \arrow[r, dotted] \arrow[d] & M \\ G/H \arrow[r, "f"] & M^H \arrow[u, hook] \end{tikzcd}$$

In this way we obtain an "inflation" homomorphism

$$\mathrm{Inf}\colon H^1(G/H, M^H) \to H^1(G, M),$$

and one verifies easily that the sequence

$$0 \to H^1(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^1(G, M) \xrightarrow{\mathrm{Res}} H^1(H, M)$$

is exact.

Cohomology of infinite Galois groups

Let k be a perfect field, and let k^{al} be an algebraic closure of k. The group G of automorphisms of k^{al} fixing the elements of k has a natural topology, called the ***Krull topology***, for which a subgroup is open if and only if it is the subgroup fixing[1] a finite extension of k. When endowed with its Krull topology, G is called the ***Galois group*** of k^{al} over k. The open subgroups of G form a neighbourhood base for 1_G. As always, open subgroups are closed, and so any intersection of open subgroups is closed; conversely, every closed subgroup is an intersection of open subgroups. The group G is compact, and so any open subgroup of G is of finite index. The usual Galois theory extends to give a one-to-one correspondence between the intermediate fields K, $k \subset K \subset k^{\mathrm{al}}$, and the *closed* subgroups of G, under which the fields of finite degree over k correspond to open subgroups of G. See FT, §7, for the details.

A G-module M is said to be ***discrete*** if the map $G \times M \to M$ is continuous relative to the discrete topology on M and the Krull topology on G. This is equivalent to requiring that

$$M = \bigcup_H M^H, \quad H \text{ open in } G,$$

[1] By which I mean "fixing each elements of".

i.e., to requiring that every element of M be fixed by the subgroup of G fixing some finite extension of k. For example, $M = k^{\mathrm{al}}$, $M = k^{\mathrm{al}\times}$, and $M = E(k^{\mathrm{al}})$ are all discrete G-modules because

$$k^{\mathrm{al}} = \bigcup K, \quad k^{\mathrm{al}\times} = \bigcup K^{\times}, E(k^{\mathrm{al}}) = \bigcup E(K)$$

where, in each case, the union runs over the finite extensions K of k contained in k^{al}.

When M is discrete, a crossed homomorphism $f\colon G \to M$ will be continuous if and only if f is constant on the cosets of some open normal subgroup H of G, so that f arises by inflation from a crossed homomorphism $G/H \to M$. Every principal crossed homomorphism is continuous because every element of M is fixed by an open normal subgroup of G.

For an infinite Galois group G and a discrete G-module M, we define $H^1(G, M)$ to be the group of continuous crossed homomorphisms $f\colon G \to M$ modulo the subgroup of principal crossed homomorphisms. With this definition

$$H^1(G, M) = \varinjlim_H H^1(G/H, M^H)$$

where H runs through the open normal subgroups of G. Explicitly, this means that:

(a) $H^1(G, M)$ is the union of the images of the inflation maps $\mathrm{Inf}\colon H^1(G/H, M^H) \to H^1(G, M)$, where H runs over the open normal subgroup of G;

(b) an element $\gamma \in H^1(G/H, M^H)$ maps to zero in $H^1(G, M)$ if and only if it maps to zero $H^1(G/H', M^{H'})$ for some open normal subgroup H' of G contained in H.

In particular, the group $H^1(G, M)$ is torsion (1.6).

The proofs of the statements in the last three paragraphs are easy, and should be written out by any reader unfamiliar with them.

EXAMPLE 1.8 (a) Proposition 1.3 shows that

$$H^1(G, k^{\mathrm{al}\times}) = \varinjlim_K H^1(\mathrm{Gal}(K/k), K^{\times}) = 0.$$

(b) For a field L and an integer $n \geq 1$, let

$$\mu_n(L) = \{\zeta \in L^{\times} \mid \zeta^n = 1\}.$$

From the exact sequence[2]

$$1 \to \mu_n(k^{\mathrm{al}}) \to k^{\mathrm{al}\times} \xrightarrow{n} k^{\mathrm{al}\times} \to 1$$

[2]We are using that k is perfect.

we obtain an exact sequence of cohomology groups

$$1 \to \mu_n(k) \to k^\times \xrightarrow{n} k^\times \to H^1(G, \mu_n(k^{\mathrm{al}}) \to 1,$$

and hence a canonical isomorphism

$$H^1(G, \mu_n(k^{\mathrm{al}})) \simeq k^\times / k^{\times n}.$$

When k is a number field and $n > 1$, this group is infinite. For example, the numbers

$$(-1)^{\varepsilon(\infty)} \prod_{p \text{ prime}} p^{\varepsilon(p)},$$

where each exponent is 0 or 1 and all but finitely many are zero, form a set of representatives for the elements of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, which is therefore an infinite-dimensional vector space over $\mathbb{F}_2$.

(c) If G acts trivially on M, then $H^1(G, M)$ is the set of continuous homomorphisms $\alpha \colon G \to M$. Because M is discrete, the kernel of such a homomorphism is open; if K is its fixed field, then α defines an injective homomorphism $\mathrm{Gal}(K/k) \to M$.

For an elliptic curve E over k, we shorten $H^i(\mathrm{Gal}(k^{\mathrm{al}}/k), E(k^{\mathrm{al}}))$ to $H^i(k, E)$.

1.9 Let E be an elliptic curve over $\mathbb{Q}$, let $\mathbb{Q}^{\mathrm{al}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and choose an algebraic closure $\mathbb{Q}_p^{\mathrm{al}}$ for $\mathbb{Q}_p$. The embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ extends to an embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$,

$$\begin{array}{ccc} \mathbb{Q}^{\mathrm{al}} & \longrightarrow & \mathbb{Q}_p^{\mathrm{al}} \\ \uparrow & & \uparrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q}_p. \end{array}$$

The action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p)$ on $\mathbb{Q}^{\mathrm{al}} \subset \mathbb{Q}_p^{\mathrm{al}}$ defines a homomorphism

$$\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}).$$

Hence any crossed homomorphism $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q}) \to E(\mathbb{Q}^{\mathrm{al}})$ defines (by composition) a crossed homomorphism $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{al}}/\mathbb{Q}_p) \to E(\mathbb{Q}_p^{\mathrm{al}})$. In this way, we obtain a homomorphism

$$H^1(\mathbb{Q}, E) \to H^1(\mathbb{Q}_p, E)$$

which (slightly surprisingly) is independent of the choice of the embedding $\mathbb{Q}^{\mathrm{al}} \hookrightarrow \mathbb{Q}_p^{\mathrm{al}}$. A similar remark applies to the cohomology groups of μ_n and E_n. See, for example, CFT, II 1.27d. In §7 below, we shall give a more natural geometric interpretation of these "localization" homomorphisms.

EXAMPLE 1.10 Let k be a perfect field with algebraic closure $\bar{k}$, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let C be a curve over k, and let $\bar{C}$ be the corresponding curve over $\bar{k}$. Let $\mathrm{Pic}(C) = \mathrm{Pic}(\bar{C})^{\Gamma}$. We examine whether $\mathrm{Pic}(C)$ is the set of divisor classes on C. From the exact sequence

$$0 \to \bar{k}(\bar{C})^{\times}/\bar{k}^{\times} \to \mathrm{Div}(\bar{C}) \to \mathrm{Pic}(\bar{C}) \to 0$$

of Γ-modules, we get an exact cohomology sequence

$$\left(\bar{k}(\bar{C})^{\times}/\bar{k}^{\times}\right)^{\Gamma} \to \mathrm{Div}(\bar{C})^{\Gamma} \to \left(\mathrm{Pic}(\bar{C})\right)^{\Gamma} \to H^1(\Gamma, \bar{k}(\bar{C})^{\times}/\bar{k}^{\times}),$$

and from the exact sequence

$$0 \to \bar{k}^{\times} \to \bar{k}(\bar{C})^{\times} \to \bar{k}(\bar{C})^{\times}/\bar{k}^{\times} \to 0,$$

we get an exact sequence

$$k(C)^{\times} \to \left(\bar{k}(\bar{C})^{\times}/\bar{k}^{\times}\right)^{\Gamma} \to H^1(\Gamma, \bar{k}(\bar{C})^{\times}) \to H^1(\Gamma, \bar{k}(\bar{C})^{\times}/\bar{k}^{\times}) \to H^2(\Gamma, \bar{k}^{\times}).$$

On combining these sequences, and using that $\mathrm{Div}(\bar{C})^{\Gamma} = \mathrm{Div}(C)$ (see p. 38) and that $H^1(\Gamma, \bar{k}(\bar{C})^{\times}) = 0$ (1.8), we obtain an exact sequence

$$k(C)^{\times} \to \mathrm{Div}(C) \to \mathrm{Pic}(C) \to H^2(\Gamma, \bar{k}^{\times}).$$

Now $H^2(\Gamma, \bar{k}^{\times})$ is the Brauer group of k (CFT, Chap. IV). Thus, $\mathrm{Pic}(C)$ is the group of divisors on C modulo principal divisors if the Brauer group of k is zero, for example, if k is finite (Wedderburn's theorem, ibid. 4.1).

2 The Selmer and Tate-Shafarevich groups

We now set $\mathbb{Q}_{\infty} = \mathbb{R}$.

LEMMA 2.1 *For every elliptic curve E over an algebraically closed field k and integer n, the map $P \mapsto nP\colon E(k) \to E(k)$ is surjective.*

PROOF. In characteristic zero, there is an elementary proof exploiting the fact that we know that $n\colon E(\mathbb{C}) \to E(\mathbb{C})$ is surjective (by III 3.10): as in the proof of (III 3.12), we may assume that $k \subset \mathbb{C}$; given a point $P \in E(k)$, in order to find a point Q such that $nQ = P$ we have to solve a pair of polynomial equations in the variables X, Y (see III 3.14); because these equations have a solution in $\mathbb{C}$, the polynomials generate a proper ideal in $k[X, Y]$, and hence have a solution in k by the Hilbert Nullstellensatz.

Alternatively, by using a little algebraic geometry we can give a proof valid in any characteristic. Because k is algebraically closed, we can identify $E(k)$ with the underlying set of E regarded as an algebraic variety. The map $n\colon E \to E$ is regular and E is connected and complete, and so the image of n is connected and closed. Therefore, it is either a point or the whole of E. The first is impossible, because not all points on E are killed by n. $\square$

From the lemma we obtain an exact sequence

$$0 \to E_n(\mathbb{Q}^{\mathrm{al}}) \to E(\mathbb{Q}^{\mathrm{al}}) \xrightarrow{n} E(\mathbb{Q}^{\mathrm{al}}) \to 0$$

and an exact cohomology sequence

$$0 \to E_n(\mathbb{Q}) \to E(\mathbb{Q}) \xrightarrow{n} E(\mathbb{Q}) \to H^1(\mathbb{Q}, E_n) \to H^1(\mathbb{Q}, E) \xrightarrow{n} H^1(\mathbb{Q}, E),$$

from which we extract the sequence

$$\boxed{0 \to E(\mathbb{Q})/nE(\mathbb{Q}) \to H^1(\mathbb{Q}, E_n) \to H^1(\mathbb{Q}, E)_n \to 0.} \tag{22}$$

Here, as usual, $H^1(\mathbb{Q}, E)_n$ is the group of elements in $H^1(\mathbb{Q}, E)$ killed by n. If $H^1(\mathbb{Q}, E_n)$ were finite, then we could deduce that $E(\mathbb{Q})/nE(\mathbb{Q})$ is finite, but it needn't be. For example, if all the points of order 2 on E have coordinates in $\mathbb{Q}$, so that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ acts trivially on $E_2(\mathbb{Q}^{\mathrm{al}}) \approx (\mathbb{Z}/2\mathbb{Z})^2$, then

$$H^1(\mathbb{Q}, E_2) \approx H^1(\mathbb{Q}, \mu_2 \times \mu_2) \simeq (\mathbb{Q}^\times/\mathbb{Q}^{\times 2}) \times (\mathbb{Q}^\times/\mathbb{Q}^{\times 2}),$$

which is infinite (see 1.8b). Instead, we proceed as follows. When we consider E as an elliptic curve over $\mathbb{Q}_p$ we obtain a similar exact sequence, and there is a commutative diagram (see 1.9):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(\mathbb{Q})/nE(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, E_n) & \longrightarrow & H^1(\mathbb{Q}, E)_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E(\mathbb{Q}_p)/nE(\mathbb{Q}_p) & \longrightarrow & H^1(\mathbb{Q}_p, E_n) & \longrightarrow & H^1(\mathbb{Q}_p, E)_n & \longrightarrow & 0.
\end{array}$$

We want to replace $H^1(\mathbb{Q}, E_n)$ with a subset that contains the image of $E(\mathbb{Q})/nE(\mathbb{Q})$ but which we shall be able to prove is finite. We do this as follows: if $\gamma \in H^1(\mathbb{Q}, E_n)$ comes from an element of $E(\mathbb{Q})$, then certainly its image γ_p in $H^1(\mathbb{Q}_p, E_n)$ comes from an element of $E(\mathbb{Q}_p)$. This suggests defining

$$\begin{aligned}
S^{(n)}(E/\mathbb{Q}) &= \{\gamma \in H^1(\mathbb{Q}, E_n) \mid \forall p, \gamma_p \text{ comes from } E(\mathbb{Q}_p)\} \\
&= \mathrm{Ker}\left(H^1(\mathbb{Q}, E_n) \to \prod_{p=2,3,5,\ldots,\infty} H^1(\mathbb{Q}_p, E)\right).
\end{aligned}$$

The group $S^{(n)}(E/\mathbb{Q})$ is called the ***Selmer group***. In the same spirit, we define the ***Tate-Shafarevich group*** to be

$$Ш(E/\mathbb{Q}) = \mathrm{Ker}\left(H^1(\mathbb{Q}, E) \to \prod_{p=2,3,5,\ldots,\infty} H^1(\mathbb{Q}_p, E)\right).$$

It is a torsion group. Later we shall give a geometric interpretation of $\text{Ш}(E/\mathbb{Q})$ which shows that it provides a measure of the failure of the Hasse principle for curves of genus 1. Also, we shall see that these definitions extend to elliptic curves over any number field.

The next lemma is as trivial to prove as it is useful.

LEMMA 2.2 *From any pair of maps of abelian groups (or modules etc.)*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

there is an exact (kernel-cokernel) sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Ker}(\alpha) \longrightarrow \operatorname{Ker}(\beta \circ \alpha) &\xrightarrow{\alpha} \operatorname{Ker}(\beta) \\ \longrightarrow \operatorname{Coker}(\alpha) \longrightarrow \operatorname{Coker}(\beta \circ \alpha) &\xrightarrow{\alpha} \operatorname{Coker}(\beta) \longrightarrow 0. \end{aligned}$$

PROOF. Exercise. □

When we apply the lemma to the maps

$$H^1(\mathbb{Q}, E_n) \to H^1(\mathbb{Q}, E)_n \to \prod_{p=2,3,\ldots,\infty} H^1(\mathbb{Q}_p, E)_n,$$

we obtain the fundamental exact sequence

$$\boxed{0 \to E(\mathbb{Q})/nE(\mathbb{Q}) \to S^{(n)}(E/\mathbb{Q}) \to \text{Ш}(E/\mathbb{Q})_n \to 0.} \qquad (23)$$

We shall prove $E(\mathbb{Q})/nE(\mathbb{Q})$ to be finite by showing that $S^{(n)}(E/\mathbb{Q})$ is finite.

NOTES The group $H^1(k, E)$ is sometimes called the ***Châtelet***, or ***Weil-Châtelet***, ***group***, and denoted $\mathrm{WC}(E/k)$ (Lang and Tate 1958; Tate 1958). I don't know who introduced the terminology for the Selmer and Tate-Shafarevich groups, but Cassels (1991, p. 109) admits responsibility for denoting the second group by the Cyrillic letter Ш (sha). "Tate-Shafarevich" is the traditional name for $\text{Ш}(E/k)$, and is the correct alphabetical order in Cyrillic, but some more recent authors reverse the order.

3 The finiteness of the Selmer group

In this section, we prove:

THEOREM 3.1 *For any elliptic curve E over a number field $\mathbb{Q}$ and any integer n, the Selmer group $S^{(n)}(E/\mathbb{Q})$ is finite (and, in fact, computable).*

Preliminaries

LEMMA 3.2 *Let E be an elliptic curve over $\mathbb{Q}_p$ with good reduction, and let n be an integer not divisible by p. A point P in $E(\mathbb{Q}_p)$ is of the form nQ for some $Q \in E(\mathbb{Q}_p)$ if and only if its image $\bar{P}$ in $E(\mathbb{F}_p)$ is of the form $n\bar{Q}$ for some $\bar{Q} \in E(\mathbb{F}_p)$.*

PROOF. As $P \mapsto \bar{P}$ is a homomorphism, the necessity is obvious, and the sufficiency follows from a diagram chase in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^1(\mathbb{Q}_p) & \longrightarrow & E(\mathbb{Q}_p) & \longrightarrow & \bar{E}(\mathbb{F}_p) & \longrightarrow & 0 \\ & & \simeq \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & E^1(\mathbb{Q}_p) & \longrightarrow & E(\mathbb{Q}_p) & \longrightarrow & \bar{E}(\mathbb{F}_p) & \longrightarrow & 0. \end{array}$$

using that the first vertical arrow is an isomorphism by (II 4.2). In detail, let $P \in E(\mathbb{Q}_p)$ be such that $\bar{P} = n\bar{Q}$; then $P - nQ$ maps to zero in $\bar{E}(\mathbb{F}_p)$, and so lies in $E^1(\mathbb{Q}_p)$. Therefore, $P - nQ = nQ'$ for some $Q' \in E^1(\mathbb{Q}_p)$, and so $P = n(Q + Q')$. □

Now we shall need a little (local) algebraic number theory. For a finite extension K of $\mathbb{Q}_p$, the integral closure $\mathcal{O}_K$ of $\mathbb{Z}_p$ in K is again a principal ideal domain with a single maximal ideal (π). Thus, $p = \text{unit} \times \pi^e$ for some e, called the ***ramification index*** of K over $\mathbb{Q}_p$. When $e = 1$, so that the maximal ideal is (p), then K is said to be ***unramified*** over $\mathbb{Q}_p$.

LEMMA 3.3 *For any finite extension k of $\mathbb{F}_p$, there exists an unramified extension K of $\mathbb{Q}_p$ of degree $[k\colon \mathbb{F}_p]$ such that $\mathcal{O}_K/p\mathcal{O}_K = k$.*

PROOF. Let α be a primitive element for k over $\mathbb{F}_p$, and let $f_0(X)$ be the minimum polynomial for α over $\mathbb{F}_p$, so that

$$k = \mathbb{F}_p[\alpha] \simeq F_p[X]/(f_0(X)).$$

For any monic polynomial $f(X) \in \mathbb{Z}_p[X]$ such that $f_0(X) = f(X) \mod p$, the field $K = \mathbb{Q}_p[X]/(f(X))$ has the required properties (see ANT, 7.41). □

REMARK 3.4 Let $K \supset \mathcal{O}_K \to k$ be as in the lemma. Let q be the order of k, so that the elements of k are the roots of $X^q - X$. Then Hensel's lemma (Theorem 2.12, Chap. I) holds for $\mathcal{O}_K$, and so all the roots of $X^q - X$ in k lift to $\mathcal{O}_K$. Therefore K contains the splitting field of $X^q - X$, and, in fact, equals it.

Let K be as in the Lemma 3.3. Because $\mathcal{O}_K$ is a principal ideal domain with p as its only prime element (up to units), every element α of $K^\times$ can be written uniquely in the form up^m with $u \in \mathcal{O}_K^\times$ and $m \in \mathbb{Z}$. Define $\operatorname{ord}_p(\alpha) = m$. Then ord_p is a homomorphism $K^\times \to \mathbb{Z}$ extending $\operatorname{ord}_p\colon \mathbb{Q}_p^\times \to \mathbb{Z}$.

The theory in II, §4, holds word-for-word with $\mathbb{Q}_p$ replaced by an unramified extension K, except that now

$$E^0(K)/E^1(K) \simeq \bar{E}^{\mathrm{ns}}(k), \quad E^n(K)/E^{n+1}(K) \simeq k.$$

Therefore, Lemma 3.2 remains valid with $\mathbb{Q}_p$ replaced by K and $\mathbb{F}_p$ by k.

Consider an elliptic curve E over $\mathbb{Q}_p$ and an n not divisible by p. Let $P \in E(\mathbb{Q}_p)$. Then $\bar{P} \in n\bar{E}(k)$ for some finite extension k of $\mathbb{F}_p$, and so $P \in nE(K)$ for any unramified extension K of $\mathbb{Q}_p$ with residue field k. We have proved:

LEMMA 3.5 *Let E be an elliptic curve over $\mathbb{Q}_p$ with good reduction, and let n be an integer not divisible by p. For any $P \in E(\mathbb{Q}_p)$, there exists a finite unramified extension K of $\mathbb{Q}_p$ such that $P \in nE(K)$.*

PROPOSITION 3.6 *Let E be an elliptic curve over $\mathbb{Q}$ with discriminant Δ, and let T be the set of prime numbers dividing $2n\Delta$. For any $\gamma \in S^{(n)}(\mathbb{Q})$ and any $p \notin T$, there exists a finite unramified extension K of $\mathbb{Q}_p$ such that γ maps to zero in $H^1(K, E_n)$.*

PROOF. From the definition of the Selmer group, we know that there exists a $P \in E(\mathbb{Q}_p)$ mapping to the image γ_p of γ in $H^1(\mathbb{Q}_p, E_n)$. Since p does not divide 2Δ, E has good reduction at p, and so there exists an unramified extension K of $\mathbb{Q}_p$ such that $P \in nE(K)$, and so γ_p maps to zero in $H^1(K, E_n)$:

$$\begin{array}{ccccc}
E(\mathbb{Q}) & \xrightarrow{n} & E(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, E_n) \\
\downarrow & & \downarrow & & \downarrow \\
E(\mathbb{Q}_p) & \xrightarrow{n} & E(\mathbb{Q}_p) & \longrightarrow & H^1(\mathbb{Q}_p, E_n) \\
\downarrow & & \downarrow & & \downarrow \\
E(K) & \xrightarrow{n} & E(K) & \longrightarrow & H^1(K, E_n).
\end{array}$$

□

Proof of the finiteness in a special case

We prove that $S^{(2)}(E/\mathbb{Q})$ is finite in the case that the points of order 2 on E have coordinates in $\mathbb{Q}$. This condition means that the equation for E has the form:

$$Y^2Z = (X - \alpha Z)(X - \beta Z)(X - \gamma Z), \quad \alpha, \beta, \gamma \in \mathbb{Q}.$$

It implies that

$$E_2(\mathbb{Q}^{\mathrm{al}}) = E_2(\mathbb{Q}) \approx (\mathbb{Z}/2\mathbb{Z})^2 = (\mu_2)^2,$$

all with the trivial action of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$, and so

$$H^1(\mathbb{Q}, E_2) \approx H^1(\mathbb{Q}, \mu_2)^2 \simeq (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2.$$

Let $\gamma \in S^{(2)}(E/\mathbb{Q}) \subset H^1(\mathbb{Q}, E_2)$. For each prime p_0 not dividing 2Δ, there exists a finite unramified extension K of $\mathbb{Q}_{p_0}$ such that γ maps to zero under the vertical arrows:

$$\begin{array}{ccc} H^1(\mathbb{Q}, E_2) & \xrightarrow{\approx} & (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2 \\ \downarrow & & \downarrow \\ H^1(K, E_2) & \xrightarrow{\approx} & (K^\times/K^{\times 2})^2. \end{array}$$

Suppose that

$$\gamma \leftrightarrow \left((-1)^{\varepsilon(\infty)} \prod\nolimits_p p^{\varepsilon(p)}, (-1)^{\varepsilon'(\infty)} \prod\nolimits_p p^{\varepsilon'(p)}\right), \quad 0 \le \varepsilon(p), \varepsilon'(p) \le 1,$$

under the top isomorphism. Now

$$\mathrm{ord}_{p_0}\left((-1)^{\varepsilon(\infty)} \prod\nolimits_p p^{\varepsilon(p)}\right) = \varepsilon(p_0),$$

and so if $(-1)^{\varepsilon(\infty)} \prod p^{\varepsilon(p)}$ is a square in K, then $\varepsilon(p_0) = 0$. Therefore the only p that can occur in the factorizations are those dividing 2Δ. This allows only finitely many possibilities for γ.

REMARK 3.7 It is possible to prove that $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite in this case without mentioning cohomology groups. Consider an elliptic curve

$$Y^2 Z = (X - \alpha Z)(X - \beta Z)(X - \gamma Z), \quad \alpha, \beta, \gamma \in \mathbb{Z}.$$

Define $\varphi_\alpha \colon E(\mathbb{Q})/2E(\mathbb{Q}) \to \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ by

$$\varphi_\alpha((x:y:z)) = \begin{cases} (x/z - \alpha)\mathbb{Q}^{\times 2} & z \neq 0, \quad x \neq \alpha z; \\ (\alpha - \beta)(\alpha - \gamma)\mathbb{Q}^\times & z \neq 0, \quad x = \alpha z \\ \mathbb{Q}^\times & (x:y:z) = (0:1:0). \end{cases}$$

Define φ_β similarly. One can prove directly that φ_α and φ_β are homomorphisms, that the kernel of $(\varphi_\alpha, \varphi_\beta) \colon E(\mathbb{Q}) \to (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2$ is $2E(\mathbb{Q})$, and that $\varphi_\alpha(P)$ and $\varphi_\beta(P)$ are represented by $\pm$ a product of primes dividing 2Δ (see Cassels 1991, Chap. 15).

Proof of the finiteness in the general case

We saw in Chapter I that $\mathbb{Q}$ has one valuation, hence one completion $\mathbb{Q}_p$, for each prime ideal (p) in $\mathbb{Z}$ and one other completion $\mathbb{R}$, which it is convenient to denote $\mathbb{Q}_\infty$. Similarly, a number field L has one valuation, hence completion, for each prime ideal of $\mathcal{O}_L$ and one valuation for each embedding of L into $\mathbb{R}$

or complex-conjugate pair of embeddings of L into $\mathbb{C}$. Write $\mathcal{P}(p)$ for the set of valuations of L extending $|\cdot|_p$. Then,

$$L \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v \in \mathcal{P}(p)} L_v$$

where L_v is the completion of L for v. Let $\mathcal{P} = \bigcup_{p=2,3,\ldots,\infty} \mathcal{P}(p)$.

For an elliptic curve E over L, we define

$$S^{(n)}(E/L) = \operatorname{Ker}\left(H^1(L, E_n) \to \prod_{v \in \mathcal{P}} H^1(L_v, E) \right).$$

For a given n, rather than prove directly that $S^{(n)}(E/\mathbb{Q})$ finite, it turns out that it is more convenient to show that $S^{(n)}(E/L)$ is finite for any suitably large L. The next lemma shows that this then gives us what we want.

LEMMA 3.8 *For any finite Galois extension L of $\mathbb{Q}$ and integer $n \geq 1$, the kernel of*

$$S^{(n)}(E/\mathbb{Q}) \to S^{(n)}(E/L)$$

is finite.

PROOF. Since $S^{(n)}(E/\mathbb{Q})$ and $S^{(n)}(E/L)$ are subgroups of $H^1(\mathbb{Q}, E_n)$ and $H^1(L, E_n)$ respectively, it suffices to prove that the kernel of

$$H^1(\mathbb{Q}, E_n) \to H^1(L, E_n)$$

is finite. But (cf. 1.7), this kernel is $H^1(\operatorname{Gal}(L/\mathbb{Q}), E_n(L))$, which is finite because both $\operatorname{Gal}(L/\mathbb{Q})$ and $E_n(L)$ are finite. □

In the proof of the finiteness in the special case, we used the following facts:

(a) $\mathbb{Q}$ contains a primitive square root of 1;
(b) $E(\mathbb{Q})_2 = E(\mathbb{Q}^{\mathrm{al}})_2$ (by assumption);
(c) for any finite set T of prime numbers, the kernel of

$$r \mapsto (\operatorname{ord}_p(r) \mod 2)\colon \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \to \bigoplus_{p \notin T} \mathbb{Z}/2\mathbb{Z}$$

is finite.

For some finite Galois extension L of $\mathbb{Q}$, L will contain a primitive nth root of 1 and $E(L)$ will contain all the points of order n in $E(\mathbb{Q}^{\mathrm{al}})$. As we explain below, the analogue of (c) for number fields follows from the three fundamental theorems proved in every course on algebraic number theory. Now, the proof of the finiteness in the special case carries over to show that $S^{(n)}(E/L)$ is finite.

REVIEW OF ALGEBRAIC NUMBER THEORY

In the following, L is a finite extension of $\mathbb{Q}$ and $\mathcal{O}_L$ is the ring of all algebraic integers in L.

Every element of $\mathcal{O}_L$ is a product of irreducible (i.e., "unfactorable") elements, but this factorization may not be unique. For example, in $\mathbb{Z}[\sqrt{-5}]$ we have

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

and 2, 3, $1+\sqrt{-5}$, $1-\sqrt{-5}$ are irreducible with no two differing by a unit. The idea of Kummer and Dedekind to remedy this problem was to enlarge the set of numbers with "ideal numbers", now called ideals, to recover unique factorization. For ideals $\mathfrak{a}$ and $\mathfrak{b}$, the set of finite sums $\sum a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$ is an ideal, denoted $\mathfrak{ab}$. One can show that $\mathcal{O}_L$ is a unique factorization domain if and only if it is a principal ideal domain.

THEOREM 3.9 (DEDEKIND) *Every ideal in $\mathcal{O}_L$ can be written uniquely as a product of prime ideals.*

PROOF. See ANT, Theorems 3.6, 3.30. □

For example, in $\mathbb{Z}[\sqrt{-5}]$,

$$(6) = (2, 1 + \sqrt{-5})(2, 1 - \sqrt{-5})(3, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}).$$

For an element $a \in \mathcal{O}_L$ and a prime ideal $\mathfrak{p}$ in $\mathcal{O}_L$, let $\mathrm{ord}_{\mathfrak{p}}(a)$ be the exponent of $\mathfrak{p}$ in the unique factorization of the ideal (a), so that

$$(a) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(a)}.$$

For $x = \frac{a}{b} \in L$, define $\mathrm{ord}_{\mathfrak{p}}(x) = \mathrm{ord}_{\mathfrak{p}}(a) - \mathrm{ord}_{\mathfrak{p}}(b)$. The ***ideal class group*** C of $\mathcal{O}_L$ (or L) is defined to be the cokernel of the homomorphism

$$\begin{array}{ccccccc} L^{\times} & \to & \bigoplus_{\mathfrak{p} \subset \mathcal{O}_L,\ \mathfrak{p} \text{ prime}} \mathbb{Z} & \to & C & \to & 0 \\ x & \mapsto & (\mathrm{ord}_{\mathfrak{p}}(x)). & & & & \end{array}$$

It is 0 if and only if $\mathcal{O}_L$ is a principal ideal domain, and so the size of C is a measure of the failure of unique factorization of elements in $\mathcal{O}_L$.

THEOREM 3.10 (FINITENESS OF THE CLASS NUMBER) *The ideal class group C is finite.*

PROOF. See ANT, Theorem 4.4. □

We next need to understand the group U of units in $\mathcal{O}_L$. For $\mathcal{O}_L = \mathbb{Z}$, $U = \{\pm 1\}$, but already for $\mathcal{O}_L = \mathbb{Z}[\sqrt{2}]$, U is infinite because $\sqrt{2}+1$ is a unit in $\mathbb{Z}[\sqrt{2}]$:

$$(\sqrt{2}+1)(\sqrt{2}-1) = 1.$$

In fact

$$\mathbb{Z}[\sqrt{2}]^\times = \{\pm(1+\sqrt{2})^n \mid n \in \mathbb{Z}\} \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

THEOREM 3.11 (DEDEKIND UNIT THEOREM) *The group U of units of $\mathcal{O}_L$ is finitely generated.*

PROOF. See Theorem 5.1. □

The full theorem gives a formula for the rank of U.

As in any commutative ring, a is a unit in $\mathcal{O}_L$ if and only if $(a) = \mathcal{O}_L$. In our case, this is equivalent to saying that $\operatorname{ord}_{\mathfrak{p}}(a) = 0$ for all prime ideals $\mathfrak{p}$, and so we have an exact sequence

$$0 \to U \to L^\times \to \bigoplus_{\mathfrak{p}} \mathbb{Z} \to C \to 0$$

with U finitely generated and C finite. The theorems reviewed imply a slightly more general result.

COROLLARY 3.12 *When T is a finite set of prime ideals in L, the groups U_T and C_T defined by the exactness of*

$$0 \to U_T \to L^\times \xrightarrow{a \mapsto (\operatorname{ord}_{\mathfrak{p}}(a))} \bigoplus_{\mathfrak{p} \notin T} \mathbb{Z} \to C_T \to 0$$

are, respectively, finitely generated and finite.

PROOF. The kernel-cokernel exact sequence (see 2.2) of

$$L^\times \to \bigoplus_{\text{all } \mathfrak{p}} \mathbb{Z} \xrightarrow{\text{project}} \bigoplus_{\mathfrak{p} \notin T} \mathbb{Z}$$

is an exact sequence

$$0 \to U \to U_T \to \bigoplus_{\mathfrak{p} \in T} \mathbb{Z} \to C \to C_T \to 0.$$

□

COMPLETION OF THE PROOF OF THE FINITENESS OF THE SELMER GROUP

According to the above discussion, the next lemma completes the proof of the finiteness of $S^{(n)}(E/L)$, and hence of $S^{(n)}(E/\mathbb{Q})$.

LEMMA 3.13 *For any finite subset T of $\mathcal{P}$ containing $\mathcal{P}(\infty)$, let N be the kernel of*

$$a \mapsto (\mathrm{ord}_{\mathfrak{p}}(a) \mod n)\colon L^{\times}/L^{\times n} \to \bigoplus_{\mathfrak{p}\notin T} \mathbb{Z}/n\mathbb{Z}.$$

Then there is an exact sequence

$$0 \to U_T/U_T^n \to N \to (C_T)_n$$

PROOF. This can be proved by a diagram chase in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_T & \longrightarrow & L^{\times} & \longrightarrow & \bigoplus_{\mathfrak{p}\notin T}\mathbb{Z} & \longrightarrow & C_T & \longrightarrow & 0 \\
 & & \downarrow n & & \downarrow n & & \downarrow n & & \downarrow n & & \\
0 & \longrightarrow & U_T & \longrightarrow & L^{\times} & \longrightarrow & \bigoplus_{\mathfrak{p}\notin T}\mathbb{Z} & \longrightarrow & C_T & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & L^{\times}/L^{\times n} & \longrightarrow & \bigoplus_{\mathfrak{p}\notin T}\mathbb{Z}/n\mathbb{Z} & & & &
\end{array}$$

In detail, let $\alpha \in L^{\times}$ represent an element of N. Then $n|\,\mathrm{ord}_{\mathfrak{p}}(\alpha)$ for all $\mathfrak{p} \notin T$, and so we can map α to the class c of $(\frac{\mathrm{ord}_{\mathfrak{p}}(\alpha)}{n})$ in C_T. Clearly $nc = 0$. If $c = 0$, then there exists a $\beta \in L^{\times}$ such that $\mathrm{ord}_{\mathfrak{p}}(\beta) = \mathrm{ord}_{\mathfrak{p}}(\alpha)/n$ for all $\mathfrak{p} \notin T$. Now α/β^n lies in U_T, and is well-defined up to an element of U_T^n. □

ASIDE 3.14 The above proof of the finiteness of the Selmer group follows that in my book Milne 1980, p. 133. It is simpler than the standard proof (e.g., Silverman 1986, pp. 190–196) which unnecessarily "translate[s] the putative finiteness of $E(L)/nE(L)$ into a statement about certain field extensions of L." However, other arguments may be more obviously algorithmic.

4 Heights; completion of the proof of the finite basis theorem

Let $P = (a_0 : \ldots : a_n) \in \mathbb{P}^n(\mathbb{Q})$. We shall say that $(a_0, \ldots, a_n)$ is a ***primitive representative*** for P if

$$a_i \in \mathbb{Z}, \quad \gcd(a_0, \ldots, a_n) = 1.$$

The ***height*** $H(P)$ of P is then defined to be

$$H(P) = \max_i |a_i|.$$

Here $|*|$ is the usual absolute value. The ***logarithmic height*** $h(P)$ of P is defined to be $\log H(P)$.[3]

Heights on $\mathbb{P}^1$

Let $F(X,Y)$ and $G(X,Y)$ be homogeneous polynomials of degree m in $\mathbb{Q}[X,Y]$, and let $V(\mathbb{Q})$ be the set of their common zeros. Then F and G define a map

$$\varphi : \mathbb{P}^1(\mathbb{Q}) \smallsetminus V(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q}), \quad (x:y) \mapsto (F(x,y) : G(x,y)).$$

PROPOSITION 4.1 *If $F(X,Y)$ and $G(X,Y)$ have no common zero in $\mathbb{P}^1(\mathbb{Q}^{\mathrm{al}})$, then there exists a constant B such that*

$$|h(\varphi(P)) - mh(P)| \leq B, \quad \textit{all } P \in \mathbb{P}^1(\mathbb{Q}).$$

PROOF. Since multiplying F and G by a nonzero constant doesn't change φ, we may suppose that F and G have integer coefficients. Let $(a:b)$ be a primitive representative for P. For any monomial cX^iY^{m-i},

$$|ca^ib^j| \leq |c| \max(|a|^m, |b|^m),$$

and so

$$|F(a,b)|, |G(a,b)| \leq C\,(\max(|a|,|b|)^m$$

with

$$C = (m+1)\max(|\text{coefficient of } F \text{ or } G|).$$

Now

$$\begin{aligned} H(\varphi(P)) &\leq \max(|F(a,b)|, |G(a,b)|) && (24)\\ &\leq C \cdot \max(|a|,|b|)^m = C \cdot H(P)^m. && (25)\end{aligned}$$

On taking logs, we obtain the inequality

$$h(\varphi(P)) \leq mh(P) + \log C.$$

The problem with proving a reverse inequality is that $F(a,b)$ and $G(a,b)$ may have a large common factor, and so the inequality (24) may be strict. We use the hypothesis that F and G have no common zero in $\mathbb{Q}^{\mathrm{al}}$ to limit this problem.

The hypothesis says that the resultant R of F and G (as homogeneous polynomials) is nonzero (I 1.26). Consider $Y^{-m}F(X,Y) = F(\frac{X}{Y},1)$ and $Y^{-m}G(X,Y) = G(\frac{X}{Y},1)$. When regarded as polynomials in the single variable $\frac{X}{Y}$, $F(\frac{X}{Y},1)$ and $G(\frac{X}{Y},1)$ have the same resultant as $F(X,Y)$ and $G(X,Y)$, and

[3]Throughout, log denotes the natural logarithm.

so (see I, §1) there are polynomials $U(\frac{X}{Y})$, $V(\frac{X}{Y}) \in \mathbb{Z}[\frac{X}{Y}]$ of degree $m-1$ such that

$$U(\tfrac{X}{Y})F(\tfrac{X}{Y},1)+V(\tfrac{X}{Y})G(\tfrac{X}{Y},1)=R.$$

On multiplying through by Y^{2m-1} and renaming $Y^{m-1}U(\frac{X}{Y})$ as $U(X,Y)$ and $Y^{m-1}V(\frac{X}{Y})$ as $V(X,Y)$, we obtain the equation

$$U(X,Y)F(X,Y)+V(X,Y)G(X,Y)=RY^{2m-1}.$$

Similarly, there are homogenous polynomials $U'(X,Y)$ and $V'(X,Y)$ of degree $m-1$ such that

$$U'(X,Y)F(X,Y)+V'(X,Y)G(X,Y)=RX^{2m-1}.$$

Substitute (a,b) for (X,Y) to obtain the equations

$$\begin{aligned} U(a,b)F(a,b)+V(a,b)G(a,b) &= Rb^{2m-1}, \\ U'(a,b)F(a,b)+V'(a,b)G(a,b) &= Ra^{2m-1}. \end{aligned} \tag{26}$$

From these equations we see that

$$\gcd(F(a,b),G(a,b)) \text{ divides } \gcd(Ra^{2m-1},Rb^{2m-1})=R.$$

As in the first part of the proof, there is a $C>0$ such that

$$U(a,b),U'(a,b),V(a,b),V'(a,b)\leq C\,(\max|a|,|b|)^{m-1}.$$

Therefore, the equations (26) show that

$$2C\,(\max|a|,|b|)^{m-1}{\cdot}\max(|F(a,b)|,|G(a,b)|)\geq|R||a|^{2m-1} \text{ and } |R||b|^{2m-1}.$$

Together with $\gcd(F(a,b),G(a,b))|R$, these inequalities imply that

$$H(\varphi(P))\geq\frac{1}{|R|}\max(|F(a,b)|,|G(a,b)|)\geq\frac{1}{2C}H(P)^m.$$

On taking logs, we obtain the inequality

$$h(\varphi(P))\geq mh(P)-\log 2C. \qquad \square$$

There is a well-defined (Veronese) map

$$(a:b),(c:d)\mapsto(ac:ad+bc:bd):\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^2.$$

LEMMA 4.2 *Let R be the image of (P,Q) under the Veronese map. Then*

$$\frac{1}{2}\leq\frac{H(R)}{H(P)H(Q)}\leq 2.$$

PROOF. Choose $(a : b)$ and $(c : d)$ to be primitive representatives of P and Q. Then

$$\begin{aligned} H(R) &\leq \max(|ac|, |ad + bc|, |bd|) \\ &\leq 2\max(|a|, |b|)\max(|c|, |d|) \\ &= 2H(P)H(Q). \end{aligned}$$

If a prime p divides both ac and bd, then either it divides a and d but not b or c, or the other way round. In either case, it doesn't divide $ad + bc$, and so $(ac, ad + bc, bd)$ is a primitive representative for R. It remains to show that

$$\max(|ac|, |ad + bc|, |bd|) \geq \frac{1}{2}(\max(|a|, |b|)(\max|c||d|),$$

but this is an elementary exercise (e.g., regard a, b, c, d as real numbers, and rescale the two pairs so that $a = 1 = c$). □

Heights on E

Let E be the elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in \mathbb{Q}, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

For $P \in E(\mathbb{Q})$, define

$$H(P) = \begin{cases} H((x(P) : z(P))) & \text{if } z(P) \neq 0 \\ 1 & \text{if } P = (0 : 1 : 0). \end{cases}$$

and

$$h(P) = \log H(P).$$

There are other definitions of h, but they differ by bounded amounts, and therefore lead to the same canonical height (see below).

LEMMA 4.3 *For any constant B, the set of $P \in E(\mathbb{Q})$ such that $h(P) < B$ is finite.*

PROOF. Certainly, for any constant B, $\{P \in \mathbb{P}^1(\mathbb{Q}) \mid H(P) \leq B\}$ is finite, but for every point $(x_0 : z_0) \in \mathbb{P}^1(\mathbb{Q})$, there are at most two points $(x_0 : y : z_0) \in E(\mathbb{Q})$, and so $\{P \in E(\mathbb{Q}) \mid H(P) \leq B\}$ is finite. □

PROPOSITION 4.4 *There exists a constant A such that*

$$|h(2P) - 4h(P)| \leq A.$$

PROOF. Let $P = (x : y : z)$ and $2P = (x_2 : y_2 : z_2)$. According to the duplication formula (p. 52),

$$(x_2 : z_2) = (F(x) : G(x))$$

where $F(X, Z)$ and $G(X, Z)$ are homogeneous polynomials of degree 4 such that

$$\begin{aligned} F(X,1) &= (3X^2+a)^2 - 8X(X^3+aX+b) \\ G(X,1) &= 4(X^3+aX+b). \end{aligned}$$

Since $X^3 + aX + b$ and its derivative $3X^2 + a$ have no common root, neither do $F(X, 1)$ and $G(X, 1)$, and so Proposition 4.1 shows that

$$|h(2P) - 4h(P)| \leq A$$

for some constant A. □

PROPOSITION 4.5 *There exists at most one function $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}$ satisfying the following conditions:*

(a) *$\hat{h}(P) - h(P)$ is bounded on $E(\mathbb{Q})$;*
(b) *$\hat{h}(2P) = 4\hat{h}(P)$.*

PROOF. If $\hat{h}$ satisfies (a) with bound B, then

$$\left|\hat{h}(2^n P) - h(2^n P)\right| \leq B.$$

If in addition it satisfies (b), then

$$\left|\hat{h}(P) - \frac{h(2^n P)}{4^n}\right| \leq \frac{B}{4^n},$$

and so $h(2^n P)/4^n$ converges to $\hat{h}(P)$. □

Any function $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}$ satisfying (a) and (b) of the proposition will be called the [4] ***canonical***, or ***Néron-Tate***, ***height function.*** According to the proof, if $\hat{h}$ exists then $\hat{h}(P)$ must be the limit of the sequence $h(2^n P)/4^n$.

LEMMA 4.6 *For any $P \in E(\mathbb{Q})$, the sequence $h(2^n P)/4^n$ is Cauchy in $\mathbb{R}$.*

PROOF. From Proposition 4.4, we know that there exists a constant A such that

$$|h(2P) - 4h(P)| \leq A$$

[4]There are different normalizations of the canonical height in the literature, one of which is twice another (see Cremona 1992, 3.4).

for all P. For $N \geq M \geq 0$ and $P \in E(\mathbb{Q})$,

$$\begin{aligned}\left|\frac{h(2^N P)}{4^N} - \frac{h(2^M P)}{4^M}\right| &= \left|\sum_{n=M}^{N-1}\left(\frac{h(2^{n+1}P)}{4^{n+1}} - \frac{h(2^n P)}{4^n}\right)\right| \\ &\leq \sum_{n=M}^{N-1}\frac{1}{4^{n+1}}|h(2^{n+1}P) - 4h(2^n P)| \\ &\leq \sum_{n=M}^{N-1}\frac{1}{4^{n+1}}A \\ &\leq \frac{A}{4^{M+1}}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) \\ &= \frac{A}{3\cdot 4^M}.\end{aligned}$$

Therefore $h(2^n P)/4^n$ is Cauchy. □

The lemma allows us to define

$$\hat{h}(P) = \lim_{n\to\infty}\frac{h(2^n P)}{4^n}, \quad \text{all } P \in E(\mathbb{Q}).$$

THEOREM 4.7 *The function* $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}$ *is a Néron-Tate height function; moreover,*

(a) *for any* $C \geq 0$, *the set* $\{P \in E(\mathbb{Q}) \mid \hat{h}(P) \leq C\}$ *is finite;*
(b) $\hat{h}(P) \geq 0$, *with equality if and only if* P *has finite order.*

PROOF. When M is taken to be zero, the inequality in the proof of (4.6) becomes

$$\left|\frac{h(2^N P)}{4^N} - h(P)\right| \leq \frac{A}{3}.$$

On letting $N \to \infty$, we find that $\hat{h}$ satisfies condition (a) of (4.5). For condition (b), note that

$$\hat{h}(2P) = \lim_{n\to\infty}\frac{h(2^{n+1}P)}{4^n} = 4\cdot\lim_{n\to\infty}\frac{h(2^{n+1}P)}{4^{n+1}} = 4\cdot\hat{h}(P).$$

Thus, $\hat{h}$ is a Néron-Tate height function.

The set of P for which $\hat{h}(P) \leq C$ is finite, because h has this property and the difference $\hat{h}(P) - h(P)$ is bounded.

Because $H(P)$ is an integer ≥ 1, $h(P) \geq 0$ and $\hat{h}(P) \geq 0$. If P is torsion, then $\{2^n P \mid n \geq 0\}$ is finite, so $\hat{h}$ is bounded on it, by D say, and $\hat{h}(P) =$

$\hat{h}(2^n P)/4^n \leq D/4^n$ for all n. On the other hand, if P has infinite order, then $\{2^n P \mid n \geq 0\}$ is infinite and $\hat{h}$ is unbounded on it. Hence $\hat{h}(2^n P) > 1$ for some n, and so $\hat{h}(P) > 4^{-n} > 0$. □

Let $f\colon M \to K$ be a function from an abelian group M into a field K of characteristic $\neq 2$. Such an f is called a ***quadratic form*** if $f(2x) = 4f(x)$ and

$$B(x,y) \stackrel{\text{def}}{=} f(x+y) - f(x) - f(y)$$

is bi-additive. Then B is symmetric, and it is the only symmetric bi-additive form $B\colon M \times M \to K$ such that $f(x) = \frac{1}{2}B(x,x)$. We shall need the following criterion:

LEMMA 4.8 *A function $f\colon M \to K$ from an abelian group into a field K of characteristic $\neq 2$ is a quadratic form if it satisfies the parallelogram*[5] *law:*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{all } x, y \in M.$$

PROOF. Let f satisfy the parallelogram law. On taking $x = y = 0$ in the parallelogram law, we find that $f(0) = 0$; on taking $x = y$ we find that $f(2x) = 4f(x)$; and on taking $x = 0$ we find that $f(-y) = f(y)$. By symmetry, it remains to show that $B(x+y,z) = B(x,z) + B(y,z)$, i.e., that

$$f(x+y+z) - f(x+y) - f(x+z) - f(y+z) + f(x) + f(y) + f(z) = 0.$$

Now four applications of the parallelogram law show that:

$$f(x+y+z) + f(x+y-z) - 2f(x+y) - 2f(z) = 0$$

$$f(x+y-z) + f(x-y+z) - 2f(x) - 2f(y-z) = 0$$

$$f(x+y+z) + f(x-y+z) - 2f(x+z) - 2f(y) = 0$$

$$2f(y+z) + 2f(y-z) - 4f(y) - 4f(z) = 0.$$

The alternating sum of these equations is (double) the required equation. □

PROPOSITION 4.9 *The height function $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}$ is a quadratic form.*

We have to prove the parallelogram law.

[5] In elementary linear algebra, the parallelogram law says that, for vectors u and v in $\mathbb{R}^n$,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

LEMMA 4.10 *There exists a constant C such that*

$$H(P_1 + P_2)H(P_1 - P_2) \leq C \cdot H(P_1)^2 \cdot H(P_2)^2$$

for all $P_1, P_2 \in E(\mathbb{Q})$.

PROOF. Let $P_1 + P_2 = P_3$ and $P_1 - P_2 = P_4$, and let $P_i = (x_i : y_i : z_i)$. Then

$$(x_3x_4 : x_3z_4 + x_4z_3 : z_3z_4) = (W_0 : W_1 : W_2)$$

where (see p. 52)

$$\begin{aligned}
W_0 &= (X_2Z_1 - X_1Z_2)^2 \\
W_1 &= 2(X_1X_2 + aZ_1Z_2)(X_1Z_2 + X_2Z_1) + 4bZ_1^4Z_2^4 \\
W_2 &= X_1^2X_2^2 - 2aX_1X_2Z_1Z_2 - 4b(X_1Z_1Z_2^2 + X_2Z_1^2Z_2) + a^2Z_1^2Z_2^2.
\end{aligned}$$

It follows that

$$H(W_0 : W_1 : W_2) \leq C \cdot H(P_1)^2 \cdot H(P_2)^2.$$

According to Lemma 4.2,

$$H(W_0 : W_1 : W_2) \geq \frac{1}{2}H(P_3)H(P_4).$$

□

LEMMA 4.11 *The canonical height function $\hat{h}\colon E(\mathbb{Q}) \to \mathbb{R}$ satisfies the parallelogram law:*

$$\hat{h}(P + Q) + \hat{h}(P - Q) = 2\hat{h}(P) + 2\hat{h}(Q).$$

PROOF. On taking logs in the previous lemma, we find that

$$h(P + Q) + h(P - Q) \leq 2h(P) + 2h(Q) + B.$$

On replacing P and Q with 2^nP and 2^nQ, dividing through by 4^n, and letting $n \to \infty$, we obtain the inequality

$$\hat{h}(P + Q) + \hat{h}(P - Q) \leq 2\hat{h}(P) + 2\hat{h}(Q).$$

Putting $P' = P + Q$ and $Q' = P - Q$ in this gives the reverse inequality:

$$\begin{aligned}
\hat{h}(P') + \hat{h}(Q') &\leq 2\hat{h}\left(\frac{P' + Q'}{2}\right) + 2\hat{h}\left(\frac{P' - Q'}{2}\right) \\
&= \frac{1}{2}\hat{h}(P' + Q') + \frac{1}{2}\hat{h}(P' - Q').
\end{aligned}$$

□

REMARK 4.12 Let K be a number field. When $\mathcal{O}_K$ is not a principal ideal domain, there might not be a primitive representative for a point[6] P of $\mathbb{P}^n(K)$ and so the definition we gave for the height of a point in $\mathbb{P}^n(\mathbb{Q})$ doesn't extend directly to number fields. Instead, we need a slightly different approach. Note that, for $c \in \mathbb{Q}^\times$,

$$\prod_{p=2,\ldots,\infty} |c|_p = 1 \quad \text{(product formula).}$$

Here $|\cdot|_p$ is the usual absolute value when $p = \infty$ and otherwise is the p-adic valuation defined in I, §2. Hence, for $P = (a_0 : a_1 : \ldots : a_n) \in \mathbb{P}^n(\mathbb{Q})$,

$$H(P) \stackrel{\text{def}}{=} \prod_{p=2,\ldots,\infty} \max_i(|a_i|_p)$$

is independent of the choice of a representative for P. Moreover, when $(a_0, \ldots, a_n)$ is chosen to be a primitive representative, then $\max_i |a_i|_p = 1$ for all $p \neq \infty$, and so $H(P) = \max_i |a_i|_\infty$, which agrees with the earlier definition. For a number field K, it is possible to "normalize" the valuations so that the product formula holds (see ANT, 8.8), and then

$$H(P) \stackrel{\text{def}}{=} \prod_v \max_i(|a_i|_v), \quad P = (a_0 : a_1 : \ldots : a_n),$$

gives a good notion of a height on $\mathbb{P}^n(K)$. With this definition, all the results of this section extend to elliptic curves over number fields.

Completion of the proof of the finite basis theorem

We prove the following slightly more precise result.

PROPOSITION 4.13 *Let $C > 0$ be such that $S \stackrel{\text{def}}{=} \{P \in E(\mathbb{Q}) \mid \hat{h}(P) \leq C\}$ contains a set of coset representatives for $2E(\mathbb{Q})$ in $E(\mathbb{Q})$; then S generates $E(\mathbb{Q})$.*

PROOF. Suppose that there exists a $Q \in E(\mathbb{Q})$ not in the subgroup generated by S. Because $\hat{h}$ takes discrete values, we may choose Q so that $\hat{h}(Q)$ is the smallest possible. From the definition of S, there exists a $P \in S$ such that $Q = P + 2R$ for some $R \in E(\mathbb{Q})$. Clearly, R can not be in the subgroup generated by S, and so $\hat{h}(R) \geq \hat{h}(Q)$. Thus,

$$\begin{aligned} 2\hat{h}(P) &= \hat{h}(P+Q) + \hat{h}(P-Q) - 2\hat{h}(Q) \\ &\geq 0 + \hat{h}(2R) - 2\hat{h}(Q) \\ &= 4\hat{h}(R) - 2\hat{h}(Q) \\ &\geq 2\hat{h}(Q), \end{aligned}$$

[6] More precisely, there might not exist $a_0, \ldots, a_n \in \mathcal{O}_K$ such that $P = (a_0 : \ldots : a_n)$ and the ideal generated by the a_i is $\mathcal{O}_K$.

which is a contradiction because $\hat{h}(P) \leq C$ and $\hat{h}(Q) > C$. □

In view of Remark 4.12, this argument works without change for any number field K.

NOTES The Néron-Tate (canonical) height function $\hat{h}$ was defined independently by Tate using the above method, and by Néron using a much more elaborate method which, however, has the advantage that it expresses $\hat{h}$ as a sum of local terms.

5 The problem of computing the rank of $E(\mathbb{Q})$

According to André Weil, one of the two oldest outstanding problems in mathematics is that of finding an algorithm for determining the group $E(\mathbb{Q})$. We know that $E(\mathbb{Q})$ is finitely generated, and so

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\mathrm{tors}} \oplus \mathbb{Z}^r,$$

for some $r \geq 0$, called the rank[7] of $E(\mathbb{Q})$. Since we know how to compute $E(\mathbb{Q})_{\mathrm{tors}}$ (see II 5.1), this amounts to finding an algorithm to find r, or better, for finding a basis for $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$. We regard $S^{(2)}(E/\mathbb{Q})$ as giving a computable upper bound for r with $\text{Ш}(E/\mathbb{Q})_2$ as the error term. The problem is to determine the image of $E(\mathbb{Q})$ in $S^{(2)}(\mathbb{Q})$.

From the commutative diagrams

$$\begin{array}{ccccccccc}
0 & \to & E(\mathbb{Q})/2^n E(\mathbb{Q}) & \to & H^1(\mathbb{Q}, E_{2^n}) & \to & H^1(\mathbb{Q}, E)_{2^n} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle 2} & & \\
0 & \to & E(\mathbb{Q})/2^{n+1} E(\mathbb{Q}) & \to & H^1(\mathbb{Q}, E_{2^{n+1}}) & \to & H^1(\mathbb{Q}, E)_{2^{n+1}} & \to & 0
\end{array}$$

we can construct a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E(\mathbb{Q})/2E(\mathbb{Q}) & \longrightarrow & S^{(2)}(E/\mathbb{Q}) & \longrightarrow & \text{Ш}(E/\mathbb{Q})_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle 2} & & \\
0 & \longrightarrow & E(\mathbb{Q})/4E(\mathbb{Q}) & \longrightarrow & S^{(4)}(E/\mathbb{Q}) & \longrightarrow & \text{Ш}(E/\mathbb{Q})_4 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle 2} & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle 2} & & \\
0 & \longrightarrow & E(\mathbb{Q})/2^n E(\mathbb{Q}) & \longrightarrow & S^{(2^n)}(E/\mathbb{Q}) & \longrightarrow & \text{Ш}(E/\mathbb{Q})_{2^n} & \longrightarrow & 0.
\end{array}$$

[7] In the literature, $E(\mathbb{Q})$ is sometimes called the ***Mordell-Weil group*** of E, and its rank is sometimes called the ***rank*** of E.

The vertical maps at left are the natural quotient maps. Let $S^{(2,n)}(E/\mathbb{Q})$ be the image of $S^{(2^n)}(E/\mathbb{Q})$ in $S^{(2)}(E/\mathbb{Q})$.

PROPOSITION 5.1 *The group $E(\mathbb{Q})/2E(\mathbb{Q})$ is contained in $\bigcap_n S^{(2,n)}(E/\mathbb{Q})$, and equals it if and only if there does not exist a nonzero element in $Ш(E/\mathbb{Q})$ divisible by all powers of 2.*

PROOF. As the left-hand vertical arrows are surjective, the image of $E(\mathbb{Q})/2E(\mathbb{Q})$ in $S^{(2)}(E/\mathbb{Q})$ equals the image of $E(\mathbb{Q})/2^nE(\mathbb{Q})$, which is contained in $S^{(2,n)}(E/\mathbb{Q})$ by the commutativity of the diagram. Conversely, let γ lie in $\bigcap_n S^{(2,n)}(E/\mathbb{Q})$, so that, for each n, there is an element $\gamma_n \in S^{(2^n)}$ mapping to γ. Let δ_n be the image of γ_n in $Ш(E/\mathbb{Q})_{2^n}$. Then $2^{n-1}\delta_n = \delta_1$ for all n, and so δ_1 is divisible by all powers of 2. If the only such element in $Ш(E/\mathbb{Q})$ is zero, then γ is in the image of $E(\mathbb{Q})/2E(\mathbb{Q})$. □

REMARK 5.2 Because $Ш(E/\mathbb{Q})$ is torsion and $Ш(E/\mathbb{Q})_2$ is finite, if there does not exist a nonzero element in $Ш(E/\mathbb{Q})$ divisible by all powers of 2, then the 2-primary component of $Ш(E/\mathbb{Q})$ is finite. If $2^{n_0-1}Ш(E/\mathbb{Q})_{2^{n_0}} = 0$, then a diagram chase shows that

$$S^{(2,n_0)}(E/\mathbb{Q}) = S^{(2,n_0+1)}(E/\mathbb{Q}) = \cdots \simeq E(\mathbb{Q})/2E(\mathbb{Q}).$$

This gives a strategy for computing r. Calculate $S^{(2)}$, and let your computer run overnight to calculate the subgroup $T(1)$ of $E(\mathbb{Q})$ generated by the points with height $h(P) \leq 10$. If $T(1)$ maps onto $S^{(2)}$ we have found r, and even a set of generators for $E(\mathbb{Q})$. If not, calculate $S^{(2^2)}$, and have the computer run overnight again to calculate the subgroup $T(2)$ of $E(\mathbb{Q})$ generated by points with height $h(P) \leq 10^2$. If the image of $T(2)$ in $S^{(2)}$ is $S^{(2,2)}$, then we have found r. If not, we continue ....

Nightmare possibility: The Tate-Shafarevich group contains a nonzero element divisible by all powers of 2, in which case "we are doomed to continue computing through all eternity" (Tate 1974, p. 193). This would happen, for example, if $Ш(E/\mathbb{Q})$ contains a copy of $\mathbb{Q}/\mathbb{Z}$. It is widely conjectured that this doesn't happen.

CONJECTURE 5.3 *The Tate-Shafarevich group is always finite.*

When the conjecture is true, then the above argument becomes an algorithm for computing $E(\mathbb{Q})$.

Until the work of Kolyvagin and Rubin about 1987 (Kolyvagin 1988a,b; Rubin 1987), the Tate-Shafarevich group was not known to be finite for a single elliptic curve over a $\mathbb{Q}$, and the conjecture is still far from being proved in that case. However, the conjecture is stated for elliptic curves over all global fields (finite extensions of $\mathbb{Q}$ or $\mathbb{F}_p(T)$), and in the function field case I proved that

Ш(E/K) is finite provided $j(E)$ lies in the ground field (Milne 1968). Later (Milne 1975) I showed that, for example, that it is finite for any curve

$$E(j): Y^2Z = X^3 - \frac{27}{4}\frac{j}{j-1728}XZ^2 - \frac{27}{4}\frac{j}{j-1728}Z^3$$

over the field $K = \mathbb{F}_p(j)$.

REMARK 5.4 (a) Cassels has shown that Ш$(E/\mathbb{Q})$ carries a nondegenerate alternating form if it is finite (Cassels 1962). Therefore, its order, if finite, is a square.[8]

(b) Define C to make the sequence

$$0 \to Ш(E/\mathbb{Q}) \to H^1(\mathbb{Q}, E) \to \bigoplus_{p=2,\ldots,\infty} H^1(\mathbb{Q}_p, E) \to C \to 0,$$

exact and endow each group with the discrete topology. When Ш$(E/\mathbb{Q})$ is finite, Cassels has shown that the Pontryagin dual of this sequence is an exact sequence

$$0 \leftarrow Ш(E/\mathbb{Q}) \leftarrow \Theta \leftarrow \prod_{p,\infty} H^1(\mathbb{Q}_p, E) \leftarrow \hat{E}(\mathbb{Q}) \leftarrow 0,$$

where $\hat{E}(\mathbb{Q})$ is the completion of $E(\mathbb{Q})$ for the topology for which the subgroups of finite index form a fundamental system of neighbourhoods of 0 (Cassels 1964). The similar sequence for abelian varieties over number fields is known as the ***Cassels-Tate dual exact sequence*** (Milne 1986b, I 6.26, II 5.6).

Explicit calculations of the rank

Computing the rank r of $E(\mathbb{Q})$ can be difficult[9] (perhaps impossible), but occasionally it is straightforward. In order to avoid the problem of having to work with a number field L other than $\mathbb{Q}$, we assume that the elliptic curve has all its points of order 2 rational over $\mathbb{Q}$:

$$E: Y^2Z = (X - \alpha Z)(X - \beta Z)(X - \gamma Z), \quad \alpha, \beta, \gamma \text{ distinct integers.}$$

[8] Recall from linear algebra that a vector space carrying a nondegenerate skew-symmetric form has even dimension provided the field is of characteristic $\neq 2$. When the form is assumed to be alternating, i.e., $\psi(x,x) = 0$ for all x, then the condition on the characteristic is unnecessary. This applies, in particular, to vector spaces over $\mathbb{F}_p$. An easy argument now shows that a finite abelian group that admits a nondegenerate alternating form

$$\varphi: A \times A \to \mathbb{Q}/\mathbb{Z}$$

has order a square. Here nondegenerate means that φ defines an isomorphism of A with its dual:

$$a \mapsto \varphi(-, a): A \to \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

[9] Of course, one can always type `ellgenerators` into Pari, and hope that someone else has done the hard work.

The discriminant of $(X-\alpha)(X-\beta)(X-\gamma)$ is

$$\Delta = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2.$$

PROPOSITION 5.5 *The rank r of $E(\mathbb{Q})$ satisfies the inequality*

$$r \le 2 \times \#\{p \mid p \text{ divides } 2\Delta\}.$$

PROOF. Since $E(\mathbb{Q}) \approx T \oplus \mathbb{Z}^r$, $T = E(\mathbb{Q})_{\text{tors}}$, we have $E(\mathbb{Q})/2E(\mathbb{Q}) \approx T/2T \oplus (\mathbb{Z}/2\mathbb{Z})^r$. Because T is finite, the kernel and cokernel of $T \xrightarrow{2} T$ have the same order, and so $T/2T \approx (\mathbb{Z}/2\mathbb{Z})^2$. We have an injection

$$E(\mathbb{Q})/2E(\mathbb{Q}) \hookrightarrow (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2,$$

and the image is contained in the product of the subgroups of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ generated by -1 and the primes where E has bad reduction, namely, those dividing 2Δ. □

It is possible to improve this estimate. Let T_1 be the set of prime numbers dividing Δ for which the reduction is nodal, and let T_2 be the set of prime numbers dividing Δ for which the reduction is cuspidal. Thus T_1 consists of the prime numbers modulo which two of the roots of

$$(X-\alpha)(X-\beta)(X-\gamma)$$

coincide, and T_2 consists of those modulo which all three coincide. Let t_1 and t_2 respectively be the numbers of elements of T_1 and T_2.

PROPOSITION 5.6 *The rank r of $E(\mathbb{Q})$ satisfies $r \le t_1 + 2t_2 - 1$.*

PROOF. Define $\varphi_\alpha : E(\mathbb{Q})/2E(\mathbb{Q}) \to \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ as in (3.7):

$$\varphi_\alpha((x:y:z)) = \begin{cases} (\frac{x}{z}-\alpha)\mathbb{Q}^{\times 2} & z \ne 0, \quad x \ne \alpha z; \\ (\alpha-\beta)(\alpha-\gamma)\mathbb{Q}^\times & z \ne 0, \quad x = \alpha z \\ \mathbb{Q}^\times & (x:y:z) = (0:1:0). \end{cases}$$

Define φ_β similarly — the map

$$P \mapsto (\varphi_\alpha(P), \varphi_\beta(P)) \colon E(\mathbb{Q})/2E(\mathbb{Q}) \to (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2$$

is injective. For each prime p, let $\varphi_p(P)$ be the element of $(\mathbb{Z}/2\mathbb{Z})^2$ whose components are

$$\operatorname{ord}_p(\varphi_\alpha(P)) \mod 2, \quad \text{and } \operatorname{ord}_p(\varphi_\beta(P)) \mod 2$$

and let $\varphi_\infty(P)$ be the element of $\{\pm\}^2$ whose components are

$$\operatorname{sign}(\varphi_\alpha(P)), \quad \text{and } \operatorname{sign}(\varphi_\beta(P)).$$

The proposition is proved by showing:

(a) if p does not divide Δ, then $\varphi_p(P) = 0$ for all P;
(b) if $p \in T_1$, then $\varphi_p(P)$ is contained in the diagonal of $\mathbb{F}_2^2$ for all P;
(c) when α, β, γ are ordered so that $\alpha < \beta < \gamma$, $\varphi_\infty(P)$ equals $(+,+)$ or $(+,-)$.

Except for $p = 2$, (a) was proved in the paragraph preceding (3.7).

We prove (b) in the case $\alpha \equiv \beta \mod p$ and $P = (x : y : 1)$, $x \neq \alpha, \beta, \gamma$. Let

$$a = \mathrm{ord}_p(x - \alpha), \quad b = \mathrm{ord}_p(x - \beta), \quad c = \mathrm{ord}_p(x - \gamma).$$

Because

$$(x - \alpha)(x - \beta)(x - \gamma)$$

is a square, $a + b + c \equiv 0 \mod 2$.

If $a < 0$, then (because $\alpha \in \mathbb{Z}$) p^{-a} occurs as a factor of the denominator of x (in its lowest terms), and it follows that $b = a = c$. Since $a + b + c \equiv 0 \mod 2$, this implies that $a \equiv b \equiv c \equiv 0 \mod 2$, and so $\varphi_p(P) = 0$. The same argument applies if $b < 0$ or $c < 0$.

If $a > 0$, then p divides the numerator of $x - \alpha$. Because p doesn't divide $(\alpha - \gamma)$, it doesn't divide $(\alpha - \gamma) + (x - \alpha) = (x - \gamma)$, and so $c = 0$. Now $a + b \equiv 0 \mod 2$ implies that $\varphi_p(P)$ lies in the diagonal of $\mathbb{F}_2^2$. A similar argument applies if $b > 0$ or $c > 0$.

The remaining cases of (b) are proved similarly.

We prove (c). Let $P = (x : y : 1)$, $x \neq \alpha, \beta, \gamma$. We may suppose that $\alpha < \beta < \gamma$, so that $(x - \alpha) > (x - \beta) > (x - \gamma)$. Then $\varphi_\infty(P) = (+,+)$, $(+,-)$, or $(-,-)$. However, because

$$(x - \alpha)(x - \beta)(x - \gamma)$$

is a square in $\mathbb{Q}$, the pair $(-,-)$ is impossible. The cases $x = \alpha$ etc. are equally easy. $\square$

EXAMPLE 5.7 The curve

$$E : Y^2 Z = X^3 - X Z^2$$

is of the above form with $(\alpha, \beta, \gamma) = (-1, 0, 1)$. The only bad prime is 2, and here the reduction is nodal. Therefore $r = 0$, and E has no point of infinite order:

$$E(\mathbb{Q}) \approx (\mathbb{Z}/2\mathbb{Z})^2.$$

ASIDE 5.8 It is an old, but still open, question whether the rank of $E(\mathbb{Q})$ is bounded or can be arbitrarily large. At present, the highest known rank is 28 (for the latest record, see `http://web.math.hr/~duje/tors/tors.html`). For elliptic curves over $k(T)$ (k a fixed finite field), it is known that the rank can be arbitrarily large (Tate and Shafarevich 1967).

EXERCISE 5.9 Do one of the following two problems (those who know the quadratic reciprocity law should do (2)).

(1) Show that $E(\mathbb{Q})$ is finite if E has equation

$$Y^2Z = X^3 - 4XZ^2.$$

Hint: Let P be a point of infinite order in $E(\mathbb{Q})$, and show that, after possibly replacing P with $P+Q$ where $2Q = 0$, $\varphi_2(P)$ is zero. Then show that $\varphi_\infty(P) = (+,+)$ — contradiction.

(2) Let E be the elliptic curve

$$Y^2Z = X^3 - p^2XZ^2$$

where p is an odd prime. Show that the rank r of $E(\mathbb{Q})$ satisfies:

$$\begin{array}{ll} r \le 2 & \text{if } p \equiv 1 \mod 8 \\ r = 0 & \text{if } p \equiv 3 \mod 8 \\ r \le 1 & \text{otherwise.} \end{array}$$

Hint: Let P be a point of infinite order in $E(\mathbb{Q})$, and show that, after possibly replacing P with $P+Q$ where $2Q = 0$, $\varphi_p(P)$ is zero.

(These are fairly standard examples. You should do them without looking them up in a book.)

6 The Néron-Tate pairing

We saw in §4, that there is a canonical $\mathbb{Z}$-bilinear pairing

$$B : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}, \quad B(x,y) = \hat{h}(x+y) - \hat{h}(x) - \hat{h}(y).$$

This pairing extends uniquely to an $\mathbb{R}$-bilinear pairing

$$B : E(\mathbb{Q}) \otimes \mathbb{R} \times E(\mathbb{Q}) \otimes \mathbb{R} \to \mathbb{R}.$$

If $\{e_1, \dots, e_r\}$ is a $\mathbb{Z}$-basis for $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$, then $\{e_1 \otimes 1, \dots, e_r \otimes 1\}$ is an $\mathbb{R}$-basis for $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$, with respect to which B has matrix $\big(B(e_i, e_j)\big)$.

THEOREM 6.1 *The bilinear pairing*

$$B \colon E(\mathbb{Q}) \otimes \mathbb{R} \times E(\mathbb{Q}) \otimes \mathbb{R} \to \mathbb{R}$$

is positive definite; in particular, it is nondegenerate.

This follows from Theorem 4.7 and the following elementary statement. By a ***lattice*** in a real vector space V, I mean the $\mathbb{Z}$-submodule generated by a basis for V (sometimes this is called a full, or complete, lattice).

LEMMA 6.2 *Let $q: V \to \mathbb{R}$ be a quadratic form on a finite-dimensional real vector space V. If there exists a lattice Λ in V such that*

(a) *for every constant C, the set $\{P \in \Lambda \mid q(P) \leq C\}$ is finite,*
(b) *the only $P \in \Lambda$ with $q(P) = 0$ is $P = 0$,*

then q is positive definite on V.

PROOF. There exists a basis for V relative to which q takes the form

$$q(x) = x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_t^2, \quad t \leq \dim V.$$

We assume $s \neq \dim V$ and derive a contradiction. We use the basis to identify V with $\mathbb{R}^n$. Let λ be the length of the shortest vector in Λ, i.e.,

$$\lambda = \inf\{q(P) \mid P \in \Lambda, P \neq 0\}.$$

The conditions (a,b) imply that $\lambda > 0$. Consider the set

$$B(\delta) = \left\{ (x_i) \in \mathbb{R}^n \;\middle|\; x_1^2 + \cdots + x_s^2 \leq \frac{\lambda}{2}, \quad x_{s+1}^2 + \cdots + x_t^2 \leq \delta \right\}.$$

The length (using q) of any vector in $B(\delta)$ is $\leq \lambda/2$, and so $B(\delta) \cap \Lambda = \{0\}$, but the volume of $B(\delta)$ can be made arbitrarily large by taking δ large, and so this violates the following famous theorem of Minkowski. □

THEOREM 6.3 (MINKOWSKI) *Let Λ be a lattice in $\mathbb{R}^n$ with fundamental parallelepiped D_0, and let B be a subset of $\mathbb{R}^n$ that is compact, convex, and symmetric in the origin. If*

$$Vol(B) \geq 2^n \, Vol(D)$$

then B contains a point of Λ other than the origin.

PROOF. We first show that a measurable set S in $\mathbb{R}^n$ with $\mathrm{Vol}(S) > \mathrm{Vol}(D_0)$ contains distinct points α, β such that $\alpha - \beta \in \Lambda$. Clearly

$$\mathrm{Vol}(S) = \sum \mathrm{Vol}(S \cap D)$$

where the sum is over all the translates of D_0 by elements of Λ. The fundamental parallelepiped D_0 will contain a unique translate (by an element of Λ) of each set $S \cap D$. Since $\mathrm{Vol}(S) > \mathrm{Vol}(D_0)$, at least two of these sets will overlap, and so there exist elements $\alpha, \beta \in S$ such that

$$\alpha - \lambda = \beta - \lambda', \quad \text{some distinct } \lambda, \lambda' \in \Lambda.$$

Then $\alpha - \beta = \lambda - \lambda' \in \Lambda \smallsetminus \{0\}$.

We apply this with $S = \frac{1}{2}B \stackrel{\text{def}}{=} \{\frac{x}{2} \mid x \in B\}$. It has volume $\frac{1}{2^n}\mathrm{Vol}(B) > \mathrm{Vol}(D_0)$, and so there exist distinct $\alpha, \beta \in B$ such that $\alpha/2 - \beta/2 \in \Lambda$. Because B is symmetric about the origin, $-\beta \in B$, and because it is convex, $(\alpha + (-\beta))/2 \in B$. □

REMARK 6.4 Systems consisting of a real vector space V, a lattice Λ in V, and a positive-definite quadratic form q on V are of great interest in mathematics — they are typically referred to simply as lattices. There exists a basis for V that identifies (V, q) with $(\mathbb{R}^n, X_1^2 + \cdots + X_n^2)$. Finding a dense packing of spheres in $\mathbb{R}^n$ centred on the points of a lattice amounts to finding a lattice Λ such that

$$\frac{\|\text{ shortest vector}\|^n}{\text{Vol(fundamental parallelopiped)}}$$

is large. Many lattices, for example, the Leech lattice, have very interesting automorphism groups. See Conway and Sloane 1993.

From an elliptic curve E over $\mathbb{Q}$, one obtains such a system, namely, $V = E(\mathbb{Q}) \otimes \mathbb{R}$, $\Lambda = E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$, $q = \hat{h}$. As far as I know, they aren't interesting — at present no elliptic curve is known with $\operatorname{rank}(E(\mathbb{Q})) > 28$. However, when the number field is replaced by a function field in one variable, one gets infinite families of very interesting lattices. We discuss this in §11 below.

7 Geometric interpretation of the cohomology groups; jacobians

For convenience, throughout this section we take k to be a perfect field, for example, a field of characteristic zero or a finite field. We let $\Gamma = \operatorname{Gal}(k^{\text{al}}/k)$. For any finite Galois extension E of k contained in k^{al}, the restriction map $\Gamma \to \operatorname{Gal}(E/k)$ is surjective. As k^{al} is the union of such E, this shows that $k^{\text{al}\Gamma} = k$: the elements of k^{al} fixed by Γ are those in k. See FT, §7, for more details.

For any elliptic curve E over k, we have an exact sequence of cohomology groups (see §2):

$$0 \to E(k)/nE(k) \to H^1(k, E_n) \to H^1(k, E)_n \to 0.$$

Here $H^1(k, E_n)$ and $H^1(k, E)$ are defined to be the groups of continuous crossed homomorphisms from Γ to $E(k^{\text{al}})_n$ and $E(k^{\text{al}})$ respectively modulo the principal crossed homomorphisms. In this section, we shall give a geometric interpretation of these groups, and hence also of the Selmer and Tate-Shafarevich groups. We shall attach to any curve W of genus 1 over k, possibly without a point with coordinates in k, an elliptic curve E, called its jacobian (variety), and we shall see that the Tate-Shafarevich group of an elliptic curve E classifies the curves of genus 1 over k with jacobian E for which the Hasse principle fails, i.e., curves having a point in each $\mathbb{Q}_p$ and in $\mathbb{R}$ without having a point in $\mathbb{Q}$.

In general, $H^1(k, *)$ classifies objects over k that become isomorphic over k^{al} to a fixed object with automorphism group $*$. We shall see several examples of this.

Principal homogeneous spaces of sets

Let A be a commutative group. A right A-set

$$(w,a) \mapsto w + a \colon W \times A \to W$$

is called a ***principal homogeneous space*** for A if $W \neq \emptyset$ and the map

$$(w,a) \mapsto (w, w+a) \colon W \times A \to W \times W$$

is bijective, i.e., if for every pair $w_1, w_2 \in W$, there is a unique $a \in A$ such that $w_1 + a = w_2$.

EXAMPLE 7.1 (a) Addition $A \times A \to A$ makes A into a principal homogeneous space for A, called the ***trivial*** principal homogeneous space.

(b) A principal homogeneous space for a vector space (for example, the universe according to Newton) is called an ***affine space***. Essentially it is a vector space without a preferred origin.

A ***morphism*** $\varphi \colon W \to W'$ of principal homogeneous spaces is simply a map A-sets. We leave it to the reader to check the following statements.

7.2 Let W and W' be principal homogeneous spaces for A.

(a) For any points $w_0 \in W$, $w_0' \in W'$, there exists a unique morphism $\varphi \colon W \to W'$ sending w_0 to w_0'.

(b) Every morphism $W \to W'$ is an isomorphism (i.e., has an inverse that is also a morphism).

7.3 (a) Let W be a principal homogeneous space over A. For any point $w_0 \in W$, there is a unique morphism $A \to W$ (of principal homogeneous spaces) sending 0 to w_0.

(b) An $a \in A$ defines an automorphism $w \mapsto w + a$ of W, and every automorphism of W is of this form for a unique $a \in A$.

Hence $\mathrm{Aut}(W) = A$. For any commutative group A, we have defined a class of objects having A as their groups of automorphisms.

Principal homogeneous spaces of curves

Let E be an elliptic curve over a field k. A ***principal homogeneous space***[10] for E is a curve W over k together with a right action of E given by a regular map

$$(w,P) \mapsto w + P \colon W \times E \to W$$

[10]The word "torsor" is also used. Principal homogeneous spaces are the analogue in arithmetic geometry of principal bundles in topology.

such that

$$(w,P)\mapsto(w,w+P)\colon W\times E\to W\times W$$

is an isomorphism of algebraic varieties. The conditions imply that, for any field $K\supset k$, $W(K)$ is either empty or is a principal homogeneous space for the group $E(K)$ (in the sense of sets). A ***morphism*** of principal homogeneous spaces for E is a regular map $\varphi\colon W\to W'$ such that

$$\begin{array}{ccc} W\times E & \longrightarrow & W \\ \big\downarrow{\scriptstyle \varphi\times\mathrm{id}_E} & & \big\downarrow{\scriptstyle\varphi} \\ W'\times E & \longrightarrow & W' \end{array}$$

commutes. The statements in the previous subsection extend mutatis mutandis to principal homogeneous spaces for elliptic curves:

7.4 Addition $E\times E\to E$ makes E into a principal homogeneous space for E. Any principal homogeneous space isomorphic to this principal homogeneous space is said to be ***trivial.***

7.5 Let W and W' be principal homogeneous spaces for E. For any field $K\supset k$ and any points $w_0\in W(K)$, $w_0'\in W'(K)$, there exists a unique morphism $\varphi\colon W\to W'$ over K sending w_0 to w_0'. Every morphism of principal homogeneous spaces is an isomorphism.

7.6 Let W be a principal homogeneous space for E. For any point $w_0\in W(k)$, there is a unique homomorphism $E\to W$ (of principal homogeneous spaces) sending 0 to w_0. Thus W is trivial if and only if $W(k)\neq\emptyset$. Since W will have a point with coordinates in some finite extension of k (this follows from the Hilbert Nullstellensatz), it becomes trivial over such an extension.

7.7 A point $P\in E(K)$ defines an automorphism $w\mapsto w+P$ of W, and every automorphism of W over K is of this form for a unique $P\in E(K)$.

The classification of principal homogeneous spaces

Let W be a principal homogeneous space for E over k, and choose a point $w_0\in W(k^{\mathrm{al}})$. As we observed on p. 19, Γ acts on $W(k^{\mathrm{al}})$. For $\sigma\in\Gamma$, $\sigma w_0=w_0+f(\sigma)$ for a unique $f(\sigma)\in E(k^{\mathrm{al}})$. Note that

$$(\sigma\tau)w_0=\sigma(\tau w_0)=\sigma(w_0+f(\tau))=\sigma w_0+\sigma(f(\tau))=w_0+f(\sigma)+\sigma f(\tau)$$
$$(\sigma\tau)w_0=w_0+f(\sigma\tau)\qquad\text{(definition of } f\text{)},$$

and so f is a crossed homomorphism $\Gamma\to E(k^{\mathrm{al}})$. Because w_0 has coordinates in a finite extension of k, f is continuous. A second point $w_1\in W(k^{\mathrm{al}})$

will define another crossed homomorphism f_1, but $w_1 = w_0 + P$ for some $P \in E(\mathbb{Q}^{\mathrm{al}})$, and so

$$\sigma w_1 = \sigma(w_0+P) = \sigma w_0 + \sigma P = w_0 + f(\sigma) + \sigma P = w_1 + f(\sigma) + \sigma P - P.$$

Hence

$$f_1(\sigma) = f(\sigma) + \sigma P - P,$$

i.e., f and f' differ by a principal crossed homomorphism. Thus the cohomology class of f depends only on W. If the cohomology class is zero, then $f(\sigma) = \sigma P - P$ for some $P \in E(k^{\mathrm{al}})$, and

$$\sigma(w_0 - P) = \sigma w_0 - \sigma P = w_0 + \sigma P - P - \sigma P = w_0 - P.$$

This implies that $w_0 - P \in W(k)$, and so W is a trivial principal homogeneous space (see 7.6).

THEOREM 7.8 *The map $W \mapsto [f]$ is a bijection*

$$\{\text{principal homogeneous spaces for } E\}/\approx \; \to H^1(k, E)$$

sending the trivial principal homogeneous space to the zero element.

PROOF. Let $\varphi\colon W \to W'$ be an isomorphism of principal homogeneous spaces for E (over k), and let $w_0 \in W(k^{\mathrm{al}})$. One checks immediately that (W, w_0) and $(W', \varphi(w_0))$ define the same crossed homomorphism, and hence the map

$$\{\text{principal homogeneous spaces for } E\} \to H^1(k, E)$$

is constant on isomorphism classes. If W and W' define the same cohomology class, we can choose w_0 and w_0' so that (W, w_0) and (W', w_0') define the same crossed homomorphism. There is a unique regular map $\varphi\colon W \to W'$ over k^{al} sending w_0 to w_0' (see 7.5). Let $w \in W(k^{\mathrm{al}})$, and write $w = w_0 + P$. Then

$$\begin{aligned}
\varphi(\sigma w) &= \varphi(\sigma(w_0 + P)) = \varphi(\sigma w_0 + \sigma P) \\
&= \varphi(w_0 + f(\sigma) + \sigma P) = w_0' + f(\sigma) + \sigma P = \sigma w_0' + \sigma P \\
&= \sigma\varphi(w),
\end{aligned}$$

which implies that the map φ is defined over k, i.e., it is defined by polynomials with coordinates in k rather than k^{al} (see I 5.5). Hence the map is one-to-one. Now let $f\colon \Gamma \to E(k^{\mathrm{al}})$ be a crossed homomorphism, and let φ_σ be the regular map $E_{k^{\mathrm{al}}} \to E_{k^{\mathrm{al}}}$ such that $\varphi_\sigma(\sigma P) = P + f(\sigma)$. Then

$$\begin{aligned}
\varphi_{\sigma\tau}(\sigma\tau P) &= P + f(\sigma\tau) \\
&= P + f(\sigma) + \sigma(f(\tau)) \\
&= \sigma(\varphi_\sigma(\sigma^{-1}P)) + f(\sigma) \\
&= \varphi_\sigma(\sigma(\varphi_\sigma(\sigma^{-1}P))) + f(\sigma),
\end{aligned}$$

and so (φ_σ) is a descent system. According to (I 5.3, I 5.4), $(E,(\varphi_\sigma))$ arises from a curve W over k, and it is easy to see that W is a principal homogeneous space for E. □

ASIDE 7.9 Write $\mathrm{WC}(E/k)$ for the set of isomorphism classes of principal homogeneous spaces for E over k. The bijection defines a commutative group structure on $\mathrm{WC}(E/k)$, which can be described as follows: for principal homogeneous spaces W and W', define $W \wedge W'$ to be the quotient of $W \times W'$ by the diagonal action of E, so that

$$(W\wedge W')(k^{\mathrm{al}}) = \left(W(k^{\mathrm{al}}) \times W'(k^{\mathrm{al}})\right)/\sim, \quad (w,w') \sim (w+P, w'+P),\ P \in E(k^{\mathrm{al}});$$

then $W \wedge W'$ has a natural structure of a principal homogeneous space, and represents the sum of W and W' in $\mathrm{WC}(E/k)$.

Geometric interpretation of $H^1(\mathbb{Q}, E_n)$

We now give a geometric interpretation of $H^1(k, E_n)$. An ***n-covering*** is a pair (W, α) consisting of a principal homogeneous space W for E and a regular map $\alpha\colon W \to E$ (defined over k) with the property: for some $w_1 \in W(k^{\mathrm{al}})$, $\alpha(w_1 + P) = nP$ for all $P \in E(k^{\mathrm{al}})$. A morphism $(W,\alpha) \to (W',\alpha')$ of n-coverings is a morphism $\varphi\colon W \to W'$ of principal homogeneous spaces such that $\alpha = \alpha' \circ \varphi$.

For $\sigma \in \Gamma$, write

$$\sigma w_1 = w_1 + f(\sigma), \quad f(\sigma) \in E(k^{\mathrm{al}}). \tag{27}$$

As before, f is a crossed homomorphism with values in $E(k^{\mathrm{al}})$. On applying α to both sides of (27) and using that

$$\alpha(\sigma w_1) = \sigma(\alpha w_1) = \sigma(\alpha(w_1 + O)) = O,$$

we find that $nf = 0$ and so f takes values in $E_n(k^{\mathrm{al}})$. The element w_1 is uniquely determined up to a replacement by $w_1 + Q$, $Q \in E_n(k^{\mathrm{al}})$, from which it follows that the class of f in $H^1(k, E_n)$ is independent of the choice of w_1.

THEOREM 7.10 *The map $(W,\alpha) \mapsto [f]$ defines a bijection*

$$\{n\text{-coverings}\}/{\approx} \to H^1(k, E_n).$$

PROOF. Write $\mathrm{WC}(E_n/k)$ for the set of n-coverings modulo isomorphism. One shows that the map "forget α" $(W,\alpha) \mapsto W$ defines a surjection

$$\mathrm{WC}(E_n/k) \to \mathrm{WC}(E/k)_n$$

and that the fibres are, in a natural way principal homogeneous spaces for $E(k)/nE(k)$. For example, if W is trivial, so that there exists a $w_0 \in W(k)$, then $\alpha(w_0) \in E(k)$; if w_0' also $\in W(k)$, then $w_0' = w_0 + P$ for some $P \in E(k)$, and $\alpha(w_0') = \alpha(w_0) + nP$, and so $\alpha(w_0)$ is well-defined as an element of $E(k)/nE(k)$. Now consider

$$\begin{tikzcd}
 & & \mathrm{WC}(E_n/k) \arrow[r, "b"] \arrow[d, "a"] & \mathrm{WC}(E/k)_n \arrow[d, "\simeq"] & \\
0 \arrow[r] & E(k)/nE(k) \arrow[r] & H^1(k, E_n) \arrow[r, "c"] & H^1(k, E)_n \arrow[r] & 0.
\end{tikzcd}$$

The diagram commutes, and so a maps the fibres of b into the fibres of c. As these fibres are principal homogeneous spaces for $E(k)/nE(k)$, a is bijective on each fibre (by 7.2), and hence is bijective on the entire sets. □

The diagram in the proof gives a geometric interpretation of the exact sequence (22).

ASIDE 7.11 For a principal homogeneous space W for E and a homomorphism $\varphi: E \to E'$, there is a well-defined principal homogeneous space $W' = \varphi_* W$ for E' and a φ-equivariant map $W \to W'$. To give an n-covering amounts to giving a principal homogeneous space W for E and a trivialization of $n_* E$.

Twists of elliptic curves

In this subsection we study the following problem:

> given an elliptic curve E_0 over k, find all elliptic curves E over k that become isomorphic to E_0 over k^{al}.

Such a curve E is often called a ***twist*** of E_0. Remember that an elliptic curve E over k has a distinguished point $O \in E(k)$. Throughout, I assume that the characteristic of k is $\neq 2, 3$, so that we can write our elliptic curve as

$$E(a,b) : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in k, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

Recall (Theorem 2.1, Chap. I) that $E(a,b)$ and $E(a',b')$ are isomorphic if and only if there exists a $c \in k^\times$ such that $a' = c^4 a$, $b' = c^6 b$; every such c defines an isomorphism

$$(x : y : z) \mapsto (c^2 x : c^3 y : z) \colon E(a,b) \to E(a',b'),$$

and all isomorphisms are of this form.

EXAMPLE 7.12 Consider an elliptic curve $E(a,b)$ over k. For any $d \in k^\times$,

$$E_d : dY^2Z = X^3 + aXZ^2 + bZ^3,$$

is an elliptic curve over k that becomes isomorphic to $E(a,b)$ over k^{al}. Indeed, after making the change of variables $dZ \leftrightarrow Z$, the equation becomes

$$Y^2Z = X^3 + \frac{a}{d^2}XZ^2 + \frac{b}{d^3}Z^3,$$

and so E_d becomes isomorphic to $E(a,b)$ over any field in which d is a square.

In order to be able to apply cohomology, we need to compute the group $\mathrm{Aut}(E,0)$ of automorphisms of E fixing the zero element. Since these maps not only send O to O', but also map straight lines in $\mathbb{P}^2$ to straight lines, they are homomorphisms. We apply Theorem 2.1, Chap. I, in the case $(a',b') = (a,b)$.

CASE $ab \neq 0$: Here we seek $c \in k^\times$ such that $c^4 = 1 = c^6$. These equations imply that $c = \pm 1$, and so the only automorphism of (E,O) other than the identity map is

$$(x : y : z) \mapsto (x : -y : z).$$

CASE $a = 0$: Here c can be any 6th root ζ of 1 in k, and the automorphisms of (E,O) are the maps

$$(x : y : z) \mapsto (\zeta^{2i}x : \zeta^{3i}y : z), \quad i = 0,1,2,3,4,5.$$

CASE $b = 0$: Here c can be any 4th root ζ of 1 in k, and the automorphisms of (E,O) are the maps

$$(x : y : z) \mapsto (\zeta^{2i}x : \zeta^{3i}y : z), \quad i = 0,1,2,3.$$

PROPOSITION 7.13 *The automorphism group of (E,O) is $\{\pm 1\}$ unless $j(E)$ is 0 or 1728, in which cases it is isomorphic to $\mu_6(k)$ or $\mu_4(k)$ respectively.*

PROOF. As $j(E) = \frac{1728(4a^3)}{4a^3+27b^2}$, this follows from the above discussion. □

For an elliptic curve E_0 over k, we write $\mathrm{Aut}_{k^{\mathrm{al}}}(E)$ for $\mathrm{Aut}(E_{k^{\mathrm{al}}})$.

REMARK 7.14 (a) Notice that the proposition is consistent with Proposition III 3.17, which says that (over $\mathbb{C}$), $\mathrm{End}(E)$ is isomorphic to $\mathbb{Z}$ or to a subring of the ring of integers in a field $\mathbb{Q}[\sqrt{-d}]$, $d > 0$. The only units in such rings are roots of 1, and only $\mathbb{Q}[\sqrt{-1}]$ and $\mathbb{Q}[\sqrt{-3}]$ contain roots of 1 other than ± 1.

(b) When we allow k to have characteristic 2 or 3, then it is still true that $\mathrm{Aut}(E,O) = \{\pm 1\}$ when $j(E) \neq 0, 1728$, but when $j = 0$ or 1728 the group of automorphisms of (E,O) can have as many as 24 elements.

Fix an elliptic curve E_0 over k, and let E be an elliptic curve over k that becomes isomorphic to E_0 over k^{al}. Choose an isomorphism $\varphi\colon E_0 \to E$ over

k^{al}. For any $\sigma \in \Gamma$, we obtain a second isomorphism $\sigma\varphi \overset{\mathrm{def}}{=} \sigma \circ \varphi \circ \sigma^{-1}\colon E_0 \to E$ over k^{al}. For example, if φ is $(x : y : z) \mapsto (c^2x : c^3y : z)$, then $\sigma\varphi$ is $(x : y : z) \mapsto ((\sigma c)^2x : (\sigma c)^3y : z)$. The two isomorphisms $\varphi, \sigma\varphi\colon E_0 \to E$ (over k^{al}) differ by an automorphism of E_0 over k^{al}:

$$\sigma\varphi = \varphi \circ \alpha(\sigma), \quad \alpha(\sigma) \in \mathrm{Aut}_{k^{\mathrm{al}}}(E_0, O).$$

Note that

$$(\sigma\tau)\varphi = \sigma(\tau\varphi) = \sigma(\varphi \circ \alpha(\tau)) = \varphi \circ \alpha(\sigma) \circ \sigma\alpha(\tau),$$

and so α is a crossed homomorphism $\Gamma \to \mathrm{Aut}_{k^{\mathrm{al}}}(E_0, O)$. Choosing a different isomorphism φ replaces $\alpha(\sigma)$ by its composite with a principal crossed homomorphism.

THEOREM 7.15 *The map $E \mapsto [\alpha]$ defines a one-to-one correspondence*

$$\{\text{elliptic curves over } k, \text{ isomorphic to } E_0 \text{ over } k^{\mathrm{al}}\}/\approx \;\overset{1:1}{\longleftrightarrow}\; H^1(\Gamma, \mathrm{Aut}_{k^{\mathrm{al}}}(E_0)).$$

PROOF. The proof is similar to that of Theorem 7.8. □

COROLLARY 7.16 *If $j(E_0) \neq 0, 1728$, then the list of twists of E_0 in Example 7.12 is complete.*

PROOF. In this case, $\mathrm{Aut}_{k^{\mathrm{al}}}(E, O) = \mu_2$, and so, according to Example 1.8, $H^1(\Gamma, \mu_2) = k^\times/k^{\times 2}$. Under the correspondence in the theorem, $E_d \leftrightarrow d \bmod k^{\times 2}$. □

REMARK 7.17 The same arguments can be used to obtain the description of the twisted multiplicative groups on p. 55. The endomorphisms of the algebraic group $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$ are the maps $t \mapsto t^m$, $m \in \mathbb{Z}$. Hence $\mathrm{End}(\mathbb{G}_m) \simeq \mathbb{Z}$ and $\mathrm{Aut}(\mathbb{G}_m) = (\mathrm{End}(\mathbb{G}_m))^\times = \{\pm 1\}$. The twisted forms of $\mathbb{G}_m$ are classified by

$$H^1(k, \{\pm 1\}) \simeq H^1(k, \mu_2) \simeq k^\times/k^{\times 2}$$

(see 1.8). The twisted multiplicative group corresponding to $a \in k^\times/k^{\times 2}$ is $\mathbb{G}_m[a]$.

REMARK 7.18 Let $\mathrm{Aut}(E)$ be the group of all automorphisms of E, not necessarily preserving O. The map $Q \mapsto t_Q$, where t_Q is the translation $P \mapsto P + Q$, identifies $E(k)$ with a subgroup of $\mathrm{Aut}(E)$. I claim that $\mathrm{Aut}(E)$ is a semi-direct product,

$$\mathrm{Aut}(E) = E(k) \rtimes \mathrm{Aut}(E, O),$$

i.e., that

(a) $E(k)$ is a normal subgroup of $\mathrm{Aut}(E)$;
(b) $E(k) \cap \mathrm{Aut}(E, O) = \{0\}$;
(c) $\mathrm{Aut}(E) = E(k) \cdot \mathrm{Aut}(E, O)$.

Let $Q \in E(k)$ and let $\alpha \in \mathrm{Aut}(E, O)$. As we noted above, α is a homomorphism, and so, for any $P \in E$,

$$(\alpha \circ t_Q \circ \alpha^{-1})(P) = \alpha(\alpha^{-1}(P) + Q) = P + \alpha(Q) = t_{\alpha(Q)}(P),$$

which proves (a). Assertion (b) is obvious. For (c), let $\gamma \in \mathrm{Aut}(E)$, and let $\gamma(0) = Q$; then $\gamma = t_Q \circ (t_{-Q} \circ \gamma)$, and $t_{-Q} \circ \gamma \in \mathrm{Aut}(E, O)$.

Curves of genus 1

Let W be a principal homogeneous space for an elliptic curve E over k. Then W becomes isomorphic to E over k^{al}, and so W is projective, nonsingular, and of genus 1 (at least over k^{al}, which implies that it is also over k). The next theorem shows that, conversely, every projective nonsingular curve W of genus 1 over k occurs as a principal homogeneous space for some elliptic curve over k.

THEOREM 7.19 *Let W be a nonsingular projective curve over k of genus 1. Then there exists an elliptic curve E_0 over k such that W is a principal homogeneous space for E_0. Moreover, E_0 is unique up to an isomorphism (over k).*

PROOF. (Sketch, assuming that the characteristic of k is not 2 or 3). By assumption, there exists an isomorphism $\varphi\colon W \to E$ from W to an elliptic curve E over k^{al}, which we may suppose to be in our standard form

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in k, \quad \Delta = 4a^3 + 27b^2 \neq 0.$$

Let $\sigma \in \Gamma$. Then $\sigma\varphi$ is an isomorphism $\sigma W \to \sigma E$. Here σW and σE are obtained from W and E by applying σ to the coefficients of the polynomials defining them (so $E = E(\sigma a, \sigma b)$). But W is defined by polynomials with coefficients in k, and so $\sigma W = W$. Therefore $E \approx W \approx \sigma E$, and so $j(E) = j(\sigma E) = \sigma j(E)$. Since this is true for all $\sigma \in \Gamma$, $j(E)$ lies in k. Now (see 2.3) there is a curve E_0 over k with $j(E_0) = j(E)$. In fact, there will be many such curves over k, and so we have to make sure we have the correct one. We choose one, E_0, and twist it to get the correct one.

Choose an isomorphism $\varphi\colon E_0 \to W$ over k^{al}, and for $\sigma \in \Gamma$, let $\sigma\varphi = \varphi \circ \alpha(\sigma)$ where $\alpha(\sigma) \in \mathrm{Aut}_{k^{\mathrm{al}}}(E_0)$. Then $\sigma \mapsto \alpha(\sigma)$ is a crossed homomorphism into $\mathrm{Aut}_{k^{\mathrm{al}}}(E_0)$, and hence defines a class $[\alpha]$ in $H^1(k, \mathrm{Aut}_{k^{\mathrm{al}}}(E_0))$. According to (7.18) there is an exact sequence

$$1 \to E_0(k^{\mathrm{al}}) \to \mathrm{Aut}_{k^{\mathrm{al}}}(E_0) \to \mathrm{Aut}_{k^{\mathrm{al}}}(E_0, O) \to 1.$$

If $[\alpha]$ lies in the subgroup $H^1(k, E_0)$ of $H^1(k, \mathrm{Aut}_{k^{\mathrm{al}}}(E_0))$, then W is a principal homogeneous space for E_0. If not, we use the image of $[\alpha]$ in $H^1(k, \mathrm{Aut}_{k^{\mathrm{al}}}(E_0, O))$ to twist E_0 to obtain a second curve E_1 over k with the same j-invariant. Now one can check that the class of the crossed homomorphism $[\alpha]$ lies in $H^1(k, E_1)$, and so W is a principal homogeneous space for E_1. □

The curve E_0 given by the theorem is called the ***jacobian*** of W. It is characterized by the following property: there exists an isomorphism $\varphi\colon E_0 \to W$ over k^{al} such that, for every $\sigma \in \Gamma$, there exists a point $Q_\sigma \in E_0(k^{\mathrm{al}})$ such that

$$(\sigma\varphi)(P) = \varphi(P + Q_\sigma), \quad \text{all } P \in E(k^{\mathrm{al}}).$$

To find it:

(a) by a change of variables over $\mathbb{Q}^{\mathrm{al}}$, obtain an isomorphism $W \approx E$ where E is an elliptic curve over $\mathbb{Q}^{\mathrm{al}}$ in standard form;
(b) write down an elliptic curve E_0 over $\mathbb{Q}$ in standard form that becomes isomorphic to E over $\mathbb{Q}$;
(c) modify E_0 if necessary so that it has the property characterizing the jacobian.

REMARK 7.20 In the above proof we used crossed homomorphisms into $\mathrm{Aut}_{k^{\mathrm{al}}}(E_0)$, which need not be commutative. However, one can still define $H^1(G, M)$ when M is not commutative. Write M multiplicatively. As in the commutative case, a crossed homomorphism is a map $f\colon G \to M$ such that $f(\sigma\tau) = f(\sigma)\cdot\sigma f(\tau)$. Call two crossed homomorphisms f and g equivalent if there exists an $m \in M$ such that $g(\sigma) = m^{-1}\cdot f(\sigma)\cdot\sigma m$, and let $H^1(G, M)$ be the set of equivalence classes of crossed homomorphisms. It is a set with a distinguished element, namely, the map $\sigma \mapsto 1$.

The classification of curves of genus 1 over $\mathbb{Q}$

We summarize the above results.

7.21 Let (E, O) be an elliptic curve over $\mathbb{Q}$. We attach to it the invariant $j(E) \in \mathbb{Q}$. Every element of $\mathbb{Q}$ occurs as the j-invariant of an elliptic curve over $\mathbb{Q}$, and two elliptic curves over $\mathbb{Q}$ have the same j-invariant if and only if they become isomorphic over $\mathbb{Q}^{\mathrm{al}}$ (II 2.1).

7.22 Fix a $j \in \mathbb{Q}$, and consider the elliptic curves (E, O) over $\mathbb{Q}$ with $j(E) = j$. Choose such an (E, O). The isomorphism classes of such curves are in natural one-to-one correspondence with the elements of $H^1(\mathbb{Q}, \mathrm{Aut}(E, O))$. For example, if $j \neq 0, 1728$, then $\mathrm{Aut}(E, O) = \mu_2$, $H^1(\mathbb{Q}, \mathrm{Aut}(E, O)) = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, and the curve corresponding to $d \in \mathbb{Q}^\times$ is the curve E_d of Example 7.12.

7.23 Fix an elliptic curve (E, O) over $\mathbb{Q}$, and consider the curves of genus 1 over $\mathbb{Q}$ having E as their jacobian. Such a curve has the structure of a principal homogeneous space for E, and every principal homogeneous space for E has E as its jacobian. The principal homogeneous spaces for E are classified by the group $H^1(\mathbb{Q}, E)$ (which is a very large group).

7.24 Every curve of genus 1 over $\mathbb{Q}$ has a jacobian, which is an elliptic curve E over $\mathbb{Q}$, and the curve is a principal homogeneous space for E.

EXERCISE 7.25 Find the jacobian of the curve

$$W : aX^3 + bY^3 + cZ^3 = 0, \quad a, b, c \in \mathbb{Q}^\times.$$

Hint: The curve $E : X^3 + Y^3 + dZ^3 = 0$, $d \in \mathbb{Q}^\times$, has the point $O : (1 : -1 : 0)$ — the pair (E, O) is an elliptic curve over $\mathbb{Q}$. It can be put in standard form by the change of variables $X = X' + Y'$, $Y = X' - Y'$.

8 Failure of the Hasse (local-global) principle

We discuss a family of curves whose Tate-Shafarevich groups are nonzero, and which therefore give examples of elliptic curves for which the Hasse principle fails.

PROPOSITION 8.1 *If* $p \equiv 1 \mod 8$, *then the* 2-*Selmer group* $S^{(2)}(E/\mathbb{Q})$ *of the elliptic curve*

$$E : Y^2Z = X^3 + pXZ^2$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

For a detailed proof of the proposition, see Silverman 1986, X 6.2 — I merely make a few remarks. The family of curves in the statement is similar to that in Exercise 5.9(2), but since only one of the points of order 2 on E has coordinates in $\mathbb{Q}$, we don't have a simple description of $H^1(\mathbb{Q}, E_2)$. Of course, one can pass to $\mathbb{Q}[\sqrt{p}]$, but it is easier to proceed as follows. Let E' be the quotient of E by the subgroup generated by its point $P = (0 : 0 : 1)$ of order 2. There are homomorphisms

$$E \xrightarrow{\phi} E' \xrightarrow{\psi} E$$

whose composite is multiplication by 2. From a study of the cohomology sequences of

$$0 \to \langle P \rangle \to E(\mathbb{Q}^{\mathrm{al}}) \xrightarrow{\phi} E'(\mathbb{Q}^{\mathrm{al}}) \to 0$$

and

$$0 \to \operatorname{Ker} \psi \to E'(\mathbb{Q}^{\mathrm{al}}) \xrightarrow{\psi} \mathbb{Q}^{\mathrm{al}}) \to 0$$

one can draw information about $E(\mathbb{Q})/2E(\mathbb{Q})$, $S^{(2)}(E/\mathbb{Q})$, $Ш(E/\mathbb{Q})_2$. For example, the kernel-cokernel sequence (2.2) of the maps

$$E(\mathbb{Q}) \xrightarrow{\phi} E'(\mathbb{Q}) \xrightarrow{\psi} E(\mathbb{Q})$$

is an exact sequence

$$E(\mathbb{Q})/\phi(E(\mathbb{Q})) \to E(\mathbb{Q})/2E(\mathbb{Q}) \to E(\mathbb{Q})/\psi(E(\mathbb{Q})) \to 0.$$

Since $E(\mathbb{Q})_2 \simeq \mathbb{Z}/2\mathbb{Z}$, it follows from (22) that

$$\mathrm{rank}(E(\mathbb{Q})) + \dim_{\mathbb{F}_2} Ш(E/\mathbb{Q})_2 = \dim_{\mathbb{F}_2} S^{(2)}(E/\mathbb{Q}) - 1.$$

Thus $r = 0, 1$, or 2. If $Ш(E/\mathbb{Q})(2)$ is finite, its order is known to be a square (see 5.4), and so, conjecturally, the only possibilities are $r = 0, 2$.

PROPOSITION 8.2 *Let E be as in (8.1). If 2 is not a fourth power modulo p, then $\mathrm{rank}(E(\mathbb{Q})) = 0$ and $Ш(E/\mathbb{Q})_2 \approx (\mathbb{Z}/2\mathbb{Z})^2$.*

We discuss the proof below.

REMARK 8.3 It is, of course, easy (for a computer) to check for any particular prime whether 2 is a fourth power modulo p, but Gauss found a more efficient test. From basic algebra, we know that the ring of Gaussian integers, $\mathbb{Z}[i]$, is a principal ideal domain. An odd prime p either remains prime in $\mathbb{Z}[i]$ or it factors as $p = (A + iB)(A - iB)$. The first case occurs exactly when $\mathbb{Z}[i]/p\mathbb{Z}[i]$ is a field extension of $\mathbb{F}_p$ of degree 2. Therefore p remains prime if and only if $\mathbb{F}_p$ doesn't contain a primitive 4th root of 1. Because $\mathbb{F}_p^\times$ is cyclic, it contains an element of order 4 if and only if 4 divides its order $p - 1$. We conclude that the odd primes p that can be expressed as $p = A^2 + B^2$ with $A, B \in \mathbb{Z}$ are exactly those such that $p \equiv 1 \bmod 4$.

Gauss showed that for a prime $p \equiv 1 \mod 8$, 2 is a 4th power modulo p if and only if $8|AB$. Therefore, p satisfies the hypotheses of the proposition if p is

$$17 = 1^2 + 4^2, \quad 41 = 5^2 + 4^2, \quad 97 = 9^2 + 4^2, \quad 193 = 7^2 + 12^2 ...$$

The proof of Gauss's criterion, which is quite elementary, can be found in Silverman 1986, X 6.6. Number theorists will wish to prove that there are infinitely many such primes p (and find their density).

It is very difficult to show directly that the rank of an elliptic curve is smaller than the bound given by the Selmer group. Instead, in this case, one exhibits 3 nontrivial elements of $Ш(E/\mathbb{Q})_2$. They are:

$$Y^2 = 4pX^4 - 1, \quad \pm Y^2 = 2pX^4 - 2.$$

One can (no doubt) check directly that these three curves are principal homogeneous spaces for $E : Y^2Z = X^3 + pZ^3$, but it can be more easily seen from the proof of Proposition 8.1 (Silverman 1986, 6.2b).

REMARK 8.4 We should explain what we mean by these curves. Consider, more generally, the curve

$$C : Y^2 = aX^4 + bX^3 + cX^2 + dX + e$$

where the polynomial on the right has no repeated roots. Assume that the characteristic is $\neq 2, 3$. Then this is a nonsingular affine curve, but its projective closure

$$\bar{C} : Y^2Z^2 = aX^4 + bX^3Z + cX^2Z^2 + dXZ^3 + eZ^4$$

is singular: on setting $Y = 1$, we obtain the equation

$$Z^2 = aX^4 + bX^3Z + cX^2Z^2 + dXZ^3 + eZ^4$$

which is visibly singular at $(0,0)$. The genus of a plane projective curve of degree d is

$$g = \frac{(d-1)(d-2)}{2} - \sum_{P \text{ singular}} \delta_P.$$

For $P = (0,0)$, $\delta_P = 2$, and so the genus of $\bar{C}$ is $3 - 2 = 1$. One can resolve the singularity to obtain a nonsingular curve C' and a regular map $C' \to \bar{C}$ which is an isomorphism except over the singular point. It is really C' that one means when one writes C.

We shall prove that the curve

$$C : Y^2 = 2 - 2pX^4$$

has no points in $\mathbb{Q}$, but has points in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p. For this we shall need to use the quadratic reciprocity law. For an integer a not divisible by the prime p, the ***Legendre symbol*** $\left(\frac{a}{p}\right)$ is $+1$ if a is a square modulo p and is -1 otherwise.

THEOREM 8.5 (QUADRATIC RECIPROCITY LAW) *For odd primes p, q,*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Moreover,

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

PROOF. The theorem surely has more published proofs than any other in mathematics. The first six complete proofs were found by Gauss. Most introductory books on number theory contain a proof. For a nonelementary proof, see ANT, p. 123. □

We now prove that

$$C : Y^2 = 2 - 2pX^4$$

has no points with coordinates in $\mathbb{Q}$. Suppose (x, y) is a point on the curve. Let $x = r/t$ with r and t integers having no common factor. Then

$$y^2 = \frac{2t^4 - 2pr^4}{t^4}.$$

The numerator and denominator on the right are integers with no common factor, and so $2t^4 - 2pr^4$ is the square of an integer, which must be even. Therefore, there exists an integer s such that

$$\boxed{2s^2 = t^4 - pr^4.}$$

Let q be an odd prime dividing s. Then $t^4 \equiv pr^4 \mod q$, and so $\left(\frac{p}{q}\right) = 1$. According to the quadratic reciprocity law, this implies that $\left(\frac{q}{p}\right) = 1$. From the quadratic reciprocity law, $\left(\frac{2}{p}\right) = 1$, and so all prime factors of s are squares modulo p. Hence s^2 is a 4th power modulo p. The equation

$$2s^2 \equiv t^4 \mod p$$

now shows that 2 is a 4th power modulo p, which contradicts our hypothesis.

We should also make sure that there is no point lurking at infinity. The projective closure of C is

$$\bar{C} : Y^2Z^2 = 2Z^4 - 2pX^4,$$

and we have just shown that $\bar{C}$ has no rational point with $Z = 1$. For $Z = 0$, there is a rational solution, namely, $(0 : 1 : 0)$, but this is the singular point $(0, 0)$ on the curve

$$Z^2 = 2Z^4 - 2pX^4.$$

On the desingularization $C' \to \bar{C}$ of $\bar{C}$, no $\mathbb{Q}$-point lies over $(0 : 1 : 0)$.

The curve C obviously has points in $\mathbb{R}$. Hensel's lemma (I 2.12) shows that C has a point in $\mathbb{Q}_q$ if its reduction modulo the prime q has a nonsingular point with coordinates in $\mathbb{F}_q$. For $q \neq 2, p$, the curve C has good reduction at q, and Corollary 9.3 below shows that it has a point with coordinates in $\mathbb{F}_p$. Therefore, C automatically has a point with coordinates in $\mathbb{Q}_q$ except possibly for q equal to 2 or p. These two primes require a more elaborate application of Hensel's lemma, which we leave to the reader.

9 Elliptic curves over finite fields

We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$, and let $\mathbb{F}_q$ (q a power of p) be the subfield of $\mathbb{F}$ whose elements are the roots of $X^q - X$ in $\mathbb{F}$. Let $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$.

In this section, I will often make use of the fact that many of the power series identities in calculus are, in fact, valid over any commutative ring. For example, we can define

$$\begin{aligned} \log(1+T) &= T - \tfrac{1}{2}T^2 + \tfrac{1}{3}T^3 - \tfrac{1}{4}T^4 + \tfrac{1}{5}T^5 - \cdots \\ \exp(T) &= 1 + T + \tfrac{1}{2!}T^2 + \cdots + \tfrac{1}{n!}T^n + \cdots \end{aligned}$$

and then

$$\begin{aligned} &\log \frac{1}{1-T} = -\log(1-T) = T + \tfrac{1}{2}T^2 + \tfrac{1}{3}T^3 + \tfrac{1}{4}T^4 + \tfrac{1}{5}T^5 + \cdots, \text{ and} \\ &\exp(\log(1+T)) = 1 + T. \end{aligned}$$

The numbers a_p

Let $E : Y^2Z = X^3 + aXZ^2 + bZ^3$ be an elliptic curve over $\mathbb{F}_p$, $p \neq 2$. There are only finitely many points on E with coordinates in $\mathbb{F}_p$, and we wish to find the number N_p of them. There is one point $(0:1:0)$ "at infinity", and the remainder are the solutions of

$$Y^2 = X^3 + aX + b$$

in $\mathbb{F}_p$. One way of counting them is to make a list of the squares in $\mathbb{F}_p$, and then check the values of $x^3 + ax + b$, $x \in \mathbb{F}_p$, against the list. For $z \in \mathbb{F}_p^\times$, let $\chi(z) = 1$ if z is a square in $\mathbb{F}_p^\times$ and -1 otherwise; extend χ to $\mathbb{F}_p$ by setting $\chi(0) = 0$. Then

$$\begin{aligned} N_p &= 1 + \sum_{x \in \mathbb{F}_p} (\chi(x^3 + ax + b) + 1) \\ &= p + 1 - a_p \end{aligned}$$

where

$$-a_p = \sum_{x \in \mathbb{F}_p} \chi(x^3 + ax + b). \tag{28}$$

As $\mathbb{F}_p^\times$ is cyclic of even order, exactly half of its elements are squares. Since there is no reason to expect that $x^3 + ax + b$ is more (or less) likely to be a square than not, we might expect that $-a_p$ is a sum of p terms randomly distributed between $+1$ and -1. The expected value of $|a_p|$ would then be of the order $\sqrt{p}$. Calculations support this when we fix $a, b \in \mathbb{Z}$ and compute a_p for the reductions of E modulo the different primes p. However, we always find (for good reduction) that

$$|a_p| < 2\sqrt{p} \tag{29}$$

whereas, if the terms in the sum (28) were truly random, then everything between 0 and $p-1$ would be possible. We shall prove the inequality (29) presently, and, in the remainder of the book, we shall see the innocuous-looking numbers a_p turn into key players.

The Frobenius map

Let C be a projective plane curve of degree d over $\mathbb{F}_p$, so that C is defined by a polynomial

$$F(X,Y,Z)=\sum_{i+j+k=d} a_{ijk}X^iY^jZ^k,\quad a_{ijk}\in\mathbb{F}_p.$$

If $P=(x:y:z)\in C(\mathbb{F})$, then

$$\sum_{i+j+k=d} a_{ijk}x^iy^jz^k=0.$$

On raising this equation to the pth power, remembering that we are in characteristic p and that $a^p=a$ for all $a\in\mathbb{F}_p$, we obtain the equation

$$\sum_{i+j+k=d} a_{ijk}x^{ip}y^{jp}z^{kp}=0,$$

which says that $(x^p:y^p:z^p)$ also lies on C. We therefore obtain a map

$$(x:y:z)\mapsto(x^p:y^p:z^p)\colon C\to C,$$

which, being defined by polynomials, is regular. It is called the ***Frobenius map.***

PROPOSITION 9.1 *(a) The degree of the Frobenius map is p.*
(b) The Frobenius map as zero acts on the tangent space at O.

PROOF. (a) From the diagram

$$\begin{tikzcd} k(x,y) \arrow[r, "(x,y)\mapsto(x^p,y^p)"] \arrow[d, dash, "2"] & k(x,y) \arrow[d, dash, "2"] \\ k(X) \arrow[r, "X\mapsto X^p"] & k(X) \end{tikzcd}$$

we see that

$$\deg\varphi\overset{\text{def}}{=}[k(x,y):k(x^p,y^p)]=[k(X):k(X^p)]=p.$$

(b) Obvious. □

Curves of genus 1 over $\mathbb{F}_p$

PROPOSITION 9.2 *For any elliptic curve E over $\mathbb{F}_p$, $H^1(\mathbb{F}_p, E) = 0$.*

PROOF. We first review the structure $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ (see, for example, FT, 7.15). There is a canonical element $a \mapsto a^p$ in Γ, called the ***Frobenius automorphism*** — denote it by σ. Then σ generates Γ topologically in the sense that Γ is the closure of the subgroup $\langle\sigma\rangle$ generated by σ.

In order to prove the proposition, we must show that every continuous crossed homomorphism $f\colon \Gamma \to E(\mathbb{F})$ is principal. Let $\varphi\colon E \to E$ be the Frobenius map. Then $\varphi - \mathrm{id}_E$ is a nonconstant regular map $E \to E$, and so the map

$$P \mapsto \varphi(P) - P\colon E(\mathbb{F}) \to E(\mathbb{F})$$

is surjective (4.23). In particular, there exists a $P \in E(\mathbb{F})$ such that $\varphi(P) - P = f(\sigma)$, i.e., such that $f(\sigma) = \sigma P - P$. Then

$$f(\sigma^2) = f(\sigma) + \sigma f(\sigma) = \sigma P - P + \sigma^2 P - \sigma P = \sigma^2 P - P,$$

$$\cdots$$

$$f(\sigma^n) = f(\sigma) + \sigma f(\sigma^{n-1}) = \sigma P - P + \sigma(\sigma^{n-1} P - P) = \sigma^n P - P.$$

Therefore f and the principal crossed homomorphism $\tau \mapsto \tau P - P$ agree on σ^n for all n. Because both are continuous, this implies that they agree on the whole of Γ. □

COROLLARY 9.3 *A nonsingular projective curve C of genus 1 over $\mathbb{F}_p$ has a point with coordinates in $\mathbb{F}_p$.*

PROOF. According to (7.19), the curve C is a principal homogeneous space for its jacobian E, and according to the Proposition, it is a trivial principal homogeneous space, i.e., $C(\mathbb{F}_p) \neq \emptyset$. □

Proof of the Riemann hypothesis for elliptic curves

THEOREM 9.4 (CONGRUENCE RIEMANN HYPOTHESIS) *For an elliptic curve E over $\mathbb{F}_p$,*

$$|\#E(\mathbb{F}_p) - (p+1)| \le 2p.$$

PROOF. The kernel of

$$\mathrm{id}_E - \varphi\colon E(\mathbb{F}) \to E(\mathbb{F})$$

is the set of points $(x\colon y\colon z)$ on E such that $(x^p\colon y^p\colon z^p) = (x\colon y\colon z)$. These are precisely the points with a representative $(x\colon y\colon z)$ having $x, y, z \in \mathbb{F}_p$. Because φ acts as zero on the tangent space at O, $\mathrm{id}_E - \varphi$ acts as the identity map, and so $\mathrm{id}_E - \varphi$ is separable (I 4.26). It follows (I 4.24) that $\#E(\mathbb{F}_p) = \deg(\mathrm{id}_E - \varphi)$.

Let
$$f(X) = X^2 + cX + d \overset{\text{def}}{=} \det(X - \varphi | T_\ell E)$$
be the characteristic polynomial of φ. Then $d = \deg(\varphi) = p$ and $c^2 \leq 4d = 4p$ (III 3.23). Now
$$\begin{aligned}\deg(\mathrm{id}_E - \varphi) &= \det(\mathrm{id}_E - \varphi | T_\ell(E)) \\ &= f(1) \\ &= 1 + c + p,\end{aligned}$$
and so
$$\left|\#E(\mathbb{F}_p) - (p+1)\right| = |c| \leq 2\sqrt{p}. \qquad \square$$

After this brief proof, we spend the rest of the section explaining why the inequality is called the (congruence) Riemann hypothesis.

ASIDE 9.5 Let E be an elliptic curve over $\mathbb{Q}$. For a prime p where E has good reduction, write $\#E_p(\mathbb{F}_p) = p + 1 - a_p$. We can regard the Riemann hypothesis as saying that $\#E_p(\mathbb{F}_p)$ is approximately $p+1$ with error term a_p with $|a_p| \leq 2\sqrt{p}$. It turns out that we can say more about the error term. In the above proof, we showed that the characteristic polynomial of φ is $X^2 - a_pX + p$. As $|a_p| \leq 2\sqrt{p}$,
$$X^2 - a_pX + p = (X - \alpha_p)(X - \bar{\alpha}_p),$$
where $\bar{\alpha}_p$ is the complex conjugate of α_p. As $\alpha_p\bar{\alpha}_p = p$, we have $|\alpha_p| = p^{\frac{1}{2}}$, and we may suppose that α_p has been chosen so that $\alpha_p = \sqrt{p}e^{i\theta(p)}$ with $0 \leq \theta(p) \leq \pi$. For a curve without complex multiplication (i.e., with endomorphism ring $\mathbb{Z}$), the Sato-Tate conjecture says that the density of primes p such that $a \leq \theta(p) \leq b$ is
$$\int_a^b \frac{2}{\pi} \sin^2 t \, dt.$$
Sato discovered this experimentally, and Tate gave a heuristic derivation of it from his conjecture on algebraic cycles applied to a product of copies of E (Tate 1965, pp. 105–106). Recently, Richard Taylor and his collaborators (Clozel, Harris, Shepherd-Barron) have announced a proof of the conjecture (Mazur 2006).

Zeta functions of number fields

First recall that the original (i.e., Riemann's) Riemann zeta function is
$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} n^{-s}, \quad s \in \mathbb{C}, \quad \Re(s) > 1. \tag{30}$$
The second equality is an expression of unique factorization:
$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} = \prod_p \left(1 + p^{-s} + (p^{-s})^2 + (p^{-s})^3 + \cdots\right);$$

on multiplying out this product, we obtain a sum of terms

$$(p_1^{-s})^{r_1}(p_2^{-s})^{r_2}\cdots(p_t^{-s})^{r_t} = (p_1^{r_1}\cdots p_t^{r_t})^{-s}.$$

Both the sum and the product in (30) converge for $\Re(s) > 1$, and so $\zeta(s)$ is holomorphic and nonzero for $\Re(s) > 1$. In fact, $\zeta(s)$ extends to a meromorphic function on the whole complex plane with a simple pole at $s = 0$. Moreover, the function $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ satisfies the functional equation $\xi(s) = \xi(1-s)$, has simple poles at $s = 0, 1$, and is otherwise holomorphic. Here $\Gamma(s)$ is the gamma function. Since $\Gamma(s)$ has poles at $s = 0, -1, -2, -3, \ldots$, this forces ζ to be zero at $s = -2n$, $n > 0$, $n \in \mathbb{Z}$. These are called the trivial zeros of the zeta function.

CONJECTURE 9.6 (RIEMANN HYPOTHESIS) *The nontrivial zeros of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$.*

This is probably the most famous problem remaining in mathematics.[11]

Dedekind extended Riemann's definition by attaching a zeta function $\zeta_K(s)$ to any number field K. He defined

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1-\mathbb{N}(\mathfrak{p})^{-s}} = \sum_{\mathfrak{a}} \mathbb{N}(\mathfrak{a})^{-s}, \quad s \in \mathbb{C}, \quad \Re(s) > 1.$$

The first product is over the nonzero prime ideals in $\mathcal{O}_K$, and the second is over all nonzero ideals in $\mathcal{O}_K$. The ***numerical norm*** $\mathbb{N}\mathfrak{a}$ of an ideal $\mathfrak{a}$ is the order of the quotient ring $\mathcal{O}_K/\mathfrak{a}$. The proof of the second equality uses that every nonzero ideal $\mathfrak{a}$ has a unique factorization $\mathfrak{a} = \prod \mathfrak{p}_i^{r_i}$ into a product of powers of prime ideals (see 3.9) and that $\mathbb{N}(\mathfrak{a}) = \prod \mathbb{N}(\mathfrak{p}_i)^{r_i}$. Note that for $K = \mathbb{Q}$, this definition gives back $\zeta(s)$. The function $\zeta_K(s)$ extends to a meromorphic function on the whole complex plane with a simple pole at $s = 1$, and a certain multiple $\xi_K(s)$ of it satisfies a functional equation $\xi_K(s) = \xi_K(1-s)$. It is conjectured that the nontrivial zeros of $\zeta_K(s)$ lie on the line $\Re(s) = \frac{1}{2}$ (generalized Riemann hypothesis).

Zeta functions of affine plane curves over finite fields

Consider an affine plane curve

$$C : f(X,Y) = 0$$

over the field $\mathbb{F}_p$. In analogy with the above definition, we define

$$\zeta(C,s) = \prod_{\mathfrak{p}} \frac{1}{1-\mathbb{N}\mathfrak{p}^{-s}}, \quad \Re(s) > 1, \tag{31}$$

[11] It is one of the seven Millennium Problems (www.claymath.org/millennium/).

where $\mathfrak{p}$ runs over the nonzero prime ideals in $\mathbb{F}_p[x,y] \stackrel{\text{def}}{=} \mathbb{F}_p[X,Y]/(f(X,Y))$. For any nonzero prime ideal $\mathfrak{p}$ of $\mathbb{F}_p[x,y]$, the quotient $\mathbb{F}_p[x,y]/\mathfrak{p}$ is finite, and we denote its order by $\mathbb{N}\mathfrak{p}$.

Because $\mathbb{F}_p[x,y]/\mathfrak{p}$ is finite and an integral domain, it is a field, and we let $\deg\mathfrak{p}$ denote its degree over $\mathbb{F}_p$. Then $\mathbb{N}\mathfrak{p} = p^{\deg\mathfrak{p}}$, which allows us to make a change of variables in the zeta function: when we define[12]

$$Z(C,T) = \prod_{\mathfrak{p}} \frac{1}{1-T^{\deg\mathfrak{p}}}, \tag{32}$$

then

$$\zeta(C,s) = Z(C,p^{-s}).$$

The product $\prod \frac{1}{1-T^{\deg\mathfrak{p}}}$ converges for all small T, but usually we simply regard it as a formal power series

$$Z(C,T) = \prod_{\mathfrak{p}} (1 + T^{\deg\mathfrak{p}} + T^{2\deg\mathfrak{p}} + T^{3\deg\mathfrak{p}} + \cdots) \in \mathbb{Z}[[T]].$$

EXAMPLE 9.7 We can realize $\mathbb{A}^1$ as the affine plane curve $Y=0$, so that

$$\mathbb{F}_p[x,y] = \mathbb{F}_p[X,Y]/(Y) = \mathbb{F}_p[X].$$

The nonzero ideals on $\mathbb{F}_p[X]$ are in one-to-one correspondence with the monic polynomials $f \in \mathbb{F}_p[X]$, and the ideal (f) is prime if and only if f is irreducible. Thus

$$Z(\mathbb{A}^1,T) = \prod_{f \text{ monic irreducible}} \frac{1}{1-T^{\deg f}}.$$

Take logs of both sides,

$$\log Z(\mathbb{A}^1,T) = -\sum_f \log(1-T^{\deg f}),$$

and then differentiate

$$\begin{aligned}\frac{Z'(\mathbb{A}^1,T)}{Z(\mathbb{A}^1,T)} &= \sum_f \frac{\deg f \cdot T^{\deg f - 1}}{1-T^{\deg f}} \\ &= \sum_f \sum_{n\geq 0} \deg f \cdot T^{(n+1)\deg f - 1}.\end{aligned}$$

In this power series, the coefficient of T^{m-1} is $\sum \deg f$, where f runs over all monic irreducible polynomials $f \in \mathbb{F}_p[T]$ of degree dividing m. These

[12]The Z is an upper case zeta.

polynomials are exactly the irreducible factors of $X^{p^m} - X$ (FT, 4.20), and so the coefficient of T^{m-1} is p^m, and we have shown that

$$\frac{Z'(\mathbb{A}^1, T)}{Z(\mathbb{A}^1, T)} = \sum p^m T^{m-1}.$$

On integrating, we find that

$$\log Z(\mathbb{A}^1, T) = \sum \frac{p^m T^m}{m} = \log \frac{1}{1 - pT}, \tag{33}$$

and so

$$Z(\mathbb{A}^1, T) = \frac{1}{1 - pT}.$$

Expressing $Z(C, T)$ in terms of the points of C

Let $N_m = \#\mathbb{A}^1(\mathbb{F}_{p^m}) = p^m$. Then (33) can be rewritten as

$$\log Z(\mathbb{A}^1, T) = \sum \frac{N_m T^m}{m}.$$

We show that a similar formula holds for any nonsingular affine curve over $\mathbb{F}_p$.

Let $C\colon f(X, Y) = 0$ be a nonsingular affine curve over $\mathbb{F}_p$, and let $\mathbb{F}_p[x, y] = \mathbb{F}_p[X, Y]/(f)$. As for $\mathbb{A}^1$,

$$\begin{aligned} \frac{Z'(C, T)}{Z(C, T)} &= \sum_{\mathfrak{p}} \frac{\deg \mathfrak{p} \cdot T^{\deg \mathfrak{p} - 1}}{1 - T^{\deg \mathfrak{p}}} \\ &= \sum_{\mathfrak{p}} \sum_{n \geq 0} \deg \mathfrak{p} \cdot T^{(n+1)\deg \mathfrak{p}} / T. \end{aligned}$$

In this power series, the coefficient of T^{m-1} is $\sum \deg \mathfrak{p}$ where $\mathfrak{p}$ runs over the nonzero prime ideals of $k[x, y]$ such that $\deg \mathfrak{p}$ divides m. But $\deg \mathfrak{p} \overset{\text{def}}{=} [\mathbb{F}_p[x, y]/\mathfrak{p} : \mathbb{F}_p]$, and so the condition $\deg \mathfrak{p} | m$ means that there exists a homomorphism $\mathbb{F}_p[C]/\mathfrak{p} \hookrightarrow \mathbb{F}_{p^m}$ (FT 4.21). There will in fact be exactly $\deg \mathfrak{p}$ such homomorphisms because $\mathbb{F}_p[C]/\mathfrak{p}$ is separable over $\mathbb{F}_p$. Conversely, every homomorphism $\mathbb{F}_p[x, y] \to \mathbb{F}_{p^m}$ factors through $\mathbb{F}_p[x, y]/\mathfrak{p}$ for some prime ideal with $\deg \mathfrak{p} | m$. Therefore, the coefficient of T^{m-1} in the above power series is the number of homomorphisms (of $\mathbb{F}_p$-algebras)

$$\mathbb{F}_p[x, y] \to \mathbb{F}_{p^m}.$$

But a homomorphism $\mathbb{F}_p[x, y] \to \mathbb{F}_{p^m}$ is determined by the images a, b of x, y, and conversely the homomorphism $P(X, Y) \mapsto P(a, b)\colon \mathbb{F}_p[X, Y] \to \mathbb{F}_{p^m}$ factors through $\mathbb{F}_p[X, Y]/(f(X, Y))$ if and only if $f(a, b) = 0$. Therefore there is a natural one-to-one correspondence

$$\{\text{homomorphisms } \mathbb{F}_p[C] \to \mathbb{F}_{p^m}\} \xleftrightarrow{1:1} C(\mathbb{F}_{p^m}).$$

We have proved:

PROPOSITION 9.8 *Let $Z(C,T)$ be the zeta function of an affine curve C over $\mathbb{F}_p$; then*

$$\log Z(C,T) = \sum_{m\geq 1} N_m \frac{T^m}{m}, \quad N_m = \#C(\mathbb{F}_{p^m}),$$
$$Z(C,T) = \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right).$$

The second equality is obtained by applying $\exp$ to the first.

Zeta functions of plane projective curves

For a projective plane curve C over $\mathbb{F}_p$, we define

$$Z(C,T) = \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right), \quad N_m = \#C(\mathbb{F}_{p^m}),$$
$$\zeta(C,s) = Z(C,p^{-s}).$$

For example,

$$N_m(\mathbb{P}^1) = N_m(\mathbb{A}^1) + 1, \text{ for all } m,$$

and so

$$\log Z(\mathbb{P}^1,T) = \log Z(\mathbb{A}^1,T) + \log \frac{1}{1-T}.$$

Therefore

$$Z(\mathbb{P}^1,T) = \frac{1}{1-T} Z(\mathbb{A}^1,T) = \frac{1}{(1-T)(1-pT)}.$$

Similarly, for E an elliptic curve over $\mathbb{F}_p$ and E^{aff} the affine curve $E \cap \{Z \neq 0\}$,

$$Z(E,T) = \frac{1}{1-T} Z(E^{\mathrm{aff}},T).$$

Counting positive divisors of fixed degree

We shall need the next result in order to prove that $Z(E,T)$ is a rational function of T.

PROPOSITION 9.9 *On an elliptic curve E over $\mathbb{F}_p$, there are exactly $\#E(\mathbb{F}_p)(p^m-1)/(p-1)$ positive divisors of degree m for any $m \geq 1$.*

PROOF. Fix a divisor D_0, and let $P(D_0)$ be the set of all positive divisors D on E such that $D \sim D_0$, i.e., $D = D_0 + (f)$ for some $f \in \mathbb{F}_p(E)^\times$.Then

$$f \mapsto D_0 + (f)\colon L(D_0) \smallsetminus \{0\} \to P(D_0)$$

is surjective, and two functions have the same image if and only if one is a constant multiple of the other. We therefore have a bijection

$$(L(D_0) \smallsetminus \{0\})/\mathbb{F}_p^\times \to P(D_0).$$

When $\deg D_0 = m \geq 1$, the Riemann-Roch theorem (I 4.13) shows that $L(D_0)$ has dimension m, and so $P(D_0)$ has $(p^m - 1)/(p-1)$ elements.

Let $\operatorname{Pic}(E) = \operatorname{Div}(E)/\{(f) \mid f \in k(E)^\times\}$ — according to (1.10) this agrees with the definition in I, §4. Because the degree of a principal divisor is zero, the degree map factors through $\operatorname{Pic}(E)$:

$$\deg\colon \operatorname{Pic} E \longrightarrow \mathbb{Z}.$$

Note that the map deg is surjective because $\mathfrak{p}_\infty \mapsto 1$. We define

$$\operatorname{Pic}^m E = \{\mathfrak{d} \in \operatorname{Pic}(E) \mid \deg \mathfrak{d} = m\}.$$

Then:

(a) the map $\operatorname{Pic}^0(E) \to \operatorname{Pic}^m(E)$, $\mathfrak{d} \mapsto \mathfrak{d} + m\mathfrak{p}_\infty$, is a bijection (obvious);
(b) there exists a bijection $E(k) \to \operatorname{Pic}^0(E)$ ((9), p. 38).

We are now able to count the positive divisor classes of degree m on E. We saw above that each class in $\operatorname{Pic}^m(E)$ has $\frac{p^m-1}{p-1}$ elements, and that there are

$$\# \operatorname{Pic}^m(E) \overset{\text{(a)}}{=} \# \operatorname{Pic}^0(E) \overset{(b)}{=} \#E(\mathbb{F}_p)$$

such classes. Therefore, altogether, there are

$$\#E(\mathbb{F}_p)\frac{p^m-1}{p-1}$$

positive divisors of degree m. □

The rationality of the zeta function of an elliptic curve

THEOREM 9.10 *Let E be an elliptic curve over $\mathbb{F}_p$. Then*

$$Z(E,T) = \frac{1 + (N_1 - p - 1)T + pT^2}{(1-T)(1-pT)}, \quad N_1 = \#E(\mathbb{F}_p).$$

PROOF. Here

$$\begin{aligned} Z(E,T) &= \frac{1}{1-T} Z(E^{\text{aff}},T) \\ &= \frac{1}{1-T}\prod_{\mathfrak{p}} \frac{1}{1-T^{\deg \mathfrak{p}}}, \end{aligned}$$

where the $\mathfrak{p}$ run through the prime ideals of

$$\mathbb{F}_p[x,y] = \mathbb{F}_p[X,Y]/(Y^2 - X^3 - aX - b).$$

On multiplying the product out we find that

$$Z(E,T) = \sum d_m T^m$$

where $d_0 = 1$ and d_m is the number of positive divisors of degree m. According to (9.9), $d_m = N_1 \frac{p^m-1}{p-1}$. Therefore,

$$\begin{aligned} Z(E,T) &= 1 + \sum_{m>0} N_1 \frac{p^m-1}{p-1} T^m \\ &= 1 + \frac{N_1}{p-1}\left(\frac{1}{1-pT} - \frac{1}{1-T}\right) \\ &= \frac{1 + (N_1 - p - 1)T + pT^2}{(1-T)(1-pT)}. \end{aligned}$$

□

REMARK 9.11 (a) Write

$$1 + (N_1 - p - 1)T + pT^2 = (1-\alpha T)(1-\beta T),$$

so that α, β are algebraic integers such that

$$N_1 - p - 1 = -\alpha - \beta, \quad \alpha\beta = p.$$

Then

$$\log Z(E,T) = \log \frac{(1-\alpha T)(1-\beta T)}{(1-T)(1-pT)} = \sum (1 + p^m - \alpha^m - \beta^m)\frac{T^m}{m},$$

and so

$$N_m(E) = 1 + p^m - \alpha^m - \beta^m.$$

Thus, if one knows N_1, one can find α and β, and the whole of the sequence

$$N_1(E), N_2(E), N_3(E), \ldots.$$

(b) With the notation in (a),

$$\zeta(E,s) = \frac{(1-\alpha p^{-s})(1-\beta p^{-s})}{(1-p^{-s})(1-p^{1-s})}.$$

It has simple poles at $s = 0$ and $s = 1$, and zeros where $p^s = \alpha$ and $p^s = \beta$. Write $s = \sigma + it$. Then $|p^s| = p^\sigma$, and the zeros of $\zeta(E,s)$ have real part $\frac{1}{2}$ if and only if α and β have absolute value $p^{\frac{1}{2}}$.

By definition α and β are the roots of a polynomial

$$1 + bT + pT^2, \quad b = N_1 - p - 1.$$

If $b^2 - 4p \leq 0$, then α and β are complex conjugates. Since their product is p, this implies that they each have absolute value $p^{\frac{1}{2}}$. Conversely, if $|\alpha| = p^{\frac{1}{2}} = |\beta|$, then

$$|N_1 - p - 1| = |\alpha + \beta| \leq 2\sqrt{p}.$$

Thus, granted Theorem 9.10, the Riemann hypothesis for E is equivalent to the statement

$$|N_1 - p - 1| \leq 2\sqrt{p}.$$

EXERCISE 9.12 Prove that the zeta function of an elliptic curve E over $\mathbb{F}_p$ satisfies the functional equation

$$\zeta(E,s) = \zeta(E, 1-s).$$

EXERCISE 9.13 Compute the zeta functions for the curve

$$E : Y^2Z + YZ^2 = X^3 - X^2Z$$

over the fields $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\mathbb{F}_7$, and verify the Riemann hypothesis in each case. How many points does the curve have over the field with 625 elements?

A brief history of zeta

The story begins, as do most stories in number theory, with Gauss.

GAUSS 1801

Consider the elliptic curve

$$E : X^3 + Y^3 + Z^3 = 0$$

over $\mathbb{F}_p$, $p \neq 3$.

If $p \not\equiv 1 \mod 3$, then 3 doesn't divide the order of $\mathbb{F}_p^\times$, and so $a \mapsto a^3$ is a bijection $\mathbb{F}_p^\times \to \mathbb{F}_p^\times$. It follows that $(x : y : z) \mapsto (x^3 : y^3 : z^3)$ is a bijection from $E(\mathbb{F}_p)$ onto $L(\mathbb{F}_p)$ where L is the line $L : X + Y + Z = 0$. Therefore $\#E(\mathbb{F}_p) = \#L(\mathbb{F}_p) = p+1$, and $a_p = 0$.

If $p \equiv 1 \mod 3$, then $4p = A^2 + 27B^2$ where A and B are integers, uniquely determined up to sign. Fix the sign of A by requiring that $A \equiv 1 \mod 3$. Then Gauss showed that $\#E(\mathbb{F}_p) = p + 1 + A$, and so $a_p = -A$. Note that

$$a_p^2 = A^2 = 4p - 27B^2,$$

and so $|a_p| \leq 2\sqrt{p}$ — the Riemann hypothesis holds for E.

For details, see Silverman and Tate 1992, IV 2. This result is typical for elliptic curves with complex multiplication; in particular, for such curves, a_p is always zero for half the primes p (contrast 9.5 for curves without complex multiplication).

EMIL ARTIN 1924

In his thesis, Artin defined the zeta function for a curve of the form

$$C : Y^2 = f(X).$$

Here $\mathbb{F}_p(C) = \mathbb{F}_p(X)[\sqrt{f}]$, and so $\mathbb{F}_p(C)$ is analogous to a quadratic extension of $\mathbb{Q}$. He proved that it is a rational function of p^{-s} and satisfies a functional equation, and he checked the Riemann hypothesis, at least for a few curves. As far as I know, Artin was not aware (at the time) of the interpretation of the zeta function in terms of the points of C.

F.K. SCHMIDT 1925–1930

For a nonsingular projective curve C over $\mathbb{F}_p$, Schmidt showed (as in the above proof for elliptic curves) that

$$Z(C,T) = \frac{P(T)}{(1-T)(1-pT)}, \quad P(T) = 1 + \cdots \in \mathbb{Z}[T], \quad \deg P(T) = 2g.$$

Moreover,

$$Z\left(\frac{1}{pT}\right) = p^{1-g}T^{2-2g}Z(T),$$

and so

$$\zeta(1-s) = p^{1-g}(p^{2-2g})^{-s}\zeta(s).$$

Write

$$P(T) = \prod(1 - \alpha_j T).$$

Then the formulas show that

$$N_m = 1 + p^m - \sum \alpha_j^m.$$

Thus, once one knows $\alpha_1, \ldots, \alpha_{2g}$, then one knows N_m for all m. However, unlike the elliptic curve case, N_1 doesn't determine the α_j's — one needs to know several of the N_m's.

HASSE 1933/34

Hasse proved that the Riemann hypothesis is true for elliptic curves.

WEIL 1940–1948

In 1940, Weil announced a proof of the Riemann hypothesis for all curves, i.e., that $|\alpha_i| = \sqrt{p}$ for $1 \leq i \leq 2g$ where α_i is above. His proof assumed the existence of a theory of algebraic geometry over arbitrary fields, including the theory of jacobian and abelian varieties, which at the time was known only over $\mathbb{C}$. He spent most of the 1940s developing the algebraic geometry he needed, and gave a detailed proof of the Riemann hypothesis for curves in a book published in 1948.

WEIL 1949

Weil studied zeta functions of some special algebraic varieties, and stated his famous "Weil conjectures". For a nonsingular projective variety V of any dimension over $\mathbb{F}_p$, one can define (as for curves)

$$\zeta(V,s) = Z(V,p^{-s}), \quad Z(V,T) = \exp\left(\sum_{m\geq 1} N_m \frac{T^m}{m}\right), \quad N_m = \#V(\mathbb{F}_{p^m}).$$

Weil conjectured that $Z(V,T)$ is a rational function of T, that a "Riemann hypothesis" holds for $\zeta(V,s)$, and that $\zeta(V,s)$ satisfies a functional equation.[13]

DWORK 1960

Dwork gave an "elementary" proof that $Z(V,T)$ is a rational function of T.

GROTHENDIECK ET AL 1963/64

Grothendieck defined étale cohomology and, with the help of M. Artin and Verdier, developed it sufficiently to prove that $Z(V,T)$ is rational and satisfies a functional equation. The rationality follows from a "Lefschetz fixed point formula", and the functional equation from a "Poincaré duality" theorem.

DELIGNE 1973

Deligne used étale cohomology to prove the remaining Weil conjecture, namely, the Riemann hypothesis. For this, he received the Fields medal.

In summary, the results of Grothendieck, Artin, Verdier, and Deligne show that, for a nonsingular projective variety V over $\mathbb{F}_p$,

$$Z(V,T) = \frac{P_1(T)P_3(T)\cdots P_{2d-1}(T)}{(1-T)P_2(T)P_4(T)\cdots P_{2d-2}(T)(1-p^dT)}$$

where $d = \dim V$ and $P_i(T) \in \mathbb{Z}[T]$. Moreover

$$Z(V,\frac{1}{p^dT}) = \pm T^{\chi} p^{d\chi/2} Z(V,T)$$

[13] It appears certain that Weil was aware at this time that the rationality and functional equation would follow from the existence of a cohomology theory admitting a Lefschetz fixed point formula and a Poincaré duality theorem for varieties over arbitrary fields, but (apparently) the idea was too outlandish for him to mention in print.

where χ is the self-intersection number of the diagonal in $V \times V$. Finally (Riemann hypothesis):

$$P_i(T) = \prod(1 - \alpha_{ij}T), \quad |\alpha_{ij}| = p^{i/2}.$$

This last statement says that $\zeta(V, s)$ has its zeros on the lines

$$\Re(s) = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2d-1}{2}$$

and its poles on the lines

$$\Re(s) = 0, 1, 2, \ldots, d.$$

EXERCISE 9.14 (a) Let E be the elliptic curve

$$E : Y^2Z = X^3 - 4X^2Z + 16Z^3.$$

Compute $N_p \stackrel{\text{def}}{=} \#E(\mathbb{F}_p)$ for all primes $3 \leq p \leq 13$ (more if you use a computer).

(b) Let $F(q)$ be the (formal) power series given by the infinite product

$$F(q) = q\prod_{n=1}^{\infty}(1 - q^n)^2(1 - q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + \cdots.$$

Calculate the coefficient M_n of q^n in $F(q)$ for $n \leq 13$ (more if you use a computer).

(c) For each prime p, compute the sum $M_p + N_p$. Formulate a conjecture as to what $M_p + N_p$ should be in general.

(d) Prove your conjecture. [This is probably very difficult, perhaps even impossible, using only the information covered in the course.]

(Your conjecture is, or, at least, should be a special case of a theorem of Eichler and Shimura. Wiles's big result is that the theorem of Eichler and Shimura applies to virtually all elliptic curves over $\mathbb{Q}$.)

10 The conjecture of Birch and Swinnerton-Dyer

Introduction

We return to the problem of computing the rank of $E(\mathbb{Q})$. Our purely algebraic approach provides only an upper bound for the rank, in terms of the Selmer group, and the difference between the upper bound and the actual rank is measured by the mysterious Tate-Shafarevich group. It is very difficult to decide whether an element of $S^{(2)}(E/\mathbb{Q})$ comes from an element of infinite order or survives to give a nontrivial element of $\text{Ш}(E/\mathbb{Q})$ — in fact, there is no (proven) algorithm for doing this. Clearly, it would be helpful to have another approach.

Let E be an elliptic curve over $\mathbb{Q}$. For each prime p where E has good reduction, I write (in contradiction to the notations of the last section)

$$N_p = \#\bar{E}(\mathbb{F}_p)$$

where $\bar{E}$ denotes the reduction of E over $\mathbb{F}_p$.

One idea is that perhaps the rank $E(\mathbb{Q})$ should be related to the orders of the groups $\bar{E}(\mathbb{F}_p)$. For any good prime p, there is a reduction map

$$E(\mathbb{Q}) \to \bar{E}(\mathbb{F}_p)$$

but, in general, this will be far from injective or surjective. For example, if $E(\mathbb{Q})$ is infinite, then so also is the kernel, and if $E(\mathbb{Q})$ is finite (and hence has order ≤ 16) then it will fail to be surjective for all large p (because $\#\bar{E}(\mathbb{F}_p) \geq p + 1 - 2\sqrt{p}$).

In the late fifties, Birch and Swinnerton-Dyer had the idea that if $E(\mathbb{Q})$ is large then this should force the N_p's to be larger than usual. Since they had access to one of the few computers then in existence, they were able to test this computationally. For P a large number (large, depending on the speed of your computer), let

$$f(P) = \prod_{p \leq P} \frac{N_p}{p}.$$

Recall that N_p is approximately p. Their calculations led them to the following conjecture.

CONJECTURE 10.1 *For each elliptic curve E over $\mathbb{Q}$, there exists a constant C such that*

$$\lim_{P \to \infty} f(P) = C \log(P)^r$$

where $r = \operatorname{rank}(E(\mathbb{Q}))$.

We write the conjecture more succinctly as

$$f(P) \sim C \log(P)^r \text{ as } P \to \infty.$$

Note the remarkable nature of this conjecture: it predicts that one can determine the rank of $E(\mathbb{Q})$ from the sequence of numbers N_p. Moreover, together with an estimate for the error term, it will provide an algorithm for finding r.

Birch and Swinnerton-Dyer were, in practice, able to predict r from this conjecture with fairly consistent success, but they found that $f(P)$ oscillates vigorously as P increases, and that there seemed to be little hope of finding C with an error of less than say 10%. Instead, they re-expressed their conjecture in terms of the zeta function of E.

The zeta function of a variety over $\mathbb{Q}$

Let V be a nonsingular projective variety over $\mathbb{Q}$. Such a variety is the zero set of a collection of homogeneous polynomials

$$F(X_0, \ldots, X_n) \in \mathbb{Q}[X_0, \ldots, X_n].$$

Scale each such polynomial so that its coefficients lie in $\mathbb{Z}$ but have no common factor, and let $\bar{F}(X_0, \ldots, X_n) \in \mathbb{F}_p[X_0, \ldots, X_n]$ be the reduction of the scaled polynomial modulo p. If the polynomials $\bar{F}$ define a nonsingular variety V_p over $\mathbb{F}_p$, then we say V has ***good reduction*** at p, or that p is ***good*** for V. All but finitely many primes will be good for a given variety.

For each good prime p we have a zeta function

$$\zeta(V_p, s) = Z(V_p, p^{-s}), \text{ with } \quad \log Z(V_p, T) = \sum \# V_p(\mathbb{F}_{p^m}) \frac{T^m}{m},$$

and we define

$$\zeta(V, s) = \prod_p \zeta(V_p, s).$$

Because the Riemann hypothesis holds for V_p, the product converges for $\Re(s) > d + 1$, $d = \dim V$ (cf. the explanation below for the L-series of an elliptic curve).

Let Pt be the point over $\mathbb{Q}$, i.e., $\mathrm{Pt} = \mathbb{A}^0 = \mathbb{P}^0$. For this variety, all primes are good, and

$$\log Z(\mathrm{Pt}_p, T) = \sum 1 \frac{T^m}{m} = \log \frac{1}{1 - T}.$$

Therefore

$$\zeta(\mathrm{Pt}, s) = \prod_p \frac{1}{1 - p^{-s}}$$

which is just the Riemann zeta function — already an interesting function.

Let $V = \mathbb{P}^n$. Again all primes are good, and

$$\# \mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1} - 1}{q - 1} = 1 + q + \cdots + q^n, \quad q = p^m,$$

from which it follows that

$$\zeta(\mathbb{P}^n, s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-n)$$

with $\zeta(s)$ the Riemann zeta function.

CONJECTURE 10.2 (HASSE-WEIL) *For any nonsingular projective variety V over $\mathbb{Q}$, $\zeta(V, s)$ can be analytically continued to a meromorphic function on the whole complex plane, and satisfies a functional equation relating $\zeta(V, s)$ with $\zeta(V, d + 1 - s)$, $d = \dim V$ (and, of course, it is expected to satisfy a Riemann hypothesis, but let's not concern ourselves with that!).*

The conjecture is widely believed to be true, but it is known in only a few cases (and the parenthetical statement is not known for any variety, not even a point!). Note that the above calculations show that, for $V = \mathbb{P}^n$, it follows from the similar statement (or conjecture) for $\zeta(s)$.

The zeta function of an elliptic curve over $\mathbb{Q}$

Let E be an elliptic curve over $\mathbb{Q}$, and let S be the set of primes where E has bad reduction. According to the above definition

$$\begin{aligned}\zeta(E,s) &= \prod_{p\notin S} \frac{1+(N_p-p-1)p^{-s}+p^{1-2s}}{(1-p^{-s})(1-p^{1-s})}\\ &= \frac{\zeta_S(s)\zeta_S(s-1)}{L_S(s)}\end{aligned}$$

where $\zeta_S(s)$ is Riemann's zeta function except that the factors corresponding to the primes in S have been omitted, and

$$L_S(E,s) = \prod_{p\notin S} \frac{1}{1+(N_p-p-1)p^{-s}+p^{1-2s}}.$$

Write

$$1+(N_p-p-1)T+pT^2 = (1-\alpha_p T)(1-\beta_p T),$$

so that

$$L_S(E,s) = \prod_p \frac{1}{1-\alpha_p p^{-s}} \frac{1}{1-\beta_p p^{-s}}.$$

As we noted on p. 151, the product $\prod_p \frac{1}{1-p^{-s}}$ converges for $\Re(s) > 1$, and so

$$\prod_p \frac{1}{1-p^{\frac{1}{2}}p^{-s}}$$

converges for $\Re(s) > \frac{3}{2}$. Because $|\alpha_p| = p^{\frac{1}{2}} = |\beta_p|$, a similar estimate shows that $L_S(E,s)$ converges for $\Re(s) > \frac{3}{2}$.

We want to add factors to $L_S(E,s)$ for the bad primes. Define

$$L_p(T) = \begin{cases} 1-a_pT+pT^2, & a_p = p+1-N_p, \quad p \text{ good}\\ 1-T, & \text{if } E \text{ has split multiplicative reduction}\\ 1+T, & \text{if } E \text{ has non-split multiplicative reduction}\\ 1, & \text{if } E \text{ has additive reduction.}\end{cases}$$

In the four cases

$$L_p(p^{-1}) = \frac{N_p}{p}, \frac{p-1}{p}, \frac{p+1}{p}, \frac{p}{p}.$$

Thus in each case, $L_p(p^{-1}) = \#E^{\text{ns}}(\mathbb{F}_p)/p$, where E^{ns} is the nonsingular part of the reduction of the elliptic curve modulo p (see the table in II, §3). Define

$$L(E,s) = \prod_p \frac{1}{L_p(p^{-s})},$$

where the product is now over all prime numbers. The ***conductor*** $N_{E/\mathbb{Q}}$ of E is defined to be $\prod_{p \text{ bad}} p^{f_p}$ where

$$f_p = \begin{cases} f_p = 1 & \text{if } E \text{ has multiplicative reduction at } p \\ f_p \geq 2 & \text{if } E \text{ has additive reduction at } p, \text{ and equals 2 if } p \neq 2, 3. \end{cases}$$

The precise definition of f_p when E has additive reduction is a little complicated. However, there is a formula of Ogg that can be used to compute it in all cases:

$$f_p = \operatorname{ord}_p(\Delta) + 1 - m_p$$

where m_p is the number of irreducible components on the Néron model (not counting multiplicities) and Δ is the discriminant of the minimum equation of E

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_{6.}Z^3.$$

Define $\Lambda(E,s) = N_{E/\mathbb{Q}}^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$. The following is a more precise version of the Hasse-Weil conjecture for the case of an elliptic curve.

CONJECTURE 10.3 *The function $\Lambda(E,s)$ can be analytically continued to a meromorphic function on the whole of $\mathbb{C}$, and it satisfies a functional equation*

$$\Lambda(E,s) = w_E\Lambda(E,2-s), \quad w_E = \pm 1.$$

There is even a recipe for what w_E should be.

For curves with complex multiplication, i.e., such that $\operatorname{End}(E) \neq \mathbb{Z}$, the conjecture was proved by Deuring in the early 1950s. This case occurs for exactly 9 values of the j-invariant. The key point is that there is always a "formula" for N_p similar to that proved by Gauss for the curve $X^3 + Y^3 + Z^3 = 0$ (see p. 157) which allows one to identify the L-series of E with an L-series of a type already defined and studied by Hecke (a "Hecke L-series") and for which one knows analytic continuation and a functional equation.

A much more important result, which we'll spend most of the rest of the book discussing, is the following. Let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}.$$

Then $\Gamma_0(N)$ acts on the complex upper half-plane, and the quotient $\Gamma_0(N)\backslash\mathbb{H}$ has the structure of a Riemann surface. An elliptic curve E over $\mathbb{Q}$ is said to be ***modular*** if there is a nonconstant map of Riemann surfaces

$$\Gamma_0(N)\backslash\mathbb{H} \to E(\mathbb{C})$$

for some N. Eichler and Shimura (in the fifties and sixties) proved a slightly weaker form of Conjecture 10.3 for modular elliptic curves (see Chap. V, §7).

Recall that a curve is said to have semistable reduction at p if it has good reduction at p or multiplicative reduction at p, i.e., if the reduced curve doesn't have a cusp. Wiles (with the help of Richard Taylor) proved that an elliptic curve with semistable reduction at all p is modular, and Breuil, Conrad, Diamond, and Taylor extended the result to *all* elliptic curves over $\mathbb{Q}$ (see Chapter V). Thus, Conjecture 10.3 is known for all elliptic curves over $\mathbb{Q}$.

REMARK 10.4 (a) Deuring's result is valid for elliptic curves with complex multiplication over any number field. The results of Eichler, Shimura, Wiles, et al. are valid only for elliptic curves over $\mathbb{Q}$. Even today, little is known about the L-series of elliptic curves over number fields other than $\mathbb{Q}$.

(b) Both results prove much more than the simple statement of Conjecture 10.3 — they succeed in identifying the function $\Lambda(E,s)$.

Statement of the conjecture of Birch and Swinnerton-Dyer

Let E be an elliptic curve over $\mathbb{Q}$. Let

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

be a minimum equation for E over $\mathbb{Q}$, and let

$$\omega = \frac{dx}{2y + a_1x + a_3}.$$

Recall (§6) that there is a canonical pairing

$$\langle\ ,\ \rangle\colon E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}, \quad \langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q).$$

CONJECTURE 10.5 (BIRCH AND SWINNERTON-DYER) *Let r be the rank of $E(\mathbb{Q})$, and let $P_1, \ldots, P_r$ be linearly independent elements of $E(\mathbb{Q})$; then*

$$L(E,s) \sim \left(\Omega \prod_{p \text{ bad}} c_p\right) \frac{[\text{Ш}(E/\mathbb{Q})]\det(\langle P_i, P_j\rangle)}{(E(\mathbb{Q}) : \sum \mathbb{Z}P_i)^2}(s-1)^r \text{ as } s \to 1,$$

where

$[*] =$ *order of* $*$ *(elsewhere written* $\#*$*);*
$\Omega = \int_{E(\mathbb{R})} |\omega|$;
$c_p = (E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p))$.

REMARK 10.6 (a) For a modular elliptic curve, all terms in the conjecture are computable except for the Tate-Shafarevich group, and, in fact, can be computed by Pari (see II, §7).

(b) The quotient

$$\frac{\det(\langle P_i, P_j\rangle)}{(E(\mathbb{Q}) : \sum \mathbb{Z}P_i)^2}$$

is independent of the choice of $P_1, \ldots, P_r$, and equals

$$\frac{\operatorname{disc}\langle\ ,\ \rangle}{[E(\mathbb{Q})_{\mathrm{tors}}]^2}$$

when they form a basis for $E(\mathbb{Q})$ modulo torsion.

(c) The integral $\int_{E(\mathbb{Q}_p)} |\omega|$ makes sense, and, in fact it is equal to $(E(\mathbb{Q}_p) : E^1(\mathbb{Q}_p))/p$. The explanation for the formula is that (see II 4.3) there is a bijection $E^1(\mathbb{Q}_p) \leftrightarrow p\mathbb{Z}_p$ under which ω corresponds to the Haar measure on $\mathbb{Z}_p$ for which $\mathbb{Z}_p$ has measure 1 and (therefore) $p\mathbb{Z}_p$ has measure $1/(\mathbb{Z}_p : p\mathbb{Z}_p) = 1/p$. Hence,

$$\int_{E(\mathbb{Q}_p)} |\omega| = (E(\mathbb{Q}_p) : E^1(\mathbb{Q}_p)) \int_{E^1(\mathbb{Q}_p)} |\omega| = \frac{(E(\mathbb{Q}_p) : E^1(\mathbb{Q}_p))}{p} = \frac{c_p N_p}{p}.$$

For any finite set S of prime numbers including all those for which E has bad reduction, define

$$L_S^*(s) = \left(\prod_{p \in S \cup \{\infty\}} \int_{E(\mathbb{Q}_p)} |\omega| \right)^{-1} \prod_{p \notin S} \frac{1}{L_p(p^{-s})}.$$

In this, $\mathbb{Q}_p = \mathbb{R}$ when $p = \infty$. When p is good,

$$L_p(p^{-1}) = \frac{N_p}{p} = \left(\int_{E(\mathbb{Q}_p)} |\omega| \right),$$

and so the behaviour of $L_S^*(s)$ near s is independent[14] of S satisfying the condition, and the conjecture of Birch and Swinnerton-Dyer can be stated as:

$$L_S^*(E, s) \sim \frac{[\text{Ш}(E/\mathbb{Q})] \operatorname{disc}\langle\ ,\ \rangle}{[E(\mathbb{Q})_{\mathrm{tors}}]^2} (s-1)^r \text{ as } s \to 1.$$

This is how Birch and Swinnerton-Dyer stated their conjecture.

ASIDE 10.7 Formally, $L_p(1) = \prod \frac{p}{N_p}$, and so Conjecture 10.5 has an air of compatibility with Conjecture 10.1. Assuming E to be modular, Goldfeld (1982) has shown that, if $\prod_{p \le x} N_p/p \sim C(\log x)^{-r}$ for some $C > 0$ and r, then

- ◇ $L(E, s) \sim C' \cdot (s-1)^r$ as $s \to 1$ *and*
- ◇ the generalized Riemann hypothesis holds for $L(E, s)$, i.e., $L(E, s) \neq 0$ for $\Re(s) > 1$.

[14] More precisely, $\lim_{s \to 1} L_S^*(s)/L_{S'}^*(s) = 1$ for any two such sets S, S'.

Needless to say, Conjecture 10.1 has not been proved for a single curve. In fact, it is *stronger* than the generalized Riemann hypothesis: for a fixed modular elliptic curve E over $\mathbb{Q}$, the following statements are equivalent,

- ◇ $\prod_{p\leq x} N_p/p \sim C(\log x)^r$ for some nonzero C and some r;
- ◇ $\psi_E(x) = o(x \log x)$, where $\psi_E(x) = \sum_{p\leq x}(\alpha_p^k + \bar{\alpha}_p^k)\log p$.

(Conrad 2005, Kuo and Murty 2005). The generalized Riemann hypothesis for $L(E, s)$ is equivalent to $\psi_E(x) = O(x(\log x)^2)$, and so $\psi_E(x) = o(x \log x)$ can be considered a deeper (but still plausible) form of it.

What is known about the conjecture of Birch and Swinnerton-Dyer

Beginning with the work of Birch and Swinnerton-Dyer (1963, 1965), a massive amount of computational evidence has accumulated in support of the conjectures: all the terms in the conjecture except Ш have been computed for thousands of curves; the value of [Ш] predicted by the computations is always a square; whenever the order of some p-primary component of [Ш] has been computed, it has agreed with the conjecture.

For a pair of isogenous elliptic curves over $\mathbb{Q}$, most of the terms in Conjecture 10.5 will differ for the two curves, but nevertheless Cassels (1965) has shown that if the conjecture is true for one curve, then it is true for the other.

For certain elliptic curves over function fields, the conjecture is known (see the next section).

However, by the mid-seventies, little progress had been made toward proving Conjecture 10.5 over $\mathbb{Q}$. As Tate (1974) put it, “This remarkable conjecture relates the behaviour of a function L at a point where it is not at present known to be defined to the order of a group Ш which is not known to be finite.”

Coatesand Wiles (1977) proved that if E has complex multiplication and $E(\mathbb{Q})$ is infinite, then $L(E, 1) = 0$.

From now on, we assume that $E/\mathbb{Q}$ is modular. Thus $L(E/\mathbb{Q}, s)$ extends to the whole complex plane, and satisfies a functional equation (Conjecture 10.3). We write w_E for the sign in the functional equation (note that $w_E = +1 \iff L(E/\mathbb{Q}, s)$ has a zero of even order at $s = 1$).

For a modular elliptic curve $E/\mathbb{Q}$ and a complex quadratic extension K of $\mathbb{Q}$, Birch (1969a, 1969b, 1970, 1975) defined a “Heegner point” $P_K \in E(K)$, and suggested that it should often be of infinite order. (For a recent account of this work, see Birch 2004.)

Gross and Zagier (1983, 1986) proved that if E is a modular elliptic curve over $\mathbb{Q}$, then

$$L'(E/K, 1) = C \cdot \hat{h}(P_K), \quad C \neq 0.$$

Thus P_K has infinite order if and only if $L'(E/K, 1) \neq 0$.

Kolyvagin (1988a, 1988b) showed that if $w_E = +1$ and P_K has infinite order for some complex quadratic extension K of $\mathbb{Q}$, then $E(\mathbb{Q})$ and Ш$(E/\mathbb{Q})$ are both finite. (For an exposition of this work, see Rubin 1989.)

Let $K = \mathbb{Q}[\sqrt{D}]$, $D < 0$, be a complex quadratic extension of $\mathbb{Q}$, and write E^K for the twist $DY^2Z = X^3 + aXZ^2 + bZ^3$ of $E : Y^2Z = X^3 + aXZ^2 + bZ^3$ — thus E^K becomes isomorphic to E over K. There is an elementary formula,

$$L(E/K, s) = L(E/\mathbb{Q}, s) \cdot L(E^K/\mathbb{Q}, s). \tag{34}$$

Bump, Friedberg, and Hoffstein (1989) showed that if $w_E = +1$, then there exists a complex quadratic field K such that $L(E^K/\mathbb{Q}, s)$ has a zero of order one at $s = 1$ (and so $L'(E/K, 1) \neq 0$ if $L(E/\mathbb{Q}, 1) \neq 0$).

On combining these results, we find that

$$L(E/\mathbb{Q}, 1) \neq 0 \Rightarrow E(\mathbb{Q}) \text{ and } Ш(E/\mathbb{Q}) \text{ are finite.}$$

In fact, Kolyvagin proves much more. For example, he shows that $[Ш(E/\mathbb{Q})]$ divides its conjectured order. To complete the proof of the conjecture of Birch and Swinnerton-Dyer, it suffices to check the its p-primary component has the correct order for a finite set of primes. This has been done for some curves.

11 Elliptic curves and sphere packings

The conjecture of Birch and Swinnerton-Dyer is expected to hold, not just for elliptic curves over $\mathbb{Q}$, but also for elliptic curves over all global fields, i.e., finite extensions of $\mathbb{Q}$ and of $\mathbb{F}_p(T)$. For functions fields, the conjecture has been proved in some important cases, and Elkies, Shioda, and others used this to show that the lattices $(E(K), \langle\ ,\ \rangle)$ arising in this way give very dense sphere packings.[15]

Let K be a finite extension of $\mathbb{F}_q(T)$ where q is a power of the prime p. There exists a nonsingular projective curve C over $\mathbb{F}_q$ such that $\mathbb{F}_q(T) = \mathbb{F}_q(C)$ (cf. Fulton 1969, p. 180). As we discussed in §9, the zeta function of C over $\mathbb{F}_q$,

$$Z(C, T) = \frac{\prod_{i=1}^{2g}(1 - \omega_i T)}{(1 - T)(1 - qT)}, \quad |\omega_i| = q^{\frac{1}{2}}, \quad g = \text{ genus}(C).$$

Now consider a ***constant*** elliptic curve E over K, i.e., a curve defined by an equation

$$E : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

with the $a_i \in \mathbb{F}_q \subset K$. Write the zeta function of E over $\mathbb{F}_q$ as

$$Z(E, T) = \frac{(1 - \alpha_1 T)(1 - \alpha_2 T)}{(1 - T)(1 - qT)}, \quad |\alpha_1| = q^{\frac{1}{2}} = |\alpha_2|.$$

[15] "One of the most exciting developments has been Elkies' (sic) and Shioda's construction of lattice packings from the Mordell-Weil groups of elliptic curves over function fields. Such lattices have a greater density than any previously known in dimensions from about 54 to 4096." Conway and Sloane 1993, p. xvi.

PROPOSITION 11.1 *The conjecture Birch and Swinnerton-Dyer for E/K is equivalent to the following statement:*

(a) *the rank r of $E(K)$ is equal to the number of pairs $(i,j)_{1\le i\le 2, 1\le j\le 2g}$ such that $\alpha_i = \omega_j$, and*
(b) *for any basis $a_1, \ldots, a_r$ of $E(K)/E(K)_{\text{tors}}$,*

$$q^g \prod_{\alpha_i \neq \omega_j} \left(1 - \frac{\omega_j}{\alpha_i}\right) = [\text{Ш}(E/K)] \left|\det\langle a_i, a_j\rangle\right|.$$

PROOF. Elementary, but omitted (see Milne 1968, §3). □

THEOREM 11.2 *In the situation of the proposition, the conjecture of Birch and Swinnerton-Dyer is true.*

PROOF. Statement (a) of (11.1) was proved by Tate (1966), and statement (b) by Milne (1968). □

In fact, the conjecture of Birch and Swinnerton-Dyer is true under the weaker hypothesis that $j(E) \in \mathbb{F}_q$ (Milne 1975), for example, for all curves of the form

$$Y^2 Z = X^3 + bZ^3, \quad b \in K.$$

Sphere packings

As we noted in (6.4) pairs consisting of a free $\mathbb{Z}$-module of finite rank L and a positive definite quadratic form q on $V \stackrel{\text{def}}{=} L \otimes \mathbb{R}$ are of great interest. We can choose a basis for V that identifies (V, q) with $(\mathbb{R}^r, X_1^2 + \cdots + X_r^2)$. The bilinear form associated with q is

$$\langle x, y\rangle = q(x+y) - q(x) - q(y).$$

Given such a pair (L, q), the numbers one needs to compute are

(a) the rank r of L;
(b) the square of the length of the shortest vector

$$m(L) = \inf_{v \in L, v \neq 0} \langle v, v\rangle;$$

(c) the discriminant of L,

$$\operatorname{disc} L = \det(\langle e_i, e_j\rangle)$$

where $e_1, \ldots, e_r$ is a basis for L.

The discriminant is independent of the choice of a basis for L. Let

$$\gamma(L) = m(L)/\operatorname{disc}(L)^{\frac{1}{r}}.$$

The volume a fundamental parallelepiped for L is $\sqrt{\operatorname{disc} L}$. The sphere packing associated with L is formed of spheres of radius $\frac{1}{2}\sqrt{m(L)}$, and therefore its density is

$$d(L) = 2^{-r} b_r \gamma(L)^{\frac{r}{2}}$$

where $b_r = \pi^{r/2}/\Gamma(\frac{r+2}{2})$ is the volume of the r-dimensional unit ball. To maximize $d(L)$, we need to maximise $\gamma(L)$.

Let E be a constant elliptic curve over a field $\mathbb{F}_q(C)$ as above, and let $L = E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}}$ with the quadratic form $q = 2\hat{h}$. If we know the numbers ω_i and α_j, then part (a) of Theorem 11.2 gives us r, and part (b) gives us an upper bound for $\operatorname{disc} L$:

$$\operatorname{disc} L = q^g \prod_{\alpha_i \neq \omega_j} \left(1 - \frac{\omega_j}{\alpha_i}\right) / [\text{Ш}] \leq q^g \prod_{\alpha_i \neq \omega_j} (1 - \frac{\omega_j}{\alpha_i}).$$

Finally, an easy, but nonelementary argument[16], shows that

$$m(L) \geq 2[C(k)]/[E(k)]$$

for all finite $k \supset \mathbb{F}_q$.

Example

Consider the curve

$$C : X^{q+1} + Y^{q+1} + Z^{q+1} = 0$$

over $\mathbb{F}_{q^2}$ (note, not over $\mathbb{F}_q$).

LEMMA 11.3

(a) *The curve C is nonsingular, of genus $g = \frac{q(q-1)}{2}$.*
(b) $\#C(\mathbb{F}_{q^2}) = q^3 + 1$.
(c) $Z(C,T) = \frac{(1+qT)^{q(q-1)}}{(1-T)(1-q^2T)}$.

PROOF. (a) The partial derivatives of the defining equation are X^q, Y^q, Z^q, and these have no common zero in $\mathbb{P}^2$. Therefore, the curve is nonsingular, and so the formula on p. 34 shows that it has genus $q(q-1)/2$.

---

[16]An element P of $E(K)$ defines a map $u: C \to E$, and $\hat{h}(P)$ is related to the degree of u. Thus, we get a lower bound for $m(L)$ in terms of the ω_i and α_j.

(b) The group $\mathbb{F}_{q^2}^{\times}$ is cyclic of order $q^2 - 1 = (q+1)(q-1)$, and $\mathbb{F}_q^{\times}$ is its subgroup of order $q-1$. Thus, $x^{q+1} \in \mathbb{F}_q^{\times}$ and $x \mapsto x^{q+1}: \mathbb{F}_{q^2}^{\times} \to \mathbb{F}_q^{\times}$ is a surjective homomorphism with kernel a cyclic group of order $q+1$. As x runs through $\mathbb{F}_{q^2}$, x^{q+1} takes the value 0 once and each nonzero value in $\mathbb{F}_q^{\times}$ exactly $q+1$ times. A similar remark applies to y^{q+1} and z^{q+1}. We can scale each solution of $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$ so that $x = 0$ or 1.

Case 1: $x = 1$, $1 + y^{q+1} \neq 0$. There are $q^2 - q - 1$ possibilities for y, and then $q+1$ possibilities for z. Hence $(q^2 - q - 1)(q+1) = q^3 - 2q - 1$ solutions.

Case 2: $x = 1$, $1 + y^{q+1} = 0$. There are $q+1$ possibilities for y, and then one for z. Hence $q+1$ solutions.

Case 3: $x = 0$. We can take $y = 1$, and then there are $q+1$ possibilities for z.

In sum, there are $q^3 + 1$ solutions.

(c) We know that

$$\#C(\mathbb{F}_q) = 1 + q^2 - \sum_{i=1}^{2g} \omega_i .$$

Therefore $\sum_{i=1}^{2g} \omega_i = q^2 - q^3 = -2gq$. Because $|\omega_i| = q$, this forces $\omega_i = -q$. □

For all q, it is known that there is an elliptic curve E over $\mathbb{F}_{q^2}$, such that $E(\mathbb{F}_{q^2})$ has $q^2 + 2q + 1$ elements (the maximum allowed by the Riemann hypothesis). For such a curve

$$Z(E,T) = \frac{(1+qT)^2}{(1-T)(1-q^2T)}.$$

PROPOSITION 11.4 *Let $L = E(K)/E(K)_{\mathrm{tors}}$ with E and $K = \mathbb{F}_q(C)$ as above. Then:*

(a) *The rank r of L is $2q(q-1)$;*
(b) *$m(L) \geq 2(q-1)$;*
(c) *$[\text{Ш}(E/K)]\,\mathrm{disc}(L) = q^{q(q-1)}$;*
(d) *$\gamma(L) \geq 2(q-1)/\sqrt{q}$.*

PROOF. (a) Since all α_i and ω_j equal $-q$, if follows from (11.2a) that the rank is $2 \times 2g = 2q(q-1)$.

(b) We have

$$m(L) \geq \frac{[C(\mathbb{F}_{q^2})]}{[E(\mathbb{F}_{q^2})]} = \frac{q^3+1}{(q^2+1)^2} > q - 2.$$

(c) This is a special case of (11.2), taking count that our field is $\mathbb{F}_{q^2}$ (not $\mathbb{F}_q$) and $g = q(q-1)/2$.

(d) Follows immediately from the preceding. □

REMARK 11.5 (a) Dummigan (1995) has obtained information on the Tate-Shafarevich group in the above, and a closely related, situation. For example, $Ш(E/K)$ is zero if $q = p$ or p^2, and has cardinality at least $p^{p^3(p-1)^3/2}$ if $q = p^3$.

(b) For $q = 2$, L is isomorphic to the lattice denoted D_4, for $q = 3$, to the Coxeter-Todd lattice K_{12}, and for $q = 3$ it is similar to the Leech lattice.

For a more detailed account of the applications of elliptic curves to lattices, see Oesterlé 1990.

EXERCISE 11.6 Consider $E : Y^2Z + YZ^2 = X^3$.

(a) Show that E is a nonsingular curve over $\mathbb{F}_2$.

(b) Compute $\#E(\mathbb{F}_4)$, $\mathbb{F}_4$ being the field with 4 elements.

(c) Let K be the field of fractions of the integral domain $\mathbb{F}_4[X,Y]/(X^3 + Y^3 + 1)$, and let $L = E(K)/E(K)_{\text{tors}}$ considered as a lattice in $V = L \otimes \mathbb{R}$ endowed with the height pairing. Compute the rank of L, $m(L)$, and $\gamma(L)$.

Chapter V

Elliptic curves and modular forms

We wish to understand the L-function $L(E/\mathbb{Q}, s)$ of an elliptic curve E over $\mathbb{Q}$, i.e., we wish to understand the sequence of numbers

$$N_2, N_3, N_5, \ldots, N_p, \ldots \qquad N_p = \#\bar{E}(\mathbb{F}_p),$$

or, equivalently, the sequence of numbers

$$a_2, a_3, a_5, \ldots, a_p, \ldots \qquad a_p = p + 1 - N_p.$$

There is no direct way of doing this. Instead, we shall see how the study of modular curves and modular forms leads to functions that are candidates for being the L-function of an elliptic curve over $\mathbb{Q}$, and then we shall see how Wiles (and others) showed that the L-functions of all elliptic curves over $\mathbb{Q}$ do in fact arise from modular forms.

1 The Riemann surfaces $X_0(N)$

Quotients of Riemann surfaces by group actions

We shall need to define Riemann surfaces as the quotients of other simpler Riemann surfaces by group actions. This can be quite complicated. The following examples will help.

EXAMPLE 1.1 Let $n \in \mathbb{Z}$ act on $\mathbb{C}$ by $z \mapsto z + n$. Topologically $\mathbb{C}/\mathbb{Z}$ is a cylinder. We can give it a complex structure as follows: let $\pi\colon \mathbb{C} \to \mathbb{C}/\mathbb{Z}$ be the quotient map; for any $P \in \mathbb{C}/\mathbb{Z}$ and $Q \in f^{-1}(P)$ we can find open neighbourhoods U of P and V of Q such that $\pi\colon V \to U$ is a homeomorphism; the coordinate neighbourhoods $(U, \pi^{-1}\colon U \to V)$ form a coordinate covering of $\mathbb{C}/\mathbb{Z}$, and so define a complex structure on $\mathbb{C}/\mathbb{Z}$.

For any open $U \subset \mathbb{C}/\mathbb{Z}$, a function $f\colon U \to \mathbb{C}$ is holomorphic for this complex structure if and only if $f \circ \pi$ is holomorphic. Thus the holomorphic functions f on $U \subset \mathbb{C}/\mathbb{Z}$ can be identified with the holomorphic functions g on $\pi^{-1}(U)$ invariant under $\mathbb{Z}$, i.e., such that $g(z+1) = g(z)$.

For example, $q(z) = e^{2\pi i z}$ defines a holomorphic function on $\mathbb{C}/\mathbb{Z}$. In fact, it gives an isomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^{\times}$ whose in inverse $\mathbb{C}^{\times} \to \mathbb{C}/\mathbb{Z}$ is (by definition) $(2\pi i)^{-1} \cdot \log$.

EXAMPLE 1.2 Let D be the open unit disk $\{z \mid |z| < 1\}$, and let Δ be a finite group acting on D. The Schwarz lemma (Cartan 1963, III.3) implies that $\mathrm{Aut}(D) = \{z \in \mathbb{C} \mid |z| = 1\} \approx \mathbb{R}/\mathbb{Z}$, and it follows that Δ is a finite cyclic group. Let $z \mapsto \zeta z$ be its generator and suppose that ζ has order m. Then z^m is invariant under Δ, and so defines a function on $\Delta\backslash D$, which in fact is a homeomorphism $\Delta\backslash D \to D$, and therefore defines a complex structure on $\Delta\backslash D$.

Let $\pi\colon D \to \Delta\backslash D$ be the quotient map. Then $f \mapsto f \circ \pi$ identifies the space of holomorphic functions on $U \subset \Delta\backslash D$ with the space of holomorphic functions on $\pi^{-1}(U)$ such that $f(\zeta z) = f(z)$ for all z, and so are of the form $f(z) = h(z^m)$ with h holomorphic. Note that if $\pi(Q) = P = 0$, then $\mathrm{ord}_P(f) = \frac{1}{m}\,\mathrm{ord}_Q(f \circ \pi)$.

Let Γ be a group acting on a Riemann surface X. A ***fundamental domain*** for Γ is a connected open subset D of X such that

(a) no two points of D lie in the same orbit of Γ;
(b) the closure $\bar{D}$ of D contains at least one element from each orbit.

For example,

$$D = \{z \in \mathbb{C} \mid 0 < \Re(z) < 1\}$$

is a fundamental domain for $\mathbb{Z}$ acting on $\mathbb{C}$ (as in 1.1), and

$$D_0 = \{z \in D \mid 0 < \arg z < 2\pi/m\}$$

is a fundamental domain for $\mathbb{Z}/m\mathbb{Z}$ acting on the unit disk (as in 1.2).

The Riemann surfaces $X(\Gamma)$

Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. We want to define the structure of a Riemann surface on the quotient $\Gamma\backslash\mathbb{H}$. This we can do, but the resulting surface will not be compact. Instead, we need to form a quotient $\Gamma\backslash\mathbb{H}^*$ where $\mathbb{H}^*$ properly contains $\mathbb{H}$.

THE ACTION OF $\mathrm{SL}_2(\mathbb{Z})$ ON THE UPPER HALF PLANE

Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H} = \{z \mid \Im(z) > 0\}$ according to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Note that $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathbb{H}$, and so the action factors through $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ so } Sz = \frac{-1}{z},$$

and

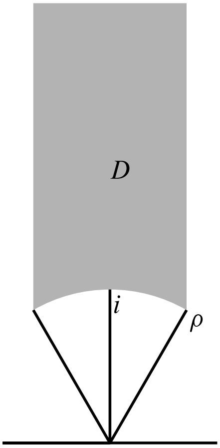
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ so } Tz = z + 1.$$

Then

$$S^2 = 1, \quad (ST)^3 = 1 \text{ in } \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}.$$

PROPOSITION 1.3 *Let*

$$D = \left\{ z \in \mathbb{H} \;\middle|\; |z| > 1, \quad -\frac{1}{2} < \Re(z) < \frac{1}{2} \right\}.$$



(a) *D is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$; moreover, two elements z and z' of the closure $\bar{D}$ of D are in the same orbit if and only if*
   i) *$\Re(z) = \pm\frac{1}{2}$ and $z' = z \pm 1$ (so $z' = Tz$ or $z = Tz'$); or*
   ii) *$|z| = 1$ and $z' = -1/z$ $(= Sz)$.*

(b) *For $z \in \bar{D}$, the stabilizer of z is $\{\pm I\}$ unless $z = i$, in which case the stabilizer is $\langle S \rangle$, or $\rho = e^{2\pi i/6}$, in which case the stabilizer is $\langle TS \rangle$, or ρ^2, in which case it is $\langle ST \rangle$.*

(c) *The group $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is generated by S and T.*

PROOF. Let $\Gamma' = \langle S, T \rangle$. One first shows that $\Gamma'\bar{D} = \mathbb{H}$, from which (a) and (b) follow easily. For (c), let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and choose a point z_0 in D. There exists a γ' in Γ' such that $\gamma z_0 = \gamma' z_0$, and it follows from (b) that $\gamma'\gamma^{-1} = \pm I$. For the details, see Serre 1973, VII 1.2, or MF 2.12. □

REMARK 1.4 Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$, and write

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\gamma_1 \cup \ldots \cup \Gamma\gamma_m \qquad \text{(disjoint union)}.$$

Then $D' = \bigcup \gamma_i D$ satisfies the conditions to be a fundamental domain for Γ, except that it won't be connected. However, it is possible to choose the γ_i so that the closure of D' is connected, in which case the interior of the closure will be a fundamental domain for Γ.

THE EXTENDED UPPER HALF PLANE

The elements of $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{P}^1(\mathbb{C})$ by projective linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z_1 : z_2) = (az_1 + bz_2 : cz_1 + dz_2).$$

Identify $\mathbb{H}$, $\mathbb{Q}$, and $\{\infty\}$ with subsets of $\mathbb{P}^1(\mathbb{C})$ according to

$$\begin{array}{rcll} z & \leftrightarrow & (z:1) & z \in \mathbb{H} \\ r & \leftrightarrow & (r:1) & r \in \mathbb{Q} \\ \infty & \leftrightarrow & (1:0) & \end{array}.$$

The action of $\mathrm{SL}_2(\mathbb{Z})$ stabilizes $\mathbb{H}^* \stackrel{\text{def}}{=} \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$: for $z \in \mathbb{H}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z : 1) = (az + b : cz + d) = (\tfrac{az+b}{cz+d} : 1);$$

for $r \in \mathbb{Q}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(r : 1) = (ar + b : cr + d) = \begin{cases} (\frac{ar+b}{cr+d} : 1) & r \neq -\frac{d}{c} \\ \infty & r = -\frac{d}{c} \end{cases};$$

and, finally,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = (a : c) = \begin{cases} (\frac{a}{c} : 1) & c \neq 0, \\ \infty & c = 0 \end{cases}.$$

Thus in passing from $\mathbb{H}$ to $\mathbb{H}^*$, we have added one additional $\mathrm{SL}_2(\mathbb{Z})$ orbit. The points in $\mathbb{H}^*$ not in $\mathbb{H}$ are often called the ***cusps***.

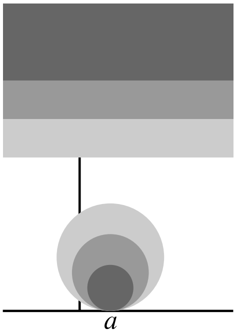
We make $\mathbb{H}^*$ into a topological space as follows: the topology on $\mathbb{H}$ is that inherited from $\mathbb{C}$; the sets

$$\{z \mid \Im(z) > M\}, \quad M > 0$$

form a fundamental system of neighbourhoods of ∞; the sets

$$\{z \mid |z - (a + ir)| < r\} \cup \{a\}$$

form a fundamental system of neighbourhoods of $a \in \mathbb{Q}$. One shows that $\mathbb{H}^*$ is Hausdorff, and that the action of $\mathrm{SL}_2(\mathbb{Z})$ is continuous.



The topology on $\Gamma\backslash\mathbb{H}^*$

Recall that if $\pi\colon X \to Y$ is a surjective map and X is a topological space, then the quotient topology on Y is that for which a set U is open if and only if

$\pi^{-1}(U)$ is open. In general the quotient of a Hausdorff space by a group action will not be Hausdorff even if the orbits are closed — one needs that distinct orbits have disjoint open neighbourhoods.

Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. The discrete group Γ acts continuously on $\mathbb{H}$, and one can show that the action is proper,[1] i.e., for any pair of points $x, y \in \mathbb{H}$, there exist neighbourhoods U of x and V of y such that

$$\{\gamma \in \Gamma \mid \gamma U \cap V \neq \emptyset\}$$

is finite. In particular, this implies that the stabilizer of any point in $\mathbb{H}$ is finite (which we knew anyway).

PROPOSITION 1.5 (a) *For any compact sets A and B of $\mathbb{H}$, $\{\gamma \in \Gamma \mid \gamma A \cap B \neq \emptyset\}$ is finite.*

(b) *Every $z \in \mathbb{H}$ has a neighbourhood U such that γU and U are disjoint for $\gamma \in \Gamma$ unless $\gamma z = z$.*

(c) *For any points x, y of $\mathbb{H}$ not in the same Γ-orbit, there exist neighbourhoods U of x and V of y such that $\gamma U \cap V = \emptyset$ for all $\gamma \in \Gamma$*

PROOF. (a) This follows easily from the fact that Γ acts continuously and properly.

(b) Let V be compact neighbourhood of z, and let S be the set of $\gamma \in \Gamma$ such that $V \cap \gamma V \neq \emptyset$ but γ doesn't fix z. From (a), we know that S is finite. For each $\gamma \in S$, choose disjoint neighbourhoods V_γ of z and W_γ of γz, and set

$$U = V \cap \bigcap_{\gamma \in S} \left(V_\gamma \cap \gamma^{-1} W_\gamma\right).$$

For $\gamma \in S$, $\gamma U \subset W_\gamma$. As W_γ is disjoint from V_γ and V_γ contains U, this implies that γU is disjoint from U.

(c) Choose compact neighbourhoods A of x and B of y, and let S be the (finite) set of $\gamma \in \Gamma$ such that $\gamma A \cap B \neq \emptyset$. Because $\gamma x \neq y$, for each $\gamma \in S$, there exist disjoint neighbourhoods U_γ and V_γ of γx and y. Now $U = A \cap \bigcap_{\gamma \in S} \gamma^{-1} U_\gamma$ and $V = B \cap \bigcap_{\gamma \in S} V_\gamma$ are neighbourhoods of x and y respectively such that γU and V are disjoint for all γ in Γ. □

COROLLARY 1.6 *The space $\Gamma \backslash \mathbb{H}$ is Hausdorff.*

PROOF. Let x and y be points of $\mathbb{H}$ not in the same Γ-orbit, and choose neighbourhoods U and V of x and y as in (c) of the last proposition. Then ΓU and ΓV are disjoint neighbourhoods of Γx and Γy. □

[1] Thus, Γ may be said to act properly continuously on $\mathbb{H}$. In the literature however, Γ is usually said to act "properly discontinuously"! See Lee 2003, p. 225, for a discussion of this terminology.

PROPOSITION 1.7 *The space $\Gamma\backslash\mathbb{H}^*$ is Hausdorff and compact.*

PROOF. After (1.6), to show that $\Gamma\backslash\mathbb{H}^*$ is Hausdorff, only requires an examination near the cusps, which we leave to the reader (cf. the next subsection). The space $\Gamma\backslash\mathbb{H}^*$ is a quotient of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$, which is compact because $\bar{D}\cup\{\infty\}$ is compact. □

The complex structure on $\Gamma_0(N)\backslash\mathbb{H}^*$

The subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that we shall be especially interested in are

$$\Gamma_0(N)=\left\{\left.\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right| c\equiv 0 \mod N\right\}.$$

We let $\Gamma_0(1)=\mathrm{SL}_2(\mathbb{Z})$.

For $z_0\in\mathbb{H}$, choose a neighbourhood V of z_0 such that

$$\gamma V\cap V\neq\emptyset\Rightarrow\gamma z_0=z_0,$$

and let $U=\pi(V)$ — it is open because $\pi^{-1}U=\bigcup\gamma V$ is open.

If the stabilizer of z_0 in $\Gamma_0(N)$ is $\pm I$, then $\pi\colon V\to U$ is a homeomorphism, with inverse φ say, and we require (U,φ) to be a coordinate neighbourhood.

If the stabilizer of z_0 in $\Gamma_0(N)$ is $\neq\{\pm I\}$, then it is a cyclic group of order $2m$ with $m=2$ or 3 (and its stabilizer in $\Gamma_0(N)/\{\pm I\}$ has order 2 or 3) — see (1.3b). The fractional linear transformation

$$\lambda:\mathbb{H}\to D,\quad z\mapsto\frac{z-z_0}{z-\bar{z}_0},$$

carries z_0 to 0 in the unit disk D. There is a well-defined map $\varphi\colon U\to\mathbb{C}$ such that $\varphi(\pi(z))=\lambda(z)^m$, and we require (U,φ) to be a coordinate neighbourhood (cf. Example 1.2).

Next consider $z_0=\infty$. Choose V to be the neighbourhood $\{z\mid\Im(z)>2\}$ of ∞, and let $U=\pi(V)$. If

$$z\in V\cap\gamma V,\quad\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N),$$

then

$$2\leq\Im(\gamma z)=\frac{\Im(z)}{|cz+d|^2}\leq\frac{1}{|c|^2\Im(z)}\leq\frac{1}{2|c|^2}$$

and so $c=0$. Therefore

$$\gamma=\pm\begin{pmatrix} 1 & m\\ 0 & 1\end{pmatrix},$$

and so there is a well-defined map $\varphi\colon U\to\mathbb{C}$ such that $\varphi(\pi(z))=e^{2\pi iz}$, and we require (U,φ) to be a coordinate neighbourhood (cf. Example 1.1).

For $z_0\in\mathbb{Q}$, we choose a $\beta\in\mathrm{SL}_2(\mathbb{Z})$ such that $\beta(z_0)=\infty$, and proceed similarly.

PROPOSITION 1.8 *The coordinate neighbourhoods defined above are compatible, and therefore define on $\Gamma_0(N)\backslash\mathbb{H}^*$ the structure of a Riemann surface.*

PROOF. Routine exercise. □

Write $X_0(N)$ for the Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$, and $Y_0(N)$ for its open subsurface $\Gamma_0(N)\backslash\mathbb{H}$.

The genus of $X_0(N)$

The genus of a Riemann surface can be computed by "triangulating" it, and using the formula

$$2-2g = V-E+F$$

where V is the number of vertices, E is the number of edges, and F is the number of faces. For example, the sphere can be triangulated by projecting out from a regular tetrahedron. Then $V=4$, $E=6$, and $F=4$, so that $g=0$ as expected.

PROPOSITION 1.9 *The Riemann surface $X_0(1)$ has genus zero.*

PROOF. One gets a fake triangulation of the sphere by taking as vertices three points on the equator, and the upper and lower hemispheres as the faces. This gives the correct genus

$$2 = 3-3+2$$

but it violates the usual definition of a triangulation, which requires that any two triangles intersect in a single side, a single vertex, or not at all. It can be made into a valid triangulation by adding the north pole as a vertex, and joining it to the three vertices on the equator.

One gets a fake triangulation of $X_0(1)$ by taking the three vertices ρ, i, and ∞ and the obvious curves joining them (two on the boundary of D and one the imaginary axis from i to ∞). It can be turned into a valid triangulation by adding a fourth point in D with real part >0, and joining it to ρ, i, and ∞. □

For a finite mapping $\pi\colon Y \to X$ of compact Riemann surfaces, the Hurwitz genus formula relates the two genuses:

$$2g_Y - 2 = (2g_X - 2)m + \sum_{Q\in Y}(e_Q - 1).$$

Here m is the degree of the mapping, so that $\pi^{-1}(P)$ has m elements except for finitely many P, and e_Q is the ramification index, so that $e_Q = 1$ unless at least two sheets come together at Q above $\pi(Q)$ in which case it is the number of such sheets.

For example, if E is the elliptic curve

$$E : Y^2Z = X^3 + aXZ^2 + bZ^3, \quad a, b \in \mathbb{C}, \quad \Delta \neq 0,$$

and π is the map

$$\infty \mapsto \infty\,, (x : y : z) \mapsto (x : z) : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

then $m = 2$ and $e_Q = 1$ except for $Q = \infty$ or one the three points of order 2 on E, in which case $e_Q = 2$. This is consistent with $E(\mathbb{C})$ having genus 1 and $\mathbb{P}^1(\mathbb{C})$ (the Riemann sphere) having genus 0.

The Hurwitz genus formula can be proved without too much difficulty by triangulating Y in such a way that the ramification points are vertices and such that the triangulation of Y lies over a triangulation of X.

Now one can compute the genus of $X_0(N)$ by studying the quotient map $X_0(N) \to X_0(1)$. The only (possible) ramification points are those $\Gamma_0(1)$-equivalent to one of i, ρ, or ∞. Explicit formulas can be found in Shimura 1971, pp. 23-25. For example, one finds that, for p a prime > 3,

$$\text{genus}(X_0(p)) = \begin{cases} n-1 & \text{if } p = 12n+1 \\ n & \text{if } p = 12n+5, 12n+7 \\ n+1 & \text{if } p = 12n+11. \end{cases}$$

Moreover,

$$\begin{aligned} g &= 0 \text{ if } N = 1, 2, 3, \ldots, 10, 12, 13, 16, 18, 25; \\ g &= 1 \text{ if } N = 11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49 \\ g &= 2 \text{ if } N = 22, 23, 26, 28, 29, 31, 37, 50 \end{aligned}$$

(Mazur 1973).

EXERCISE 1.10 (a) For a prime p, show that the natural action of $\Gamma_0(p)$ on $\mathbb{P}^1(\mathbb{Q})$ has only two orbits, represented by 0 and $\infty = (1 : 0)$. Deduce that $X_0(p) \smallsetminus Y_0(p)$ has exactly two elements.

(b) Define $\Delta(z) = \Delta(\mathbb{Z}z + \mathbb{Z})$ (see Chap. III), so that Δ is a basis for the $\mathbb{C}$-vector space of cusp forms of weight 12 for $\Gamma_0(1)$. Define $\Delta_{11}(z) = \Delta(11z)$, and show that it is a cusp form of weight 12 for $\Gamma_0(11)$. Deduce that $\Delta \cdot \Delta_{11}$ is a cusp form of weight 24 for $\Gamma_0(11)$.

(c) Assume Jacobi's formula,

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

($q = e^{2\pi i z}$), and that $\mathcal{S}_2(\Gamma_0(11))$ has dimension 1. Show that

$$F(z) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{11n})^2,$$

is a cusp form of weight 2 for $\Gamma_0(11)$. [Hint: Let f be a nonzero element of $\mathcal{S}_2(\Gamma_0(11))$, and let $g = \Delta \cdot \Delta_{11}$. Show that f^{12}/g is holomorphic on $\mathbb{H}^*$ and invariant under $\Gamma_0(1)$, and is therefore constant (because the only holomorphic functions on a compact Riemann surface are the constant functions). The only real difficulty is in handling the cusp 0, since I have more-or-less ignored cusps other than ∞.]

2 $X_0(N)$ as an algebraic curve over $\mathbb{Q}$

In the last section, we defined compact Riemann surfaces $X_0(N)$. A general theorem states that any compact Riemann surface X can be identified with the set of complex points of a unique nonsingular projective algebraic curve C over $\mathbb{C}$. However, in general C can't be defined over $\mathbb{Q}$ (or even $\mathbb{Q}^{\mathrm{al}}$) — consider for example a Riemann surface $\mathbb{C}/\Lambda$ as in Chapter III whose j-invariant is transcendental — and when C can be defined over $\mathbb{Q}$, in general, it can't be defined in any canonical way — consider an elliptic curve E over $\mathbb{C}$ with $j(E) \in \mathbb{Q}$.

In this section, we shall see that $X_0(N)$ has the remarkable property that it *is* the set of complex points of a *canonical* curve over $\mathbb{Q}$.

Modular functions

For a connected compact Riemann surface X, the meromorphic functions on X form a field of transcendence degree 1 over $\mathbb{C}$. We shall determine this field for $X = X_0(N)$.

For a subgroup Γ of finite index in $\mathrm{SL}_2(\mathbb{Z})$, the meromorphic functions on $\Gamma\backslash\mathbb{H}^*$ are called the ***modular functions*** for Γ. If $\pi\colon \mathbb{H} \to \Gamma\backslash\mathbb{H}^*$ is the quotient map, then $g \mapsto \pi \circ g$ identifies the modular functions for Γ with the functions f on $\mathbb{H}$ such that

(a) f is meromorphic on $\mathbb{H}$;
(b) for any $\gamma \in \Gamma$, $f(\gamma z) = f(z)$;
(c) f is meromorphic at the cusps (i.e., at the points of $\mathbb{H}^* \smallsetminus \mathbb{H}$).

The meromorphic functions on $X_0(1)$

Let S be the Riemann sphere $S = \mathbb{C}\cup\{\infty\}$ (better, $S = \mathbb{P}^1(\mathbb{C}) = \mathbb{A}^1(\mathbb{C})\cup\{(1:0)\}$. The meromorphic functions on S are the rational functions of z, and the automorphisms of S are the fractional-linear transformations,

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad - bc \neq 0.$$

In fact, $\mathrm{Aut}(S) = \mathrm{PGL}_2(\mathbb{C}) \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$. Moreover, given two sets of distinct points on S, $\{P_1, P_2, P_3\}$ and $\{Q_1, Q_2, Q_3\}$, there is a unique fractional-linear transformation sending each P_i to Q_i. (The proof of the last statement is an easy exercise in linear algebra: given two sets $\{L_1, L_2, L_3\}$ and $\{M_1, M_2, M_3\}$ of distinct lines through the origin in $\mathbb{C}^2$, there is a linear transformation carrying each L_i to M_i, and the linear transformation is unique up to multiplication by a nonzero constant.)

Recall that ∞, i, and ρ are points in $\mathbb{H}^*$. We use the same symbols to denote their images in $X_0(1)$.

PROPOSITION 2.1 *There exists a unique meromorphic function J on $X_0(1)$ that is holomorphic except at ∞, where it has a simple pole, and takes the values*

$$J(i) = 1, \quad J(\rho) = 0.$$

Moreover, the meromorphic functions on $X_0(1)$ are the rational functions of J.

PROOF. We saw in the last section that $X_0(1)$ is isomorphic (as a Riemann surface) to the Riemann sphere S. Let $f\colon X_0(1) \to S$ be an isomorphism, and let P, Q, R be the images of ρ, i, ∞. There is a unique fractional-linear transformation L sending P, Q, R to $0, 1, \infty$, and the composite $L \circ f$ has the required properties. If J' is a second such function, then the composite $J' \circ J^{-1}$ is an automorphism of S fixing $0, 1, \infty$, and so is the identity map. Under this isomorphism, the function z on S corresponds to the function J on $X_0(1)$. □

We wish to identify the function J. Recall from Chapter III that, for a lattice Λ in $\mathbb{C}$,

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda, \omega \neq 0} \frac{1}{\omega^{2k}},$$

and

$$G_{2k}(z) = G_{2k}(\mathbb{Z}z + \mathbb{Z}), \quad g_4(z) = 60G_4(z), \quad g_6(z) = 140G_6(z), \quad z \in \mathbb{H}.$$

Then $(\wp, \wp')$ maps $\mathbb{C}/\mathbb{Z}z + \mathbb{Z}$ onto the elliptic curve

$$Y^2Z = 4X^3 - g_4(z)XZ^2 - g_6(z)Z^3, \quad \Delta = g_4(z)^3 - 27g_6(z)^2 \neq 0,$$

whose j-invariant is

$$j(z) = \frac{1728 g_4(z)^3}{\Delta}.$$

From their definitions, it is clear that $G_{2k}(z)$, $\Delta(z)$, and $j(z)$ are invariant under $T\colon z \mapsto z + 1$, and so can be expressed in terms of the variable $q = e^{2\pi i z}$. In

Serre 1973, VII, Equations (23,33,42), one can find the following expansions:

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \quad \sigma_k(n) = \sum_{d|n} d^k, \tag{35}$$

$$\Delta = (2\pi)^{12}(q - 24q^2 + 252q^3 - 1472q^4 + \cdots), \tag{36}$$

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \sum_{n=3}^{\infty} c(n)q^n, \quad c(n) \in \mathbb{Z}. \tag{37}$$

The proof of the formula for $G_{2k}(z)$ is elementary, and the others follow from it together with elementary results on $\zeta(2k)$. The factor 1728 was traditionally included in the formula for j so that it has residue 1 at infinity.

The function j is invariant under $\mathrm{SL}_2(\mathbb{Z})$, because $j(z)$ depends only on the lattice $\mathbb{Z}z + \mathbb{Z}$. Moreover:

- ◇ $j(\rho) = 0$, because $\mathbb{C}/\mathbb{Z}\rho + \mathbb{Z}$ has complex multiplication by $\rho^2 = \sqrt[3]{1}$, and therefore is of the form $Y^2 = X^3 + b$, which has j-invariant 0;
- ◇ $j(i) = 1728$, because $\mathbb{C}/\mathbb{Z}i + \mathbb{Z}$ has complex multiplication by i, and therefore is of the form $Y^2 = X^3 + aX$.

Consequently $j = 1728J$, and we can restate (2.1) as:

PROPOSITION 2.2 *The function j is the unique meromorphic function on $X_0(1)$ that is holomorphic except at ∞, where it has a simple pole, and takes the values*

$$j(i) = 1728, \quad j(\rho) = 0.$$

In particular j defines an isomorphism from $X_0(1)$ onto the Riemann sphere, and so the field of meromorphic functions on $X_0(N)$ is $\mathbb{C}(j)$.

The meromorphic functions on $X_0(N)$

Define j_N to be the function on $\mathbb{H}$ such that $j_N(z) = j(Nz)$. For $\gamma \in \Gamma_0(1)$, one is tempted to say

$$j_N(\gamma z) \stackrel{\text{def}}{=} j(N\gamma z) = j(\gamma N z) = j(Nz) \stackrel{\text{def}}{=} j_N(z),$$

but, this is false in general, because $N\gamma z \neq \gamma N z$. However, it is true that $j_N(\gamma z) = j_N(z)$ if $\gamma \in \Gamma_0(N)$. In fact, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, so that $c = Nc'$ with $c' \in \mathbb{Z}$. Then

$$j_N(\gamma z) = j\left(\frac{Naz + Nb}{cz + d}\right) = j\left(\frac{a(Nz) + Nb}{c'(Nz) + d}\right) = j(\gamma' N z)$$

where $\gamma' = \begin{pmatrix} a & Nb \\ c' & b \end{pmatrix} \in \Gamma_0(1)$, so

$$j(\gamma' Nz) = j(Nz) = j_N(z).$$

Thus, we see that j_N is invariant under $\Gamma_0(N)$, and therefore defines a meromorphic function on $X_0(N)$.

THEOREM 2.3 *The field of meromorphic functions on $X_0(N)$ is $\mathbb{C}(j, j_N)$.*

PROOF. The curve $X_0(N)$ is a covering of $X_0(1)$ of degree $m = (\Gamma_0(1) : \Gamma_0(N))$. The general theory implies that the field of meromorphic functions on $X_0(N)$ has degree m over $\mathbb{C}(j)$, but we shall prove this again. Let $\{\gamma_1 = 1, ..., \gamma_m\}$ be a set of representatives for the right cosets of $\Gamma_0(N)$ in $\Gamma_0(1)$, so that,

$$\Gamma_0(1) = \bigsqcup_{i=1}^{m} \Gamma_0(N)\gamma_i \quad \text{(disjoint union).}$$

Right multiplication by a $\gamma \in \Gamma_0(1)$ permutes the cosets $\Gamma_0(N)\gamma_i$, and so $\{\gamma_1\gamma, ..., \gamma_m\gamma\}$ is also a set of representatives for the right cosets of $\Gamma_0(N)$ in $\Gamma_0(1)$.

If $f(z)$ is a modular function for $\Gamma_0(N)$, then $f(\gamma_i z)$ depends only on the coset $\Gamma_0(N)\gamma_i$. Hence the functions $\{f(\gamma_i\gamma z)\}$ are a permutation of the functions $\{f(\gamma_i z)\}$, and any symmetric polynomial in the $f(\gamma_i z)$ is invariant under $\Gamma_0(1)$; since such a polynomial obviously satisfies the other conditions, it is a modular function for $\Gamma_0(1)$, and hence a rational function of j. Therefore $f(z)$ satisfies a polynomial of degree m with coefficients in $\mathbb{C}(j)$, namely, $\prod(Y - f(\gamma_i z))$. Since this holds for every meromorphic function on $X_0(N)$, we see that the field of such functions has degree at most m over $\mathbb{C}(j)$ (apply the primitive element theorem, FT, 5.1).

Next I claim that all the $f(\gamma_i z)$ are conjugate to $f(z)$ over $\mathbb{C}(j)$: for let $F(j, Y)$ be the minimum polynomial of $f(z)$ over $\mathbb{C}(j)$, so that $F(j, Y)$ is monic and irreducible when regarded as a polynomial in Y with coefficients in $\mathbb{C}(j)$; on replacing z with $\gamma_i z$ and remembering that $j(\gamma_i z) = j(z)$, we find that $F(j(z), f(\gamma_i z)) = 0$, which proves the claim.

If we can show that the functions $j(N\gamma_i z)$ are distinct, then it will follow that the minimum polynomial of j_N over $\mathbb{C}(j)$ has degree m, and that the field of meromorphic functions on $X_0(N)$ has degree m over $\mathbb{C}(j)$, and is generated by j_N.

Suppose $j(N\gamma_i z) = j(N\gamma_j z)$ for some $i \neq j$. Recall (2.2) that j defines an isomorphism $\Gamma_0(1)\backslash\mathbb{H}^* \to S_2$ (Riemann sphere), and so

$$j(N\gamma_i z) = j(N\gamma_j z) \text{ all } z \Rightarrow \exists\gamma \in \Gamma_0(1) \text{ such that } N\gamma_i z = \gamma N\gamma_j z \text{ all } z,$$

and this implies that

$$\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \gamma_i = \pm\gamma \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \gamma_j.$$

Hence $\gamma_i\gamma_j^{-1} \in \Gamma_0(1) \cap \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(1) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_0(N)$, which contradicts the fact that γ_i and γ_j lie in different cosets. □

We saw in the proof that the minimum polynomial of j_N over $\mathbb{C}(j)$ is

$$F(j,Y) = \prod_{i=1}^{m}(Y - j(N\gamma_i z)).$$

The symmetric polynomials in the $j(N\gamma_i z)$ are holomorphic on $\mathbb{H}$. As they are rational functions of $j(z)$, they must in fact be polynomials in $j(z)$, and so $F_N(j,Y) \in \mathbb{C}[j,Y]$ (rather than $\mathbb{C}(j)[Y]$).

On replacing j with the variable X, we obtain a polynomial $F_N(X,Y) \in \mathbb{C}[X,Y]$,

$$F_N(X,Y) = \sum c_{r,s}X^rY^s, \quad c_{r,s} \in \mathbb{C}, \quad c_{0,m} = 1.$$

I claim that $F_N(X,Y)$ is the unique polynomial of degree $\leq m$ in Y, with $c_{0,m} = 1$, such that

$$F_N(j, j_N) = 0.$$

In fact, $F_N(X,Y)$ generates the ideal in $\mathbb{C}[X,Y]$ of all polynomials $G(X,Y)$ such that $G(j, j_N) = 0$, from which the claim follows.

PROPOSITION 2.4 *The polynomial $F_N(X,Y)$ has coefficients in $\mathbb{Q}$.*

PROOF. We know that

$$j(z) = q^{-1} + \sum_{n=0}^{\infty} c(n)q^n, \quad c(n) \in \mathbb{Z}.$$

When we substitute this into the equation

$$F(j(z), j(Nz)) = 0,$$

and equate coefficients of powers of q, we obtain a set of linear equations for the $c_{r,s}$ with coefficients in $\mathbb{Q}$, and when we adjoin the equation

$$c_{0,m} = 1,$$

then the system determines the $c_{r,s}$ uniquely. Because the system of linear equations has a solution in $\mathbb{C}$, it also has a solution in $\mathbb{Q}$ (look at ranks of matrices); because the solution is unique, the solution in $\mathbb{C}$ must in fact lie in $\mathbb{Q}$. Therefore $c_{r,s} \in \mathbb{Q}$. □

ASIDE 2.5 The polynomial $F_N(X,Y)$ was introduced by Kronecker more than 100 years ago. It is known to be symmetric in X and Y. For $N=2$, it is

$$X^3+Y^3-X^2Y^2+1488XY(X+Y)-162000(X^2+Y^2)$$
$$+40773375XY+8748000000(X+Y)-157464000000000.$$

It was computed for $N=3,5,7$ by Smith (1878), Berwick (1916), and Herrmann (1974) respectively. At this point the humans gave up, and left it to the computers, which found F_{11} in 1984. This last computation took about 20 hours on a VAX-780, and the result is a polynomial with coefficients up to 10^{60} that takes five pages to write out. It is important to know that the polynomial exists; fortunately, it is not important to know what it is.

The curve $X_0(N)$ over $\mathbb{Q}$

Let C_N be the affine curve over $\mathbb{Q}$ with equation $F_N(X,Y)=0$, and let $\bar{C}_N$ be the projective curve defined by F_N made homogeneous. Then $z\mapsto (j(z),j(Nz))$ is a map $X_0(N)\smallsetminus\Xi\to C_N(\mathbb{C})$, where Ξ is the set where j or j_N has a pole. This map extends uniquely to a map $X_0(N)\to\bar{C}_N(\mathbb{C})$, which is an isomorphism except over the singular points of $\bar{C}_N$, and the pair $(X_0(N),X_0(N)\to\bar{C}_N(\mathbb{C}))$ is uniquely determined by $\bar{C}_N$ (up to a unique isomorphism): it is the canonical "desingularization" of $\bar{C}_N$ over $\mathbb{C}$.

Now consider $\bar{C}_N$ over $\mathbb{Q}$. There is a canonical desingularization $X\to\bar{C}_N$ over $\mathbb{Q}$, i.e., a projective nonsingular curve X over $\mathbb{Q}$, and a regular map $X\to\bar{C}_N$ that is an isomorphism except over the singular points of $\bar{C}_N$, and the pair $(X,X\to\bar{C}_N)$ is uniquely determined by $\bar{C}_N$ (up to unique isomorphism). When we pass to the $\mathbb{C}$-points, we see that $(X(\mathbb{C}),X(\mathbb{C})\to\bar{C}_N(\mathbb{C}))$ has the property characterizing $(X_0(N),X_0(N)\to\bar{C}_N(\mathbb{C}))$, and so there is a unique isomorphism of Riemann surfaces $X_0(N)\to X(\mathbb{C})$ compatible with the maps to $\bar{C}_N(\mathbb{C})$.

In summary, we have a well-defined curve X over $\mathbb{Q}$, a regular map $\gamma\colon X\to\bar{C}_N$ over $\mathbb{Q}$, and an isomorphism $X_0(N)\to X(\mathbb{C})$ whose composite with $\gamma(\mathbb{C})$ is (outside a finite set) $z\mapsto(j(z),j(Nz))$.

In future, we'll often use $X_0(N)$ to denote the curve X over $\mathbb{Q}$ — it should be clear from the context whether we mean the curve over $\mathbb{Q}$ or the Riemann surface. The affine curve $X_0(N)\smallsetminus\{\text{cusps}\}\subset X_0(N)$ is denoted $Y_0(N)$; thus $Y_0(N)(\mathbb{C})=\Gamma_0(1)\backslash\mathbb{H}$.

REMARK 2.6 The curve $F_N(X,Y)=0$ is highly singular, because, without singularities, formula (7), p. 34, would predict much too high a genus.

The points on the curve $X_0(N)$

Since we can't write down an equation for $X_0(N)$ as a projective curve over $\mathbb{Q}$, we would at least like to know what its points are in any field containing $\mathbb{Q}$. This we can do.

We first look at the complex points of $X_0(N)$, i.e., at the Riemann surface $X_0(N)$. In this case, there is a diagram

$$\begin{tikzcd}
\{(E,S)\}/\approx \arrow[r, leftrightarrow] \arrow[d] & \{(\Lambda,S)\}/\mathbb{C}^\times \arrow[r, leftrightarrow] \arrow[d] & \Gamma_0(N)\backslash M/\mathbb{C}^\times \arrow[r, leftrightarrow] \arrow[d] & \Gamma_0(N)\backslash\mathbb{H} \arrow[d] \\
\{E\}/\approx \arrow[r, leftrightarrow, "(20)"] & \mathcal{L}/\mathbb{C}^\times \arrow[r, leftrightarrow, "(18)"] & \Gamma_0(1)\backslash M/\mathbb{C}^\times \arrow[r, leftrightarrow, "(18)"] & \Gamma_0(1)\backslash\mathbb{H}
\end{tikzcd}$$

whose terms we now explain. All the symbols $\leftrightarrow$ are natural bijections. The bottom row combines maps in Chapter III.

Recall that M is the subset of $\mathbb{C}\times\mathbb{C}$ of pairs (ω_1,ω_2) such that $\Im(\omega_1/\omega_2) > 0$ (so $M/\mathbb{C}^\times \subset \mathbb{P}^1(\mathbb{C})$), and that the bijection $M/\mathbb{C}^\times \to \mathbb{H}$ sends (ω_1,ω_2) to ω_1/ω_2. The rest of the right hand square is now obvious.

Recall that $\mathcal{L}$ is the set of lattices in $\mathbb{C}$, and that the lattices defined by two pairs in M are equal if and only if the pairs lie in the same $\Gamma_0(1)$-orbit. Thus in passing from an element of M to its $\Gamma_0(1)$-orbit we are forgetting the basis and remembering only the lattice. In passing from an element of M to its $\Gamma_0(N)$-orbit, we remember a little of the basis, for suppose

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(N).$$

Then

$$\begin{aligned}
\omega_1' &= a\omega_1 + b\omega_2\\
\omega_2' &= c\omega_1 + d\omega_2 \equiv d\omega_2 \mod N\Lambda.
\end{aligned}$$

Hence

$$\tfrac{1}{N}\omega_2' \equiv \tfrac{d}{N}\omega_2 \mod \Lambda.$$

Note that because $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ has determinant 1, $\gcd(d,N) = 1$, and so $\frac{1}{N}\omega_2'$ and $\frac{1}{N}\omega_2$ generate the same cyclic subgroup S of order N in $\mathbb{C}/\Lambda$. Therefore, the map

$$(\omega_1,\omega_2) \mapsto (\Lambda(\omega_1,\omega_2), \langle \tfrac{1}{N}\omega_2\rangle)$$

defines a bijection from $\Gamma_0(N)\backslash M$ to the set of pairs consisting of a lattice Λ in $\mathbb{C}$ and a cyclic subgroup S of $\mathbb{C}/\Lambda$ of order N. Now $(\Lambda,S) \mapsto (\mathbb{C}/\Lambda, S)$ defines a one-to-one correspondence between this last set and the set of isomorphism classes of pairs (E,S) consisting of an elliptic curve over $\mathbb{C}$ and a cyclic subgroup S of $E(\mathbb{C})$ of order N. An isomorphism $(E,S) \to (E',S')$ is an isomorphism $E \to E'$ carrying S into S'.

Note that the quotient of E by S,

$$E/S \simeq \mathbb{C}/\Lambda(\omega_1, \tfrac{1}{N}\omega_2),$$

and that $\omega_1/(\frac{1}{N}\omega_2) = N\frac{\omega_1}{\omega_2}$. Thus, if $j(E) = j(z)$, then $j(E/S) = j(Nz)$.

Now, for any field $k \supset \mathbb{Q}$, define $\mathcal{E}_0(N)(k)$ to be the set of isomorphism classes of pairs E consisting of an elliptic curve E over k and a cyclic subgroup $S \subset E(k^{\mathrm{al}})$ of order N stable under $\mathrm{Gal}(k^{\mathrm{al}}/k)$ — thus the subgroup S is defined over k but not necessarily its individual elements. The above remarks show that there is a canonical bijection

$$\mathcal{E}_0(N)(\mathbb{C})/{\approx} \to Y_0(N)$$

whose composite with the map $Y_0(N) \to C_N(\mathbb{C})$ is $(E, S) \mapsto (j(E), j(E/S))$. Here $Y_0(N)$ denotes the Riemann surface $\Gamma_0(N)\backslash\mathbb{H}$.

THEOREM 2.7 *For any field $k \supset \mathbb{Q}$, there is a map*

$$\mathcal{E}_0(N)(k) \to Y_0(N)(k),$$

functorial in k, such that

(a) *the composite*
$$\mathcal{E}_0(N)(k) \to Y_0(N)(k) \to C_N(k)$$
is $(E, S) \mapsto (j(E), j(E/S))$;

(b) *for all k, $\mathcal{E}_0(N)(k)/{\approx} \to Y_0(N)(k)$ is surjective, and for all algebraically closed k it is bijective.*

The map being functorial in k means that for every homomorphism $\sigma\colon k \to k'$ of fields, the diagram

$$\begin{array}{ccc}
\mathcal{E}_0(N)(k') & \longrightarrow & Y_0(N)(k') \\
\uparrow{\scriptstyle\sigma} & & \uparrow{\scriptstyle\sigma} \\
\mathcal{E}_0(N)(k) & \longrightarrow & Y_0(N)(k)
\end{array}$$

commutes. In particular, $\mathcal{E}_0(N)(k^{\mathrm{al}}) \to Y_0(N)(k^{\mathrm{al}})$ commutes with the actions of $\mathrm{Gal}(k^{\mathrm{al}}/k)$. Since $Y_0(N)(k^{\mathrm{al}})^{\mathrm{Gal}(k^{\mathrm{al}}/k)} = Y_0(N)(k)$, this implies that

$$\boxed{Y_0(N)(k) = (\mathcal{E}_0(N)(k^{\mathrm{al}})/{\approx})^{\mathrm{Gal}(k^{\mathrm{al}}/k)}}$$

for any field $k \supset \mathbb{Q}$.

This description of the points can be extended to $X_0(N)$ by adding to $\mathcal{E}_0(N)$ certain "degenerate" elliptic curves.

I now sketch the proof of Theorem 2.7. The bijection $\mathcal{E}_0(N)(\mathbb{C})/{\approx} \simeq Y_0(N)(\mathbb{C})$ defines an action of $\mathrm{Aut}(\Omega/\mathbb{Q})$ on $Y_0(N)(\mathbb{C})$, which one can show is continuous and regular, and so defines a model $E_0(N)$ of $Y_0(N)$ over $\mathbb{Q}$ (see I 5.5). Over $\mathbb{C}$, we have regular maps

$$E_0(N) \to Y_0(N) \to C_N$$

whose composite (on points) is $(E, S) \mapsto (j(E), j(E/S))$. As this map commutes with the automorphisms of $\mathbb{C}$, the regular map $E_0(N) \to C_N$ is defined over $\mathbb{Q}$ (loc. cit.), and therefore so also is $E_0(N) \to Y_0(N)$. As it is an isomorphism on the $\mathbb{C}$-points, it is an isomorphism (I 4.24). One can show that, for any field $k \subset \mathbb{C}$, the map $\mathcal{E}_0(N)(k) \to E_0(N)(k)$ is surjective with fibres equal to the geometric isomorphism classes of pairs (E, S), where two pairs are geometrically isomorphic if they become isomorphic over $\mathbb{C}$ (equivalently k^{al}).

Variants

For our applications to elliptic curves, we shall only need to use the quotients of $\mathbb{H}^*$ by the subgroups $\Gamma_0(N)$, but quotients by other subgroups are also of interest. For example, let

$$\begin{aligned}\Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a \equiv 1 \equiv d \mod N, \quad c \equiv 0 \mod N \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N \right\}.\end{aligned}$$

The quotient $X_1(N) = \Gamma_1(N) \backslash \mathbb{H}^*$ again defines a curve, also denoted $X_1(N)$, over $\mathbb{Q}$, and there is a theorem similar to (2.7) but with $\mathcal{E}_1(N)(k)$ the set of pairs (E, P) consisting of an elliptic curve E over k and a point $P \in E(k)$ of order N.

In this case, the map

$$\mathcal{E}_1(N)(k)/\approx \; \to Y_1(N)(k)$$

is a bijection whenever $4|N$. The curve $X_1(N)$ has genus 0 exactly for $N = 1, 2, \ldots, 10, 12$. Since $X_1(N)$ has a point with coordinates in $\mathbb{Q}$ for each of these N (there does exist an elliptic curve over $\mathbb{Q}$ with a point of that order — see II, §5), $X_1(N)$ is isomorphic to $\mathbb{P}^1$ (see I, §2), and so $X_1(N)$ has infinitely many rational points. Therefore, for $N = 1, 2, \ldots, 10, 12$, there are infinitely many elliptic curves over $\mathbb{Q}$ with a point of order N with coordinates in $\mathbb{Q}$. Mazur showed, that for all other N, $Y_0(N)$ is empty, and so these are the only possible orders for a point on an elliptic curve over $\mathbb{Q}$ (see II 5.11).

3 Modular forms

It is difficult to construct functions on $\mathbb{H}$ invariant under a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. One strategy is to construct functions, not invariant under Γ, but transforming in a certain fixed manner. Two functions transforming in the same manner will be invariant under Γ. This idea suggests the notion of a modular form.

Definition of a modular form

DEFINITION 3.1 Let Γ be a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$. A ***modular form*** for Γ of weight[2] $2k$ is a function $f\colon \mathbb{H} \to \mathbb{C}$ such that

(a) f is holomorphic on $\mathbb{H}$;

(b) for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(\gamma z) = (cz+d)^{2k} f(z)$;

(c) f is holomorphic at the cusps.

Recall that the cusps are the points of $\mathbb{H}^*$ not in $\mathbb{H}$. Since Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$, $T^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ is in Γ for some integer $h > 0$, which we may take to be as small as possible. Then condition (b) implies that $f(T^h z) = f(z)$, i.e., that $f(z+h) = f(z)$, and so

$$f(z) = f^*(q), \quad q = e^{2\pi i z/h},$$

and f^* is a function on a neighbourhood of $0 \in \mathbb{C}$, with 0 removed. To say that f is holomorphic at ∞ means that f^* is holomorphic at 0, and so

$$f(z) = \sum_{n \geq 0} c(n) q^n, \quad q = e^{2\pi i z/h}.$$

For a cusp $r \neq \infty$, choose a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\infty) = r$, and then the requirement is that $f \circ \gamma$ be holomorphic at ∞. It suffices to check the condition for one cusp in each Γ-orbit.

A modular form is called a ***cusp form*** if it is zero at the cusps. For example, for the cusp ∞ this means that

$$f(z) = \sum_{n \geq 1} c(n) q^n, \quad q = e^{2\pi i z/h}.$$

REMARK 3.2 Note that, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$d\gamma z = d\frac{az+b}{cz+d} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} dz = (cz+d)^{-2} dz.$$

Thus condition (3.1b) says that $f(z)(dz)^k$ is invariant under the action of Γ.

Write $\mathcal{M}_{2k}(\Gamma)$ for the vector space of modular forms of weight $2k$, and $\mathcal{S}_{2k}(\Gamma)$ for the subspace[3] of cusp forms. A modular form of weight 0 is a

[2] k and $-k$ are also used.

[3] The $\mathcal{S}$ is for "Spitzenform", the German name for cusp form. The French name is "forme parabolique".

holomorphic modular function (i.e., a holomorphic function on the compact Riemann surface $X(\Gamma)$), and is therefore constant: $\mathcal{M}_0(\Gamma) = \mathbb{C}$. The product of modular forms of weight $2k$ and $2k'$ is a modular form of weight $2(k + k')$, which is a cusp form if one of the two forms is a cusp form. Therefore $\bigoplus_{k\geq 0} \mathcal{M}_{2k}(\Gamma)$ is a graded $\mathbb{C}$-algebra.

PROPOSITION 3.3 *Let π be the quotient map $\mathbb{H}^* \to \Gamma_0(N)\backslash\mathbb{H}^*$, and for any holomorphic differential ω on $\Gamma_0(N)\backslash\mathbb{H}^*$, set $\pi^*\omega = f dz$. Then $\omega \mapsto f$ is an isomorphism from the space of holomorphic differentials on $\Gamma_0(N)\backslash\mathbb{H}^*$ to $\mathcal{S}_2(\Gamma_0(N))$.*

PROOF. The only surprise is that f is necessarily a cusp form rather than just a modular form. I explain what happens at ∞. Recall (p. 179) that there is a neighbourhood U of ∞ in $\Gamma_0(N)\backslash\mathbb{H}^*$ and an isomorphism $q: U \to D$ (some disk) such that $q \circ \pi = e^{2\pi i z}$. Consider the differential $g(q)dq$ on U. Its inverse image on $\mathbb{H}$ is

$$g(e^{2\pi i z})d(e^{2\pi i z}) = 2\pi i \cdot g(e^{2\pi i z}) \cdot e^{2\pi i z} dz = 2\pi i f dz$$

where $f(z) = g(e^{2\pi i z}) \cdot e^{2\pi i z}$. If g is holomorphic at 0, then

$$g(q) = \sum_{n\geq 0} c(n)q^n,$$

and so the q-expansion of f is $q\sum_{n\geq 0} c(n)q^n$, which is zero at ∞. □

COROLLARY 3.4 *The $\mathbb{C}$-vector space $\mathcal{S}_2(\Gamma_0(N))$ has dimension equal to the genus of $X_0(N)$.*

PROOF. It is part of the theory surrounding the Riemann-Roch theorem that the holomorphic differential forms on a compact Riemann surface form a vector space equal to the genus of the surface. □

Hence, there are explicit formulas for the dimension of $\mathcal{S}_2(\Gamma_0(N))$ — see p. 180. For example, it is zero for $N \leq 10$, and has dimension 1 for $N = 11$. In fact, the Riemann-Roch theorem gives formulas for the dimension of $\mathcal{S}_{2k}(\Gamma_0(N))$ for all N.

The modular forms for $\Gamma_0(1)$

In this section, we find the $\mathbb{C}$-algebra $\bigoplus_{k\geq 0} \mathcal{M}_{2k}(\Gamma_0(1))$.

We first explain a method of constructing functions satisfying (3.1b). As before, let $\mathcal{L}$ be the set of lattices in $\mathbb{C}$, and let $F: \mathcal{L} \to \mathbb{C}$ be a function such that

$$F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda), \quad \lambda \in \mathbb{C}, \quad \Lambda \in \mathcal{L}.$$

Then

$$\omega_2^{2k} F(\Lambda(\omega_1, \omega_2))$$

depends only on the ratio $\omega_1 : \omega_2$, and so there is a function $f(z)$ defined on $\mathbb{H}$ such that

$$\omega_2^{2k} F(\Lambda(\omega_1, \omega_2)) = f(\omega_1/\omega_2) \quad \text{whenever } \Im(\omega_1/\omega_2) > 0.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\Lambda(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = \Lambda(\omega_1, \omega_2)$ and so

$$f(\tfrac{az+b}{cz+d}) = (cz+d)^{-2k} F(\Lambda(z,1)) = (cz+d)^{-2k} f(z).$$

When we apply this remark to the Eisenstein series

$$G_{2k}(\Lambda) = \sum_{\omega \in \Lambda, \omega \neq 0} \frac{1}{\omega^{2k}},$$

we find that the function $G_{2k}(z) \overset{\text{def}}{=} G_{2k}(\Lambda(z,1))$ satisfies (3.1b). In fact:

PROPOSITION 3.5 *For all $k > 1$, $G_{2k}(z)$ is a modular form of weight $2k$ for $\Gamma_0(1)$, and Δ is a cusp form of weight 12.*

PROOF. We know that $G_{2k}(z)$ is holomorphic on $\mathbb{H}$, and the formula (35) shows that it is holomorphic at ∞, which is the only cusp for $\Gamma_0(1)$ (up to $\Gamma_0(1)$-equivalence). The statement for Δ is obvious from its definition $\Delta = g_4(z)^3 - 27g_4(z)^2$, and its q-expansion (36). □

THEOREM 3.6 *The $\mathbb{C}$-algebra $\bigoplus_{k\geq 0} \mathcal{M}_{2k}(\Gamma_0(1))$ is generated by G_4 and G_6, and G_4 and G_6 are algebraically independent over $\mathbb{C}$. Therefore*

$$\mathbb{C}[G_4, G_6] \xrightarrow{\simeq} \bigoplus_{k\geq 0} \mathcal{M}_{2k}(\Gamma_0(1)), \quad \mathbb{C}[G_4, G_6] \simeq \mathbb{C}[X, Y]$$

(isomorphisms of graded $\mathbb{C}$-algebras if X and Y are given weights 4 and 6 respectively). Moreover,

$$f \mapsto f \cdot \Delta \colon \mathcal{M}_{2k-12}(\Gamma_0(1)) \to \mathcal{S}_{2k}(\Gamma_0(1))$$

is a bijection.

PROOF. Straightforward — see Serre 1973, VII.3.2. □

Therefore, for $k \geq 0$,

$$\dim \mathcal{M}_{2k}(\Gamma_0(N)) = \begin{cases} [k/6] & \text{if } k \equiv 1 \mod 6 \\ [k/6]+1 & \text{otherwise.} \end{cases}$$

Here $[x]$ is the largest integer $\leq x$.

THEOREM 3.7 (JACOBI) *There is the following formula:*

$$\Delta = (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = e^{2\pi i z}.$$

PROOF. Let

$$F(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

From the theorem, we know that the space of cusp forms of weight 12 has dimension 1, and therefore if we can show that $F(z)$ is such a form, then we'll know it is a multiple of Δ, and it will be follow from the formula on (36) that the multiple is $(2\pi)^{12}$.

Because $\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, to verify the conditions in (3.1), it suffices to verify that F transforms correctly under T and S. For T this is obvious from the way we have defined F, and for S it amounts to checking that

$$F(-1/z) = z^{12} F(z).$$

This is trickier than it looks, but there are short (two-page) elementary proofs — see for example, Serre 1973, VII.4.4. □

4 Modular forms and the L-series of elliptic curves

In this section, I shall discuss how the L-series classify the elliptic curves over $\mathbb{Q}$ up to isogeny, and then I shall explain how the work of Hecke, Petersson, and Atkin-Lehner leads to a list of candidates for the L-series of such curves, and hence suggests a classification of the isogeny classes.

Dirichlet series

A ***Dirichlet series*** is a series of the form

$$f(s) = \sum_{n \geq 1} a(n) n^{-s}, \quad a(n) \in \mathbb{C}, \quad s \in \mathbb{C}.$$

The simplest example of such a series is, of course, the Riemann zeta function $\sum_{n \geq 1} n^{-s}$. If there exist positive constants A and b such that $|\sum_{n \leq x} a(n)| \leq Ax^b$ for all large x, then the series for $f(s)$ converges to an analytic function on the half-plane $\Re(s) > b$.

It is important to note that the function $f(s)$ determines the $a(n)$s, i.e., if $\sum a(n) n^{-s}$ and $\sum b(n) n^{-s}$ are equal as functions of s on some half-plane, then

$a(n) = b(n)$ for all n. In fact, by means of the Mellin transform and its inverse (see 4.3 below), f determines, and is determined by, a function $g(q)$ convergent on some disk about 0, and $g(q) = \sum a(n)q^n$. Therefore, the claim follows from the similar statement for power series.

We shall be especially interested in Dirichlet series that are equal to Euler products, i.e., those that can be expressed as

$$f(s) = \prod_p \frac{1}{1 - P_p(p^{-s})}$$

where each P_p is a polynomial and the product is over the prime numbers.

Dirichlet series arise in two essentially different ways: from analysis and from arithmetic geometry and number theory. One of the *big* problems in mathematics is to show that the second set of Dirichlet series is a subset of the first, and to identify the subset. This is a major theme in the Langlands program, and the rest of the book will be concerned with explaining how work of Wiles and others succeeds in identifying the L-series of all elliptic curves over $\mathbb{Q}$ with certain L-series attached to modular forms.

The L-series of an elliptic curve

Recall that for an elliptic curve E over $\mathbb{Q}$, we define

$$L(E,s) = \prod_{p \text{ good}} \frac{1}{1 - a_p p^{-s} + p^{1-s}} \cdot \prod_{p \text{ bad}} \frac{1}{1 - a_p p^{-s}}$$

where

$$a_p = \begin{cases} p + 1 - N_p & p \text{ good;} \\ 1 & p \text{ split nodal;} \\ -1 & p \text{ nonsplit nodal;} \\ 0 & p \text{ cuspidal.} \end{cases}$$

Recall also that the conductor $N = N_{E/\mathbb{Q}}$ of $\mathbb{Q}$ is $\prod_p p^{f_p}$ where $f_p = 0$ if E has good reduction at p, $f_p = 1$ if E has nodal reduction at p, and $f_p \geq 2$ otherwise (and $= 2$ unless $p = 2, 3$).

On expanding out the product (cf. below), we obtain a Dirichlet series

$$L(E,s) = \sum a_n n^{-s}.$$

This series has, among others, the following properties:

(a) (Rationality) Its coefficients a_n lie in $\mathbb{Q}$.

(b) (Euler product) It can be expressed as an Euler product; in fact, that's how it is defined.

(c) (Functional equation) *Conjecturally* it can be extended analytically to a meromorphic function on the whole complex plane that satisfies the functional equation

$$\Lambda(E,s) = w_E\,\Lambda(E,2-s), \quad w_E = \pm 1,$$

where $\Lambda(E,s) = N_{E/\mathbb{Q}}^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s)$.

L-series and isogeny classes

Recall (p. 49) that two elliptic curves E and E' are said to be isogenous if there exists a nonconstant regular map from one to the other, and that isogeny is an equivalence relation.

An isogeny $E \to E'$ defines a homomorphism $E(\mathbb{Q}) \to E'(\mathbb{Q})$ which, in general, will be neither injective nor surjective, but which does have a finite kernel and cokernel. Therefore, the ranks of $E(\mathbb{Q})$ and $E'(\mathbb{Q})$ are the same, but their torsion subgroups will, in general, be different. Surprisingly, isogenous curves over a finite field do have the same number of points.

THEOREM 4.1 *Let E and E' be elliptic curves over $\mathbb{Q}$. If E and E' are isogenous, then $N_p(E) = N_p(E')$ for all good p. Conversely, if $N_p(E) = N_p(E')$ for sufficiently many good p, then E is isogenous to E'.*

PROOF. The fact that allows us to show that $N_p(E) = N_p(E')$ when E and E' are isogenous is that $N_p(E)$ is the degree of a map $E \to E$, in fact, it is the degree of $\varphi - 1$ where φ is the Frobenius map (see the proof of Theorem 9.4). An isogeny $\alpha\colon E \to E'$ induces an isogeny $\alpha_p\colon E_p \to E'_p$ on the reductions of the curves modulo p, which commutes with the Frobenius map: if

$$\alpha(x:y:z) = (P(x,y,z) : Q(x,y,z) : R(x,y,z)), \quad P,Q,R \in \mathbb{F}_p[X,Y,Z],$$

then

$$(\alpha\varphi)(x:y:z) = (P(x^p,y^p,z^p),\ldots) \text{ and}$$
$$(\varphi\alpha)(x:y:z) = (P(x,y,z)^p,\ldots),$$

which the characteristic p binomial theorem shows to be equal. Because the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi-1} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ E' & \xrightarrow{\varphi-1} & E' \end{array}$$

commutes, we see that

$$\deg\alpha\cdot\deg(\varphi-1) = \deg(\varphi-1)\cdot\deg\alpha,$$

so,

$$\deg\alpha\cdot N_p(E)=N_p(E')\cdot\deg\alpha,$$

and we can cancel $\deg\alpha$.

The converse is much more difficult. It was conjectured by Tate about 1963, and proved under some hypotheses by Serre. It was proved in general by Faltings in his paper on Mordell's conjecture (Faltings 1983). □

Faltings's result gives an effective procedure for deciding whether two elliptic curves over $\mathbb{Q}$ are isogenous: there is a constant P such that if $N_p(E)=N_p(E')$ for all good $p\leq P$, then E and E' are isogenous. This has been made into an effective algorithm. In practice, if your computer fails to find a p with $N_p(E)\neq N_p(E')$ in a few minutes you can be very confident that the curves are isogenous.

It is not quite obvious, but it follows from the theory of Néron models, that isogenous elliptic curves have the same type of reduction at every prime. Therefore, isogenous curves have exactly the same L-series and the same conductor. Because the L-series is determined by, and determines the N_p, we have the following corollary.

COROLLARY 4.2 *Two elliptic curves E and E' are isogenous if and only if $L(E,s)=L(E',s)$.*

We therefore have a one-to-one correspondence:

$$\{\text{isogeny classes of elliptic curves over }\mathbb{Q}\}\leftrightarrow\{\text{ certain }L\text{-series}\}.$$

In the remainder of this section we shall identify the L-series arising from elliptic curves over $\mathbb{Q}$ (in fact, we'll even identify the L-series of the elliptic curves with a fixed conductor).

Since we shall be classifying elliptic curves only up to isogeny, it is worth noting that a theorem of Shafarevich implies that there are only finitely many isomorphism classes of elliptic curves over $\mathbb{Q}$ with a given conductor, hence only finitely many in each isogeny class — see Silverman 1986, IX.6.

The L-series of a modular form

Let f be a cusp form of weight $2k$ for $\Gamma_0(N)$. By definition, it is invariant under $z\mapsto z+1$ and it is zero at the cusp ∞, and so can be expressed

$$f(s)=\sum_{n\geq 1}c(n)q^n,\quad q=e^{2\pi iz},\quad c(n)\in\mathbb{C}.$$

The ***L-series*** of f is the Dirichlet series

$$L(f,s)=\sum c(n)n^{-s},\quad s\in\mathbb{C}.$$

A rather rough estimate shows that $|c(n)| \leq Cn^k$ for some constant C, and so this Dirichlet series is convergent for $\Re(s) > k + 1$.

REMARK 4.3 Let $f \sum_{n\geq 1} c(n)q^n$ be cusp form. The ***Mellin transform*** of f (more accurately, of the function $y \mapsto f(iy)\colon \mathbb{R}_{>0} \to \mathbb{C}$) is defined to be

$$g(s) = \int_0^\infty f(iy)y^s \frac{dy}{y}.$$

Ignoring questions of convergence, we find that

$$\begin{aligned} g(s) &= \int_0^\infty \sum_{n=1}^\infty c(n)e^{-2\pi ny} y^s \frac{dy}{y} \\ &= \sum_{n=1}^\infty c_n \int_0^\infty e^{-t}(2\pi n)^{-s} t^s \frac{dt}{t} \quad (t = 2\pi ny) \\ &= (2\pi)^{-s}\Gamma(s) \sum_{n=1}^\infty c(n)n^{-s} \\ &= (2\pi)^{-s}\Gamma(s)L(f,s). \end{aligned}$$

For the experts, the Mellin transform is the version of the Fourier transform appropriate for the multiplicative group $\mathbb{R}_{>0}$.

Modular forms whose L-series have a functional equations

Let $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Then

$$\alpha_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha_N^{-1} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1/N \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix},$$

and so conjugation by α_N preserves $\Gamma_0(N)$. Define

$$(w_N f)(z) = (\sqrt{N}z)^{2k} f(-1/z).$$

Then w_N preserves $\mathcal{S}_{2k}(\Gamma_0(N))$ and $w_N^2 = 1$. Therefore the only possible eigenvalues for w_N are ± 1, and $\mathcal{S}_{2k}(\Gamma_0(N))$ is a direct sum of the corresponding eigenspaces $\mathcal{S}_{2k} = \mathcal{S}_{2k}^{+1} \oplus \mathcal{S}_{2k}^{-1}$.

THEOREM 4.4 (HECKE) *Let $f \in \mathcal{S}_{2k}(\Gamma_0(N))$ be a cusp form in the ε-eigenspace, $\varepsilon = 1$ or -1. Then f extends analytically to a holomorphic function on the whole complex plane, and satisfies the functional equation*

$$\Lambda(f,s) = \varepsilon(-1)^k \Lambda(f, k-s),$$

where

$$\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s).$$

PROOF. We omit the proof — it involves only fairly straightforward analysis (see Knapp, p. 270). □

Thus we see that, for $k = 2$, $L(f,s)$ has exactly the functional equation we hope for the L-series $L(E,s)$ of an elliptic curve E.

Modular forms whose L-functions are Euler products

Write

$$q\prod_{1}^{\infty}(1-q^n)^{24} = \sum \tau(n)q^n.$$

The function $n \mapsto \tau(n)$ is called the ***Ramanujan τ-function.*** Ramanujan conjectured that it had the following properties:

(a) $|\tau(p)| \leq 2p^{11/2}$;

(b) $\begin{cases} \tau(mn) = \tau(m)\tau(n) & \text{if } \gcd(m,n)=1; \\ \tau(p)\cdot\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1}) & \text{if } p \text{ is prime and } n \geq 1. \end{cases}$

Conjecture (a) was proved by Deligne: he first showed that $\tau(p) = \alpha + \beta$ where α and β occur as the reciprocal roots of a "$P_{11}(T)$" (see p. 160), and so (a) became a consequence of his proof of the Riemann hypothesis.

Conjecture (b) was proved by Mordell in 1917 in a paper in which he introduced the first examples of Hecke operators. Consider a modular form f of weight $2k$ for $\Gamma_0(N)$ (e.g., $\Delta = (2\pi)^{12}q\prod(1-q^n)^{24}$, which is a modular form of weight 12 for $\Gamma_0(1)$), and write

$$L(f,s) = \sum_{n\geq 0} c(n)n^{-s}.$$

PROPOSITION 4.5 *The Dirichlet series $L(f,s)$ has an Euler product expansion of the form*

$$L(f,s) = \prod_{p|N}\frac{1}{1-c(p)p^{-s}}\prod_{\gcd(p,N)=1}\frac{1}{1-c(p)p^{-s}+p^{2k-1-s}}$$

if (and only if)

$$(*)\begin{cases} c(mn) = c(m)c(n) & \textit{if } \gcd(m,n)=1; \\ c(p)\cdot c(p^r) = c(p^{r+1}) + p^{2k-1}c(p^{r-1}),\ r\geq 1, & \textit{if } p \textit{ is prime to } N; \\ c(p^r) = c(p)^r,\ r\geq 1, & \textit{if } p|N. \end{cases}$$

PROOF. For a prime p not dividing N, define

$$L_p(s) = \sum c(p^m)p^{-ms} = 1 + c(p)p^{-s} + c(p^2)(p^{-s})^2 + \cdots.$$

By inspection, the coefficient of $(p^{-s})^r$ in the product

$$(1 - c(p)p^{-s} + p^{2k-1}p^{-s})L_p(s)$$

is

$$\begin{array}{lll} 1 & \text{for} & r = 0 \\ 0 & \text{for} & r = 1 \\ & \cdots & \\ c(p^{r+1}) - c(p)c(p^r) + p^{2k-1}c(p^{r-1}) & \text{for} & r + 1. \end{array}$$

Therefore

$$L_p(s) = \frac{1}{1 - c(p)p^{-s} + p^{2k-1-s}}$$

if and only if the second equation in (*) holds.

Similarly,

$$L_p(s) \stackrel{\text{def}}{=} \sum c(p^r)p^{-rs} = \frac{1}{1 - c(p)p^{-s}}$$

if and only if the third equation in (*) holds.

If $n \in \mathbb{N}$ factors as $n = \prod p_i^{r_i}$, then the coefficient of $(p^{-s})^n$ in $\prod L_p(s)$ is $\prod c(p_i^{r_i})$, which equals $c(n)$ if and only if (*) holds. □

REMARK 4.6 The proposition says that $L(f, s)$ is equal to an Euler product of the above form if and only if $n \mapsto c(n)$ is weakly multiplicative and if the $c(p^m)$ satisfy a suitable recurrence relation. Note that $(*)$, together with the normalization $c(1) = 1$, shows that the $c(n)$ are determined by the $c(p)$ for p prime.

Hecke defined linear maps (the ***Hecke operators***)

$$T(n)\colon \mathcal{S}_{2k}(\Gamma_0(N)) \to \mathcal{S}_{2k}(\Gamma_0(N)), \quad n \geq 1,$$

and proved the following theorems.

THEOREM 4.7 *The maps $T(n)$ have the having the following properties:*

(a) $T(mn) = T(m)T(n)$ *if* $\gcd(m, n) = 1$;
(b) $T(p) \cdot T(p^r) = T(p^{r+1}) + p^{2k-1}T(p^{r-1})$ *if p doesn't divide N;*
(c) $T(p^r) = T(p)^r$, $r \geq 1$, $p|N$;
(d) *all $T(n)$ commute.*

PROOF. See the next subsection. □

THEOREM 4.8 *Let f be a cusp form of weight $2k$ for $\Gamma_0(N)$ that is simultaneously an eigenvector for all $T(n)$, say $T(n)f = \lambda(n)f$, and let*

$$f(z) = \sum_{n=1}^{\infty} c(n)q^n, \quad q = e^{2\pi i z}.$$

Then

$$c(n) = \lambda(n)c(1).$$

PROOF. See the next subsection. □

Note that $c(1) \neq 0$, because otherwise $c(n) = 0$ for all n, and so $f = 0$.

COROLLARY 4.9 *Let f be as in Theorem 4.8, and normalize f so that $c(1) = 1$. Then*

$$L(f,s) = \prod_{p|N} \frac{1}{1-c(p)p^{-s}} \prod_{\gcd(p,N)=1} \frac{1}{1-c(p)p^{-s}+p^{2k-1-s}}.$$

PROOF. Apply Proposition 4.5. □

EXAMPLE 4.10 Since $\mathcal{S}_{12}(\Gamma_0(1))$ has dimension 1, Δ must be an eigenform for all $T(n)$, which implies (b) of Ramanujan's conjecture.

Definition of the Hecke operators

I first explain the definition of the Hecke operators for the full group $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. Recall that we have canonical bijections

$$\mathcal{L}/\mathbb{C}^\times \leftrightarrow \Gamma_0(1)\backslash M/\mathbb{C}^\times \leftrightarrow \Gamma_0(1)\backslash\mathbb{H}.$$

Moreover, the equation

$$f(z) = F(\Lambda(z,1))$$

defines a one-to-one correspondence between

(a) functions $F\colon\mathcal{L}\to\mathbb{C}$ such that $F(\lambda\Lambda) = \lambda^{-2k}F(\Lambda)$, $\quad \lambda\in\mathbb{C}^\times$;

(b) functions $f\colon\mathbb{H}\to\mathbb{C}$ such that $f(\gamma z) = (cz+d)^{2k}f(z)$, $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$.

We'll work first with $\mathcal{L}$.

Let $\mathcal{D}$ be the free abelian group generated by the $\Lambda\in\mathcal{L}$; thus an element of $\mathcal{D}$ is a finite sum

$$\sum n_\Lambda[\Lambda], \quad n_\Lambda\in\mathbb{Z}, \quad \Lambda\in\mathcal{L},$$

and two such sums $\sum n_\Lambda[\Lambda]$ and $\sum n'_\Lambda[\Lambda]$ are equal if and only if $n_\Lambda = n'_\Lambda$ for all Λ.

For $n\geq 1$, define maps

$$T(n)\colon\mathcal{D}\to\mathcal{D}, \quad [\Lambda]\mapsto \sum_{(\Lambda:\Lambda')=n}[\Lambda']$$

and

$$R(n)\colon\mathcal{D}\to\mathcal{D}, \quad [\Lambda]\mapsto[n\Lambda].$$

PROPOSITION 4.11 (a) $T(mn) = T(m) \circ T(n)$ *if* $\gcd(m,n) = 1$*;*
(b) $T(p^r) \circ T(p) = T(p^{r+1}) + pR(p) \circ T(p^{r-1})$.

PROOF. (a) For a lattice Λ,

$$T(mn)[\Lambda] = \sum [\Lambda''] \quad \text{(sum over } \Lambda'' \text{, with } (\Lambda : \Lambda'') = mn),$$
$$T(m) \circ T(n)[\Lambda] = \sum [\Lambda'']$$
$$\text{(sum over pairs } (\Lambda', \Lambda'') \text{ with } (\Lambda : \Lambda') = n,\ (\Lambda' : \Lambda'') = m).$$

But if Λ'' is a lattice of index mn, then Λ/Λ'' is a commutative group of order mn with $\gcd(m,n) = 1$, and so has a unique subgroup of order m. The inverse image of this subgroup in Λ will be the unique lattice $\Lambda' \supset \Lambda''$ such that $(\Lambda' : \Lambda'') = m$. Thus the two sums are the same.

(b) For a lattice Λ,

$$T(p^r) \circ T(p)[\Lambda] = \sum [\Lambda'']$$
$$\text{(sum over pairs } (\Lambda', \Lambda'') \text{ with } (\Lambda : \Lambda') = p,\ (\Lambda' : \Lambda'') = p^r),$$
$$T(p^{r+1})[\Lambda] = \sum [\Lambda''] \text{ (sum over } \Lambda'' \text{ with } (\Lambda : \Lambda'') = p^{r+1});$$
$$pR(p) \circ T(p^{n-1})[\Lambda] = p \cdot \sum R(p)[\Lambda'] \text{ (sum over } \Lambda' \text{ with } (\Lambda : \Lambda') = p^{r-1})$$
$$= p \cdot \sum [\Lambda''] \text{ (over } \Lambda'' \subset p\Lambda \text{ with } (p\Lambda : \Lambda'') = p^{r-1}).$$

Each of these is a sum of lattices Λ'' of index p^{r+1} in Λ. Fix such a lattice Λ'', and let a be the number of times that $[\Lambda'']$ occurs in the first sum, and b the number of times it occurs in the third sum. It occurs exactly once in the second sum, and so we have to prove that

$$a = 1 + pb.$$

There are two cases to consider.

The lattice Λ'' *is not contained in* $p\Lambda$**.** In this case, $b = 0$, and a is the number of lattices Λ' such that $(\Lambda : \Lambda') = p$ and $\Lambda' \supset \Lambda''$. Such lattices are in one-to-one correspondence with the subgroups of $\Lambda/p\Lambda$ of index p containing the image $\bar{\Lambda}''$ of Λ'' in $\Lambda/p\Lambda$. But $(\Lambda : p\Lambda) = p^2$ and $\Lambda/p\Lambda \neq \bar{\Lambda}'' \neq 0$, and so there is only one such subgroup, namely $\bar{\Lambda}''$ itself. Therefore there is only one possible Λ', namely $p\Lambda + \Lambda''$, and so $a = 1$.

The lattice $\Lambda'' \supset p\Lambda$**.** Here $b = 1$. Every lattice Λ' of index p in Λ contains $p\Lambda$, hence also Λ'', and the number of such Λ''s is the number of lines through the origin in $\Lambda/p\Lambda \approx \mathbb{F}_p^2$, i.e., the number of points in $\mathbb{P}^1(\mathbb{F}_p)$, which is $p + 1$ as required. □

COROLLARY 4.12 *For any m and n,*

$$T(m) \circ T(n) = \sum d \cdot R(d) \circ T(mn/d^2)$$

(the sum is over the positive divisors d of $\gcd(m,n)$).

PROOF. Prove by induction on s that

$$T(p^r)T(p^s) = \sum_{i \le r,s} p^i \cdot R(p^i) \circ T(p^{r+s-2i}),$$

and then apply (a) of the proposition. □

COROLLARY 4.13 *Let $\mathcal{H}$ be the $\mathbb{Z}$-subalgebra of $\mathrm{End}(\mathcal{D})$ generated by $T(p)$ and $R(p)$ for p prime; then $\mathcal{H}$ is commutative, and it contains $T(n)$ for all n.*

PROOF. Obvious from the proposition. □

Let F be a function $\mathcal{L} \to \mathbb{C}$. We can extend F by linearity to a function $F\colon \mathcal{D} \to \mathbb{C}$,

$$F(\sum n_\Lambda [\Lambda]) = \sum n_\Lambda F(\Lambda).$$

For any linear map $T\colon \mathcal{D} \to \mathcal{D}$, we define $T \cdot F$ to be the function $\mathcal{L} \to \mathbb{C}$ such that $T \cdot F(\Lambda) = F(T\Lambda)$. For example,

$$(T(n) \cdot F)(\Lambda) = \sum_{(\Lambda:\Lambda')=n} F(\Lambda'),$$

and if $F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda)$, then

$$R(n) \cdot F = n^{-2k} F.$$

PROPOSITION 4.14 *If $F\colon \mathcal{L} \to \mathbb{C}$ has the property that $F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda)$ for all λ, Λ, then so also does $T(n) \cdot F$, and*

(a) $T(mn) \cdot F = T(m) \cdot T(n) \cdot F$ *if* $\gcd(m,n) = 1$;
(b) $T(p) \cdot T(p^r) \cdot F = T(p^{r+1}) \cdot F + p^{1-2k} T(p^{r-1}) \cdot F$.

PROOF. Immediate consequence of Proposition 4.11. □

Now let $f(z)$ be a modular form of weight $2k$, and let F be the associated function on $\mathcal{L}$. We define $T(n) \cdot f$ to be the function on $\mathbb{H}$ associated with $n^{2k-1} \cdot T(n) \cdot F$. Thus

$$(T(n) \cdot f)(z) = n^{2k-1}(T(n) \cdot F)(\Lambda(z,1)).$$

Theorem 4.7 in the case $N = 1$ follows easily from the Proposition. To prove Theorem 4.8 we need an explicit description of the lattices of index n in a fixed lattice.

Write $M_2(\mathbb{Z})$ for the ring of 2×2 matrices with coefficients in $\mathbb{Z}$.

LEMMA 4.15 *For any $A \in M_2(\mathbb{Z})$, there exists a $U \in M_2(\mathbb{Z})^\times$ such that*

$$UA = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad ad = n, \quad a \geq 1, \quad 0 \leq b < d.$$

Moreover, the integers a, b, d are uniquely determined.

PROOF. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and suppose $ra + sc = a'$ where $a' = \gcd(a, c)$. Then $\gcd(r, s) = 1$, and so there exist e, f such that $re + sf = 1$. Now

$$\begin{pmatrix} r & s \\ -f & e \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

and $\det\begin{pmatrix} r & s \\ -f & e \end{pmatrix} = 1$. Now apply the appropriate elementary row operations to get UA into the required form. For the uniqueness, note that multiplication by such a U doesn't change the greatest common divisor of the entries in any column, and so a is uniquely determined. Now d is uniquely determined by the equation $ad = n$, and b is obviously uniquely determined modulo d. □

For the lattice $\Lambda(z, 1)$, the sublattices of index n are exactly the lattices $\Lambda(az + b, d)$ where (a, b, d) runs through the triples in the lemma. Therefore

$$(T(n) \cdot f)(z) = n^{2k-1} \sum_{a,b,d} d^{-2k} f(\frac{az + b}{d})$$

where the sum is over the same triples. On substituting this into the q-expansion

$$f = \sum_{m \geq 1} c(m) q^m$$

one finds (after a little work) that

$$T(n) \cdot f = c(n)q + \cdots.$$

Therefore, if $T(n) \cdot f = \lambda(n) f$, then

$$\lambda(n) c(1) = c(n).$$

This proves Theorem 4.8 in the case $N = 1$.

When $N \neq 1$, the theory of the Hecke operators is much the same, only a little more complicated. For example, instead of $\mathcal{L}$, one must work with the set of pairs (Λ, S) where $\Lambda \in \mathcal{L}$ and S is a cyclic subgroup of order N in $\mathbb{C}/\Lambda$. This is no problem for the $T(n)$s with $\gcd(n, N) = 1$, but the $T(p)$s with $p|N$

have to be treated differently.[4] For example, Proposition 4.14(b) holds only for the p that don't divide N; if p divides N, then $T(p^r)\cdot F = T(p)^r \cdot F$, $r \geq 1$.

It follows from Corollary 4.9 that the problem of finding cusp forms f whose L-series have Euler product expansions becomes a problem of finding simultaneous eigenforms for the linear maps $T(n)\colon \mathcal{S}_{2k}(\Gamma_0(N)) \to \mathcal{S}_{2k}(\Gamma_0(N))$. Hecke had trouble doing this because, not having taken a good course in linear algebra, he didn't know the spectral theorem.

Linear algebra: the spectral theorem

Recall that a ***hermitian form*** on a vector space V is a mapping $\langle\ ,\ \rangle\colon V\times V \to \mathbb{C}$ such that $\langle v,w\rangle = \overline{\langle w,v\rangle}$ and $\langle\ ,\ \rangle$ is linear in one variable and conjugate-linear in the other. Such a form is said to be ***positive-definite*** if $\langle v,v\rangle > 0$ whenever $v \neq 0$. A linear map $\alpha\colon V \to V$ is ***self-adjoint*** (or ***hermitian***) relative to $\langle\ ,\ \rangle$ if

$$\langle \alpha v, w\rangle = \langle v, \alpha w\rangle, \qquad \text{all } v, w.$$

THEOREM 4.16 (SPECTRAL THEOREM) *Let V be a finite-dimensional complex vector space with a positive-definite hermitian form $\langle\ ,\ \rangle$.*

(a) *Any self-adjoint linear map $\alpha\colon V \to V$ is diagonalizable, i.e., V is a direct sum of eigenspaces for α.*
(b) *Let $\alpha_1, \alpha_2, \ldots$ be a sequence of commuting self-adjoint linear maps $V \to V$; then V has a basis of consisting of vectors that are eigenvectors for all α_i.*

PROOF. (a) Because $\mathbb{C}$ is algebraically closed, α has an eigenvector e_1. Let V_1 be $(\mathbb{C}e_1)^{\perp}$. Then V_1 is stable under α, and so contains an eigenvector e_2. Let $V_2 = (\mathbb{C}e_1 \oplus \mathbb{C}e_2)^{\perp}$ etc..

(b) Now suppose $V = \bigoplus V(\lambda_i)$ where the λ_i are the distinct eigenvalues of α_1. Because α_2 commutes with α_1, it stabilizes each $V(\lambda_i)$, and so each $V(\lambda_i)$ can be decomposed into a direct sum of eigenspaces for α_2. Continuing in this fashion, we arrive at a decomposition $V = \bigoplus V_j$ such that each α_i acts as a scalar on each V_j. Choose bases for each V_j, and take their union. □

This suggests that we should look for a hermitian form on $\mathcal{S}_{2k}(\Gamma_0(N))$ for which the $T(n)$'s are self-adjoint.

The Petersson inner product

As Poincaré pointed out,[5] the unit disk forms a model for hyperbolic geometry: if one defines a "line" to be a segment of a circle orthogonal to the circumference

[4] In the literature, the $T(p)$s with $p|N$ are sometimes denoted $U(p)$.
[5] True, but others pointed it out earlier.

of the disk, angles to be the usual angles, and distances in terms of cross-ratios, one obtains a geometry that satisfies all the axioms for Euclidean geometry except that given a point P and a line ℓ, there exist *more* than one line through P not meeting ℓ. The map $z \mapsto \frac{z-i}{z+i}$ sends the upper-half plane onto the unit disk, and, being fractional-linear, maps circles and lines to circles and lines (collectively, not separately) and preserves angles. Therefore the upper half-plane is also a model for hyperbolic geometry. The group $\mathrm{PSL}_2(\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is the group of transformations preserving distances and orientation, and therefore plays the same role as the group of orientation preserving affine transformations of the Euclidean plane. The next proposition shows that the measure $\mu(U) = \iint_U \frac{dxdy}{y^2}$ plays the same role as the measure $\iint_U dxdy$ on sets in the Euclidean plane — it is invariant under transformations in $\mathrm{PGL}_2(\mathbb{R})$.

PROPOSITION 4.17 *Define* $\mu(U) = \displaystyle\iint_U \frac{dxdy}{y^2}$*; then* $\mu(\gamma U) = \mu(U)$ *for all* $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

PROOF. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\frac{d\gamma}{dz} = \frac{1}{(cz+d)^2}, \quad \Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}.$$

The next lemma shows that

$$\gamma^*(dxdy) = \left|\frac{d\gamma}{dz}\right|^2 dxdy \quad (z = x + iy),$$

and so

$$\gamma^*\left(\frac{dxdy}{y^2}\right) = \frac{dxdy}{y^2}.$$

□

LEMMA 4.18 *For any holomorphic function* $w(z)$*, the map* $z \mapsto w(z)$ *multiplies areas by* $|w'(z)|^2$.

PROOF. Write $w(z) = u(x,y) + iv(x,y)$, so that $z \mapsto w(z)$ is the map

$$(x,y) \mapsto (u(x,y), v(x,y)),$$

whose jacobian is

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - v_x u_y.$$

On the other hand, $w'(z) = u_x + iv_x$, so that

$$|w'(z)|^2 = u_x^2 + v_x^2.$$

The Cauchy-Riemann equations state that $u_x = v_y$ and $v_x = -u_y$, and so the two expressions agree. □

If f and g are modular forms of weight $2k$ for $\Gamma_0(N)$, then

$$f(z)\cdot\overline{g(z)}y^{2k}$$

is invariant under $\mathrm{SL}_2(\mathbb{R})$, which suggests defining

$$\langle f, g\rangle = \iint_D f\bar{g}y^{2k}\frac{dxdy}{y^2}$$

for D a fundamental domain for $\Gamma_0(N)$ — the above discussion shows that (assuming the integral converges) $\langle f, g\rangle$ will be independent of the choice of D.

THEOREM 4.19 (PETERSSON) *The above integral converges provided at least one of f or g is a cusp form. It therefore defines a positive-definite hermitian form on the vector space $\mathcal{S}_{2k}(\Gamma_0(N))$ of cusp forms. The Hecke operators $T(n)$ are self-adjoint for all n relatively prime to N.*

PROOF. Fairly straightforward calculus — see Knapp, p. 280. □

On putting the theorems of Hecke and Petersson together, we find that there exists a decomposition

$$\mathcal{S}_{2k}(\Gamma_0(N)) = \bigoplus V_i$$

of $\mathcal{S}_{2k}$ into a direct sum of orthogonal subspaces V_i, each of which is a simultaneous eigenspace for all $T(n)$ with $\gcd(n, N) = 1$. The $T(p)$ for $p|N$ stabilize each V_i and commute, and so there does exist at least one f in each V_i that is also an eigenform for the $T(p)$ with $p|N$. If we scale f so that $f = q + \sum_{n\geq 2} c(n)q^n$, then

$$L(f, s) = \prod_p \frac{1}{1 - c(p)p^{-s} + p^{2k-1-2s}} \prod_{p|N} \frac{1}{1 - c_p p^{-s}}$$

where the first product is over the primes not dividing N, and the second is over those dividing N.

The operator w_N is self-adjoint for the Petersson product, and does commute with the $T(n)$s with $\gcd(n, N) = 1$, and so each V_i decomposes into orthogonal eigenspaces

$$V_i = V_i^{+1} \oplus V_i^{-1}$$

for w_N. Unfortunately, w_N doesn't commute with the $T(p)$s, $p|N$, and so the decomposition is not necessarily stable under these $T(p)$s. Thus, the results above do not imply that there is a single f that is simultaneously an eigenvector for w_N (and hence has a functional equation by 4.4) and for *all* $T(n)$ (and hence is equal to an Euler product 4.9).

New forms: the theorem of Atkin and Lehner

The problem left by the last subsection has a simple remedy. If $M|N$, then $\Gamma_0(M) \supset \Gamma_0(N)$, and so $\mathcal{S}_{2k}(\Gamma_0(M)) \subset \mathcal{S}_{2k}(\Gamma_0(N))$. Recall that the N turns up in the functional equation for $L(f, s)$, and so it is not surprising that we run into trouble when we mix fs of "level" N with fs that are really of level $M|N$, $M < N$.

The way out of the problem is to define a cusp form that is in some subspace $\mathcal{S}_{2k}(\Gamma_0(M))$, $M|N$, $M < N$, to be ***old***. The old forms form a subspace $\mathcal{S}_{2k}^{\text{old}}(\Gamma_0(N))$ of $\mathcal{S}_{2k}(\Gamma_0(N))$, and the orthogonal complement $\mathcal{S}_{2k}^{\text{new}}(\Gamma_0(N))$ is called the space of ***new forms***. It is stable under all the operators $T(n)$ and w_N, and so $\mathcal{S}_{2k}^{\text{new}}$ decomposes into a direct sum of orthogonal subspaces W_i,

$$\mathcal{S}_{2k}^{\text{new}}(\Gamma_0(N)) = \bigoplus W_i$$

each of which is a simultaneous eigenspace for all $T(n)$ with $\gcd(n, N) = 1$. Since the $T(p)$ for $p|N$ and w_N each commute with the $T(n)$ for $\gcd(n, N) = 1$, each stabilizes each W_i.

THEOREM 4.20 (ATKIN-LEHNER 1970) *The spaces W_i in the above decomposition all have dimension* 1.

It follows that each W_i is also an eigenspace for w_N and $T(p)$, $p|N$. Each W_i contains (exactly) one cusp form f whose q-expansion is of the form $q + \sum_{n \geq 2} c(n)q^n$. For this form, $L(f, s)$ has an Euler product expansion, and $\Lambda(f, s)$ satisfies a functional equation

$$\Lambda(f, s) = \varepsilon \Lambda(f, 2 - s)$$

where $\varepsilon = \pm 1$ is the eigenvalue of w_N acting on W_i. If the $c(n) \in \mathbb{Z}$, then $\Lambda(f, s)$ is a candidate for being the L-function of an elliptic curve E over $\mathbb{Q}$.

EXERCISE 4.21 Let α, β, γ be nonzero relatively prime integers such that

$$\alpha^\ell + \beta^\ell = \gamma^\ell,$$

where ℓ is a prime $\neq 2, 3$, and consider the elliptic curve

$$E : Y^2 Z = X(X - \alpha^\ell Z)(X - \gamma^\ell Z).$$

(a) Show that E has discriminant $\Delta = 16\alpha^{2\ell}\beta^{2\ell}\gamma^{2\ell}$.

(b) Show that if p does not divide $\alpha\beta\gamma$, then E has good reduction at p.

(c) Show that if p is an odd prime dividing $\alpha\beta\gamma$, then E has at worst nodal reduction at p.

(d) Show that (the minimal equation for) E has at worst nodal reduction at 2.

[After possibly re-ordering α, β, γ, we may suppose, first that γ is even, and then that $\alpha^\ell \equiv 1 \mod 4$. Make the change of variables $x = 4X$, $y = 8Y + 4X$, and verify that the resulting equation has integer coefficients.]

Note that (b),(c),(d) show that the conductor N of E divides $\prod_{p|\alpha\beta\gamma} p$, and hence is much smaller than Δ. This is enough to show that E doesn't exist, but the enthusiasts may wish to verify that $N = \prod_{p|\alpha\beta\gamma} p$. [Hint: First show that if p doesn't divide c_4, then the equation is minimal at p.]

5 Statement of the main theorems

Recall that to an elliptic curve E over $\mathbb{Q}$, we have attached an L-series $L(E,s) = \sum a_n n^{-s}$ that has coefficients $a_n \in \mathbb{Z}$, can be expressed as an Euler product, and (conjecturally) satisfies a functional equation (involving $N_{E/\mathbb{Q}}$, the conductor on E). Moreover, isogenous elliptic curves have the same L-series. We therefore have a map

$$E \mapsto L(E,s) \colon \{\text{ elliptic curves}/\mathbb{Q}\}/\sim \; \to \{\text{ Dirichlet series}\}.$$

An important theorem of Faltings (1983) shows that the map is injective: two elliptic curves are isogenous if they have the same L-function.

On the other hand, the theory of Hecke and Petersson, together with the theorem of Atkin and Lehner, shows that the subspace $\mathcal{S}_2^{\text{new}}(\Gamma_0(N)) \subset \mathcal{S}_2(\Gamma_0(N))$ of new forms decomposes into a direct sum

$$\mathcal{S}_2^{\text{new}}(\Gamma_0(N)) = \bigoplus W_i$$

of one-dimensional subspaces W_i that are simultaneous eigenspaces for all the $T(n)$'s with $\gcd(n,N) = 1$. Because they have dimension 1, each W_i is also an eigenspace for w_N and for the $T(p)$ with $p|N$. An element of one of the subspaces W_i, i.e., a simultaneous eigenforms in $\mathcal{S}_2^{\text{new}}(\Gamma_0(N))$, is traditionally called a ***newform***, and I'll adopt this terminology.

In each W_i there is exactly one form $f_i = \sum c(n)q^n$ with $c(1) = 1$ (said to be normalized). Because f_i is an eigenform for all the Hecke operators, it has an Euler product, and because it is an eigenform for w_N, it satisfies a functional equation. If the $c(n)$s are[6] in $\mathbb{Z}$, then $L(f_i, s)$ is a candidate for being the L-function of an elliptic curve over $\mathbb{Q}$.

CONJECTURE 5.1 (MODULARITY, OR TANIYAMA, OR ...) *A Dirichlet L-series $\sum c(n)n^{-s}$, $c(n) \in \mathbb{Z}$, is the L-series $L(E,s)$ of an elliptic curve over $\mathbb{Q}$ with conductor N if and only if it is the L-series $L(f,s)$ of a normalized newform for $\Gamma_0(N)$.*

The next theorem proves the "if".

THEOREM 5.2 (EICHLER-SHIMURA) *Let $f = \sum c(n)q^n$ be a normalized newform for $\Gamma_0(N)$. If all $c(n) \in \mathbb{Z}$, then there exists an elliptic curve E_f of conductor N such that $L(E_f, s) = L(f,s)$.*

[6] In the next section, we shall see that the $c(n)$s automatically lie in some finite extension of $\mathbb{Q}$, and that if they lie in $\mathbb{Q}$ then they lie in $\mathbb{Z}$

The early forms of the theorem were less precise — in particular, they predate the work of Atkin and Lehner in which newforms were defined.

The theorem of Eichler-Shimura has two parts: given f, construct the curve E_f (up to isogeny); having constructed E_f, prove that $L(E_f, s) = L(f, s)$. I'll discuss the two parts in Sections 6 and 7.

After the theorem of Eichler-Shimura, to prove Conjecture 5.1, it remains to show that every elliptic curve E arises from a modular form f — such an elliptic curve is said to be ***modular***.

In a set of problems circulated to those attending the famous 1955 Tokyo and Nikko conference[7], Taniyama asked (in somewhat vague form)[8] whether every elliptic curve was modular. In the ensuing years, this question was apparently discussed by various people, including Shimura, who however published nothing about it.

One can ask whether *every* Dirichlet L-series $L(s) = \sum a_n n^{-s}$, $a_n \in \mathbb{Z}$, equal to an Euler product (of the same type as $L(E, s)$), and satisfying a functional equation (of the same type as $L(E, s)$) must automatically be of the form $L(f, s)$. Regarding this, Weil (1967) proved something only a little weaker. Let $\chi\colon (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$, $\gcd(n, N) = 1$, be a homomorphism, and extend χ to a map $\mathbb{Z} \to \mathbb{C}$ by setting $\chi(m) = \chi(m \mod n)$ if m and n are relatively prime and $= 0$ otherwise. Define

$$L_\chi(s) = \sum \chi(n) a_n n^{-s}, \quad \Lambda_\chi(s) = \left(\frac{m}{2\pi}\right)^{-s} \Gamma(s) L_\chi(s).$$

Weil showed that if all the functions $\Lambda_\chi(s)$ satisfy a functional equation relating $\Lambda_\chi(s)$ and $\Lambda_\chi(2k - s)$ (and some other mild conditions), then $L(s) = L(f, s)$ for some cusp form f of weight $2k$ for $\Gamma_0(N)$. Weil also stated Conjecture 5.1 (as an exercise!) — this was its first appearance in print.

Weil's result showed that if $L(E, s)$ and its twists satisfy a functional equation of the correct form, then E is modular. Since the Hasse-Weil conjecture was widely believed, Weil's paper (for the first time) gave a strong reason for believing Conjecture 5.1, i.e., it made (5.1) into a conjecture rather than a question.[9] Also, for the first time it related the level N of f to the conductor of E, and so made it possible to test the conjecture numerically: list all the f's for $\Gamma_0(N)$, list all isogeny classes of elliptic curves over $\mathbb{Q}$ with conductor N, and see whether they match. A small industry grew up to do just that.

For several years, the conjecture was referred to as Weil's conjecture. Then, after Taniyama's question was rediscovered, it was called the Taniyama-Weil

[7] They included Artin, Brauer, Chevalley, Deuring, Iwasawa, Nagata, Néron, Satake, Serre, Shimura, Taniyama, Weil . . . !

[8] "Let C be an elliptic curve defined over an algebraic number field k, and $L_C(s)$ denote the L-function of C over k. The problem is to ask if it is possible to prove Hasse's conjecture [i.e., 10.2] for C . . . by finding a suitable automorphic form from which $L_C(s)$ may be obtained?"

[9] Weil was careful to make this distinction.

conjecture. Finally, after Lang adopted it as one of his pet projects[10], it became unsafe to call it anything other than the Shimura-Taniyama conjecture. More recently, authors have referred to it as the modularity conjecture (now theorem).

In a lecture in 1985, Frey suggested that the curve in Exercise 4.21, defined by a counterexample to Fermat's Last Theorem, should not be modular. This encouraged Serre to rethink some old conjectures of his, and formulate two conjectures, one of which implies that Frey's curve is indeed not modular. In 1986, Ribet proved sufficient of Serre's conjectures to be able to show that Frey's curve can't be modular. I'll discuss this work in Section 9.

Thus, at this stage (1986) it was known that Conjecture 5.1 for semistable elliptic curves over $\mathbb{Q}$ implies Fermat's Last Theorem, which inspired Wiles to attempt to prove Conjecture 5.1. After a premature announcement in 1993, Wiles proved in 1994 (with the help of R. Taylor) that all semistable elliptic curves over $\mathbb{Q}$ are modular. Recall that semistable just means that the curve doesn't have cuspidal reduction at any prime. Breuil, Conrad, Diamond, and Taylor improved the theorem so that it now says that all elliptic curves E over $\mathbb{Q}$ is modular. In other words, the map

$$f \mapsto E_f : \{f\} \to \{E \text{ over } \mathbb{Q}\}/\sim$$

is surjective. I'll discuss the strategy of Wiles's proof in Section 8.

6 How to get an elliptic curve from a cusp form

Not long after Newton and Leibniz developed calculus, mathematicians[11] discovered that they couldn't evaluate integrals of the form

$$\int \frac{dx}{\sqrt{f(x)}}$$

where $f(x) \in \mathbb{R}[x]$ is a cubic or quartic polynomial without a repeated factor. In fact, such an integral can't be evaluated in terms of elementary functions. Thus, they were forced to treat them as new functions and to study their properties. For example, Euler showed that

$$\int_a^{t_1} \frac{dx}{\sqrt{f(x)}} + \int_a^{t_2} \frac{dx}{\sqrt{f(x)}} = \int_a^{t_3} \frac{dx}{\sqrt{f(x)}}$$

[10] To the great benefit of the Xerox Co., as Weil put it — I once made some of the points in the above paragraph to Lang and received a 40 page response. The most accurate account of the history of the conjecture is to found in Serre 2001.

[11] In 1655, Wallis attempted to compute the length of an arc of an elliptic curve, and found such an integral. It is because of this connection with ellipses that such integrals were called elliptic, and it is because of their connection with elliptic integrals that elliptic curves were so-named. The literature on elliptic integrals and their history is vast.

where t_3 is a rational function of t_1, t_2. The explanation for this lies with elliptic curves.

Consider the elliptic curve $Y^2 = f(X)$ over $\mathbb{R}$, and the differential one-form $\omega = \frac{1}{y}dx + 0dy$ on $\mathbb{R}^2$. As we learn in calculus, to integrate ω over a segment of the elliptic curve, we should parametrize the curve. Assume that the segment $\gamma(a,t)$ of the elliptic curve over $[a,t]$ can be smoothly parametrized by x. Thus $x \mapsto (x, \sqrt{f(x)})$ maps the interval $[a,t]$ smoothly onto the segment $\gamma(a,t)$, and

$$\int_{\gamma(a,t)} \frac{dx}{y} = \int_a^t \frac{dx}{\sqrt{f(x)}}.$$

Hence, the elliptic integral can be regarded as an integral over a segment of an elliptic curve.

A key point, which I'll explain shortly, is that the restriction of ω to E is translation invariant, i.e., if t_Q denotes the map $P \mapsto P + Q$ on E, then $t_Q^*\omega = \omega$ (on E). Hence

$$\int_{\gamma(a,t)} \omega = \int_{\gamma(a+x(Q),t+x(Q))} \omega$$

for any $Q \in E(\mathbb{R})$ (here $x(Q)$ is the x-coordinate of Q). Now Euler's theorem becomes the statement

$$\int_{\gamma(a,t_1)} \omega + \int_{\gamma(a,t_2)} = \int_{\gamma(a,t_1)} \omega + \int_{\gamma(t_1,t_3)} \omega = \int_{\gamma(a,t_3)} \omega$$

where t_3 is determined by

$$(t_2, \sqrt{f(t_2)}) - (a, \sqrt{f(a)}) + (t_1, \sqrt{f(t_1)}) = (t_3, \sqrt{f(t_3)})$$

(difference and sum for the group structure on $E(\mathbb{R})$).

Thus the study of elliptic integrals leads to the study of elliptic curves.

Differentials on Riemann surfaces

A differential one-form on an open subset of $\mathbb{C}$ is simply an expression $\omega = fdz$, with f a meromorphic function. Given a smooth curve γ

$$t \mapsto z(t) \colon [a,b] \to \mathbb{C}, \quad [a,b] = \{t \in \mathbb{R} \mid a \le t \le b\},$$

we can form the integral

$$\int_\gamma \omega = \int_a^b f(z(t)) \cdot z'(t) \cdot dt \in \mathbb{C}.$$

Now consider a compact Riemann surface X. If ω is a differential one-form on X and (U_i, z_i) is a coordinate neighbourhood for X, then $\omega|U_i = f_i(z_i)dz_i$.

If (U_j, z_j) is a second coordinate neighbourhood, so that $z_j = w(z_i)$ on $U_i \cap U_j$, then
$$f_i(z_i)dz_i = f_j(w(z_i))w'(z_i)dz_i$$
on $U_i \cap U_j$. Thus, to give a differential one-form on X is to give differential one-forms $f_i dz_i$ on each U_i, satisfying the above equation on the overlaps. For any (real) curve $\gamma\colon I \to X$ and differential one-form ω on X, the integral $\int_\gamma \omega$ makes sense.

A differential one-form is ***holomorphic*** if it is represented on the coordinated neighbourhoods by forms $f dz$ with f holomorphic.

It is an important fact (already noted) that the holomorphic differential one-forms on a Riemann surface of genus g form a complex vector space $\Omega^1(X)$ of dimension g.

For example, the Riemann sphere S has genus 0 and so should have no nonzero holomorphic differential one-forms. Note that dz is holomorphic on $\mathbb{C} = S \smallsetminus \{\text{ north pole}\}$, but that $z = 1/z'$ on $S \smallsetminus \{\text{ poles}\}$, and so $dz = -\frac{1}{z'^2}dz'$, which has a pole at the north pole. Hence dz does not extend to a holomorphic differential one-form on the whole of S.

An elliptic curve has genus 1, and so the holomorphic differential one-forms on it form a vector space of dimension 1. It is generated by $\omega = \frac{dx}{2y}$ (more accurately, the restriction of $\frac{1}{2y}dx + 0dy$ to $E^{\text{aff}}(\mathbb{C}) \subset \mathbb{C}^2$). Here I'm assuming that E has equation
$$Y^2Z = X^3 + aXZ^2 + bZ^3, \quad \Delta \neq 0.$$
Note that, on E^{aff},
$$2ydy = (3x^2 + a)dx,$$
and so
$$\frac{dx}{2y} = \frac{dy}{3x^2 + a}$$
where both are defined. As $\Delta \neq 0$, the functions $2y$ and $3x^2 + a$ have no common zero, and so ω is holomorphic on E^{aff}. One can check that it also holomorphic at the point at infinity.

For any $Q \in E(\mathbb{C})$, the translate $t_Q^*\omega$ of ω is also holomorphic, and so $t_Q^*\omega = c\omega$ for some $c \in \mathbb{C}$. Now $Q \mapsto c\colon E(\mathbb{C}) \to \mathbb{C}$ is a holomorphic function on $\mathbb{C}$, and all such functions are constant (see III 2.2). Since the function takes the value 1 when $Q = 0$, it is 1 for all Q, and so ω is invariant under translation. Alternatively, one can simply note that the inverse image of ω under the map
$$(x, y) \mapsto (\wp(z), \wp'(z)), \quad \mathbb{C} \smallsetminus \Lambda \to E^{\text{aff}}(\mathbb{C})$$
is
$$\frac{d\wp(z)}{2\wp'(z)} = \frac{dz}{2},$$
which is clearly translation invariant on $\mathbb{C}$, because $d(z + c) = dz$.

The jacobian variety of a Riemann surface

Consider an elliptic curve over E and a nonzero holomorphic differential one-form ω. We choose a point $P_0 \in E(\mathbb{C})$ and try to define a map

$$P \mapsto \int_{P_0}^{P} \omega \colon E(\mathbb{C}) \to \mathbb{C}.$$

This is not well-defined because the value of the integral depends on the path we choose from P_0 to P — nonhomotopic paths may give different answers. However, if we choose a basis $\{\gamma_1, \gamma_2\}$ for $H_1(E(\mathbb{C}), \mathbb{Z})$ (equivalently, a basis for $\pi_1(E(\mathbb{C}), P_0)$), then the integral is well-defined modulo the lattice Λ in $\mathbb{C}$ generated by

$$\int_{\gamma_1} \omega, \quad \int_{\gamma_2} \omega.$$

In this way, we obtain an isomorphism

$$P \mapsto \int_{P_0}^{P} \omega \colon E(\mathbb{C}) \to \mathbb{C}/\Lambda.$$

Note that this construction is inverse to that in III, §3.

Jacobi and Abel made a similar construction for any compact Riemann surface X. Suppose X has genus g, and let $\omega_1, \ldots, \omega_g$ be a basis for the vector space $\Omega^1(X)$ of holomorphic one-forms on X. Choose a point $P_0 \in X$. Then there is a smallest lattice Λ in $\mathbb{C}^g$ such that the map

$$P \mapsto \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_g \right) : X \to \mathbb{C}^g/\Lambda$$

is well-defined. By a lattice in $\mathbb{C}^g$, I mean the free $\mathbb{Z}$-submodule of rank $2g$ generated by a basis for $\mathbb{C}^g$ regarded as a real vector space. The quotient $\mathbb{C}^g/\Lambda$ is a complex manifold, called the ***jacobian variety*** $\mathrm{Jac}(X)$ of X, which can be considered to be a higher-dimensional analogue of $\mathbb{C}/\Lambda$. Note that it is a commutative group.

We can make the definition of $\mathrm{Jac}(X)$ more canonical. Let $\Omega^1(X)^\vee$ be the dual of $\Omega^1(X)$ as a complex vector space. For any $\gamma \in H_1(X, \mathbb{Z})$,

$$\omega \mapsto \int_\gamma \omega$$

is an element of $\Omega^1(X)^\vee$, and in this way we obtain an injective homomorphism

$$H_1(X, \mathbb{Z}) \hookrightarrow \Omega^1(X)^\vee,$$

which (one can prove) identifies $H_1(X, \mathbb{Z})$ with a lattice in $\Omega^1(X)^\vee$. Define

$$\mathrm{Jac}(X) = \Omega^1(X)^\vee / H_1(X, \mathbb{Z}).$$

When we fix a $P_0 \in X$, any $P \in X$ defines an element

$$\omega \mapsto \int_{P_0}^{P} \omega \mod H_1(X, \mathbb{Z})$$

of $\mathrm{Jac}(X)$, and so we get a map $X \to \mathrm{Jac}(X)$. The choice of a different P_0 gives a map that differs from the first only by a translation.

Construction of the elliptic curve over $\mathbb{C}$

We apply the above theory to the Riemann surface $X_0(N)$. Let π be the map $\pi\colon \mathbb{H} \to X_0(N)$ (not quite onto). For any $\omega \in \Omega^1(X_0(N))$, $\pi^*\omega = f dz$ where $f \in \mathcal{S}_2(\Gamma_0(N))$, and the map $\omega \mapsto f$ is a bijection

$$\Omega^1(X_0(N)) \to \mathcal{S}_2(\Gamma_0(N))$$

(see 3.3). The Hecke operator $T(n)$ acts on $\mathcal{S}_2(\Gamma_0(N))$, and hence on the vector space $\Omega^1(X_0(N))$ and its dual.

PROPOSITION 6.1 *There is a canonical action of $T(n)$ on $H_1(X_0(N), \mathbb{Z})$, which is compatible with the map $H_1(X_0(N), \mathbb{Z}) \to \Omega^1(X_0(N))^\vee$. In other words, the action of $T(n)$ on $\Omega^1(X_0(N))^\vee$ stabilizes its sublattice $H_1(X_0(N), \mathbb{Z})$, and therefore induces an action on the quotient* $\mathrm{Jac}(X_0(N))$.

PROOF. One can give an explicit set of generators for $H_1(X_0(N), \mathbb{Z})$, explicitly describe an action of $T(n)$ on them, and then explicitly verify that this action is compatible with the map $H_1(X_0(N), \mathbb{Z}) \to \Omega^1(X_0(N))^\vee$. Alternatively, as we discuss in the next section, there are more geometric reasons why the $T(n)$ should act on $\mathrm{Jac}(X_0(N))$. □

REMARK 6.2 From the action of $T(n)$ on $H_1(X_0(N), \mathbb{Z}) \approx \mathbb{Z}^{2g}$ we get a characteristic polynomial $P(Y) \in \mathbb{Z}[Y]$ of degree $2g$. What is its relation to the characteristic polynomial $Q(Y) \in \mathbb{C}[Y]$ of $T(n)$ acting on $\Omega^1(X)^\vee \approx \mathbb{C}^g$? The obvious guess is that $P(Y)$ is the product of $Q(Y)$ with its complex conjugate $\overline{Q(Y)}$. The proof that this is so is an exercise in linear algebra. See the next section.

Now let $f = \sum c(n)q^n$ be a normalized newform for $\Gamma_0(N)$ with $c(n) \in \mathbb{Z}$. The map

$$\alpha \mapsto \alpha(f)\colon \Omega^1(X_0(N))^\vee \to \mathbb{C}$$

identifies $\mathbb{C}$ with the largest quotient of $\Omega^1(X_0(N))^\vee$ on which each $T(n)$ acts as multiplication by $c(n)$. The image of $H_1(X_0(N), \mathbb{Z})$ is a lattice Λ_f, and we set $E_f = \mathbb{C}/\Lambda_f$ — it is an elliptic curve over $\mathbb{C}$. Note that we have constructed maps

$$X_0(N) \to \mathrm{Jac}(X_0(N)) \to E_f.$$

The inverse image of the differential on E_f represented by dz is the differential on $X_0(N)$ represented by $f dz$.

Construction of the elliptic curve over $\mathbb{Q}$

We briefly explain why the above construction in fact gives an elliptic curve over $\mathbb{Q}$. There will be a few more details in the next section.

For a compact Riemann surface X, we defined

$$\mathrm{Jac}(X) = \Omega^1(X)^\vee / H_1(X, \mathbb{Z}) \approx \mathbb{C}^g/\Lambda, \quad g = \text{genus} X.$$

This is a complex manifold, but as in the case of an elliptic curve, it is possible to construct enough functions on it to embed it into projective space, and so realize it as a projective algebraic variety.

Now suppose X is a nonsingular projective curve over an field k. Weil showed (as part of the work mentioned on p. 159) that it is possible to attach to X a projective algebraic variety $\mathrm{Jac}(X)$ over k, which, in the case $k = \mathbb{C}$ becomes the variety defined in the last paragraph. There is again a map $X \to \mathrm{Jac}(X)$, well-defined up to translation by the choice of a point $P_0 \in X(k)$. The variety $\mathrm{Jac}(X)$ is an ***abelian variety***, i.e., not only is it projective, but it also has a group structure. (An abelian variety of dimension 1 is an elliptic curve.)

In particular, there is such a variety attached to the curve $X_0(N)$ defined in Section 2. Moreover (see the next section), the Hecke operators $T(n)$ define endomorphisms of $\mathrm{Jac}(X_0(N))$. Because it has an abelian group structure, any integer m defines an endomorphism of $\mathrm{Jac}(X_0(N))$, and we define E_f to be the largest "quotient" of $\mathrm{Jac}(X_0(N))$ on which $T(n)$ and $c(n)$ agree for all n relatively prime to N. One can prove that this operation of "passing to the quotient" commutes with change of the ground field, and so in this way we obtain an elliptic curve over $\mathbb{Q}$ that becomes equal over $\mathbb{C}$ to the curve defined in the last subsection. On composing $X_0(N) \to \mathrm{Jac}(X_0(N))$ with $\mathrm{Jac}(X_0(N)) \to E_f$ we obtain a map $X_0(N) \to E_f$. In summary:

THEOREM 6.3 *Let $f = \sum c(n)q^n$ be a newform in $\mathcal{S}_2(\Gamma_0(N))$, normalized to have $c(1) = 1$, and assume that all $c(n) \in \mathbb{Z}$. Then there exists an elliptic curve E_f and a map $\alpha\colon X_0(N) \to E_f$ with the following properties:*

(a) *α factors uniquely through $\mathrm{Jac}(X_0(N))$,*

$$X_0(N) \to \mathrm{Jac}(X_0(N)) \to E_f,$$

and the second map realizes E_f as the largest quotient of $\mathrm{Jac}(X_0(N))$ on which the endomorphisms $T(n)$ and $c(n)$ of $\mathrm{Jac}(X_0(N))$ agree.

(b) *The inverse image of an invariant differential ω on E_f under $\mathbb{H} \to X_0(N) \to E_f$ is a nonzero rational multiple of $f dz$.*

7 Why the L-Series of E_f agrees with the L-Series of f

In this section we sketch a proof of the identity of Eichler and Shimura relating the Hecke correspondence $T(p)$ to the Frobenius map, and hence the L-series of E_f to that of f.

The ring of correspondences of a curve

Let X and X' be projective nonsingular curves over a field k which, for simplicity, we take to be algebraically closed.

A ***correspondence*** T between X and X', written $T\colon X \vdash X'$, is a pair of finite surjective regular maps

$$X \overset{\alpha}{\leftarrow} Y \overset{\beta}{\rightarrow} X'.$$

It can be thought of as a many-valued map $X \to X'$ sending a point $P \in X(k)$ to the set $\{\beta(Q_i)\}$ where the Q_i run through the elements of $\alpha^{-1}(P)$ (the Q_i need not be distinct). Better, recall that $\mathrm{Div}(X)$ is the free abelian group on the set of points of X, so that an element of $\mathrm{Div}(X)$ is a finite formal sum

$$D = \sum n_P[P], \quad n_P \in \mathbb{Z}, \quad P \in X(k).$$

A correspondence T then defines a map

$$\mathrm{Div}(X) \to \mathrm{Div}(X'), \quad [P] \mapsto \sum_i [\beta(Q_i)].$$

(notations as above). This map multiplies the degree of a divisor by $\deg(\alpha)$. It therefore sends the divisors of degree zero on X into the divisors of degree zero on X', and one can show that it sends principal divisors to principal divisors. Hence it defines a map $T\colon J(X) \to J(X')$ where

$$J(X) \overset{\text{def}}{=} \mathrm{Div}^0(X)/\{\text{ principal divisors}\}.$$

We define the ***ring of correspondences*** $\mathcal{A}(X)$ on X to be the subring of $\mathrm{End}(J(X))$ generated by the maps defined by correspondences.

If T is the correspondence

$$X \overset{\beta}{\leftarrow} Y \overset{\alpha}{\rightarrow} X,$$

then the transpose T^{tr} of T is the correspondence

$$X \overset{\alpha}{\leftarrow} Y \overset{\beta}{\rightarrow} X.$$

A morphism $\alpha\colon X \to X'$ can be thought of as a correspondence

$$X \leftarrow \Gamma \rightarrow X'$$

where $\Gamma \subset X \times X'$ is the graph of α and the maps are the projections. The transpose of a morphism α is the many valued map $P \mapsto \alpha^{-1}(P)$.

REMARK 7.1 Let U and U' be the curves obtained from X and X' by removing a finite number of points. Then, it follows from the theory of algebraic curves, that a regular map $\alpha: U \to U'$ extends *uniquely* to a regular map $\bar{\alpha}: X \to X'$: take $\bar{\alpha}$ to be the regular map whose graph is the Zariski closure of the graph of α. On applying this remark twice, we see that a correspondence $U \vdash U'$ extends uniquely to a correspondence $X \vdash X'$ (cf. also I 4.18).

REMARK 7.2 Let
$$X \overset{\alpha}{\leftarrow} Y \overset{\beta}{\to} X'.$$
be a correspondence $T: X \vdash X'$. For any regular function f on X', we define $T(f)$ to be the regular function $P \mapsto \sum f(\beta Q_i)$ on X (notation as above). Similarly, T will define a homomorphism $\Omega^1(X') \to \Omega^1(X)$.

The Hecke correspondence

For $p \nmid N$, the Hecke correspondence $T(p): Y_0(N) \to Y_0(N)$ is defined to be
$$Y_0(N) \overset{\alpha}{\leftarrow} Y_0(pN) \overset{\beta}{\to} Y_0(N)$$
where α is the obvious projection map and β is the map induced by $z \mapsto pz: \mathbb{H} \to \mathbb{H}$.

On points, it has the following description. Recall that a point of $Y_0(pN)$ is represented by a pair (E, S) where E is an elliptic curve and S is a cyclic subgroup of E of order pN. Because $p \nmid N$, any such subgroup decomposes uniquely into subgroups of order N and p, $S = S_N \times S_p$. The map α sends the point represented by (E, S) to the point represented by (E, S_N), and β sends it to the point represented by $(E/S_p, S/S_p)$. Since E_p has $p+1$ cyclic subgroups, the correspondence is $1 : p+1$.

The unique extension of $T(p)$ to a correspondence $X_0(N) \to X_0(N)$ acts on $\Omega^1(X_0(N)) = \mathcal{S}_2(\Gamma_0(N))$ as the Hecke correspondence defined in Section 4. This description of $T(p)$, $p \nmid N$, makes sense, and is defined on, the curve $X_0(N)$ over $\mathbb{Q}$. Similar remarks apply to the $T(p)$ for $p|N$.

The Frobenius map

Let C be a curve defined over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. If C is defined by equations
$$\sum a_{i_0 i_1 \cdots} X_0^{i_0} X_1^{i_1} \cdots = 0,$$
then we let $C^{(p)}$ be the curve defined by the equations
$$\sum a_{i_0 i_1 \cdots}^p X_0^{i_0} X_1^{i_1} \cdots = 0,$$

and we let the ***Frobenius map*** $\varphi_p\colon C \to C^{(p)}$ send the point $(b_0 : b_1 : b_2 : \ldots)$ to $(b_0^p : b_1^p : b_2^p : \ldots)$. If C is defined over $\mathbb{F}_p$, then $C = C^{(p)}$ and φ_p is the Frobenius map defined earlier.

Recall that a nonconstant morphism $\alpha\colon C \to C'$ of curves defines an inclusion $\alpha^*\colon k(C') \hookrightarrow k(C)$ of function fields, and that the degree of α is defined to be $[k(C) : \alpha^* k(C')]$. The map α is said to be ***separable*** or ***purely inseparable*** according as $k(C)$ is a separable or purely inseparable extension of $\alpha^* k(C')$. If the separable degree of $k(C)$ over $\alpha^* k(C')$ is m, then the map $C(k^{\mathrm{al}}) \to C'(k^{\mathrm{al}})$ is $m : 1$, except over the finite set where it is ramified.

PROPOSITION 7.3 *The Frobenius map $\varphi_p\colon C \to C^{(p)}$ is purely inseparable of degree p, and any purely inseparable map $\varphi\colon C \to C'$ of degree p (of complete nonsingular curves) factors as*

$$C \xrightarrow{\varphi_p} C^{(p)} \xrightarrow{\approx} C'.$$

PROOF. For $C = \mathbb{P}^1$, this is obvious, and the general case follows because $\mathbb{F}(C)$ is a separable extension of $\mathbb{F}(T)$. See Silverman 1986, II 2.12, for the details. □

Brief review of the points of order p on elliptic curves

Let E be an elliptic curve over an algebraically closed field k. The map $p\colon E \to E$ (multiplication by p) is of degree p^2. If k has characteristic zero, then the map is separable, which implies that its kernel has order p^2. If k has characteristic p, the map is never separable: either it is purely inseparable (and so E has no points of order p) or its separable and inseparable degrees are p (and so E has p points of order dividing p). The first case occurs for only finitely many values of j.

The Eichler-Shimura relation

The curve $X_0(N)$ and the Hecke correspondence $T(p)$ are defined over $\mathbb{Q}$. For almost all primes $p \nmid N$, $X_0(N)$ will reduce to a nonsingular curve $\tilde{X}_0(N)$.[12] For such a prime p, the correspondence $T(p)$ defines a correspondence $\tilde{T}(p)$ on $\tilde{X}_0(N)$.

THEOREM 7.4 *For a prime p where $X_0(N)$ has good reduction,*

$$\tilde{T}(p) = \varphi_p + \varphi_p^{\mathrm{tr}}.$$

(Equality in the ring $\mathcal{A}(\tilde{X}_0(N))$ of correspondences on $\tilde{X}_0(N)$ over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.)

[12] In fact, it is known that $X_0(N)$ has good reduction for all primes $p \nmid N$, but this is hard to prove. It is easy to see that $X_0(N)$ does not have good reduction at primes dividing N.

PROOF. We sketch a proof that they agree as many-valued maps on an open subset of $\tilde{X}_0(N)$.

Over $\mathbb{Q}_p^{\mathrm{al}}$ we have the following description of $T(p)$ (see above): a point P on $Y_0(N)$ is represented by a homomorphism of elliptic curves $\alpha\colon E \to E'$ with cyclic kernel of order N; let $S_0, \ldots, S_p$ be the subgroups of order p in E; then $T_p(P) = \{Q_0, \ldots, Q_p\}$ where Q_i is represented by $E/S_i \to E'/\alpha(S_i)$.

Consider a point $\tilde{P}$ on $\tilde{X}_0(N)$ with coordinates in $\mathbb{F}$ — by Hensel's lemma it will lift to a point on $X_0(N)$ with coordinates in $\mathbb{Q}_p^{\mathrm{al}}$. Ignoring a finite number of points of $\tilde{X}_0(N)$, we can suppose $\tilde{P} \in \tilde{Y}_0(N)$ and hence is represented by a map $\tilde{\alpha}\colon \tilde{E} \to \tilde{E}'$ where $\alpha\colon E \to E'$ has cyclic kernel of order N. By ignoring a further finite number of points, we may suppose that $\tilde{E}$ has p points of order dividing p.

Let $\alpha\colon E \to E'$ be a lifting of $\tilde{\alpha}$ to $\mathbb{Q}_p^{\mathrm{al}}$. The reduction map $E_p(\mathbb{Q}_p^{\mathrm{al}}) \to \tilde{E}_p(\mathbb{F}_p^{\mathrm{al}})$ has a kernel of order p. Number the subgroups of order p in E so that S_0 is the kernel of this map. Then each S_i, $i \neq 0$, maps to a subgroup of order p in $\tilde{E}$.

The map $p\colon \tilde{E} \to \tilde{E}$ has factorizations

$$\tilde{E} \xrightarrow{\varphi} \tilde{E}/S_i \xrightarrow{\psi} \tilde{E}, \quad i = 0, 1, \ldots, p.$$

When $i = 0$, φ is a purely inseparable map of degree p (it is the reduction of the map $E \to E/S_0$ — it therefore has degree p and has zero kernel), and so ψ must be separable of degree p (we are assuming $\tilde{E}$ has p points of order dividing p). Proposition 7.3 shows that there is an isomorphism $\tilde{E}^{(p)} \to \tilde{E}/S_0$. Similarly $\tilde{E}'^{(p)} \approx \tilde{E}'/S_0$. Therefore Q_0 is represented by $\tilde{E}^{(p)} \to \tilde{E}'^{(p)}$, which also represents $\varphi_p(P)$.

When $i \neq 0$, φ is separable (its kernel is the reduction of S_i), and so ψ is purely inseparable. Therefore $\tilde{E} \approx \tilde{E}_i^{(p)}$, and similarly $\tilde{E}' \approx \tilde{E}_i'^{(p)}$, where $\tilde{E}_i/\tilde{E}/S_i$ and $\tilde{E}_i' = \tilde{E}'/S_i$. It follows that $\{Q_1, \ldots, Q_p\} = \varphi_p^{-1}(P) = \varphi_p^{\mathrm{tr}}(P)$. □

The zeta function of an elliptic curve revisited

Recall (III 3.21) that, for an elliptic curve $E = \mathbb{C}/\Lambda$ over $\mathbb{C}$, the degree of a nonzero endomorphism of E is the determinant of α acting on Λ. More generally (III 3.22), for an elliptic curve E over an algebraically closed field k, and ℓ be a prime not equal to the characteristic of k,

$$\deg \alpha = \det(\alpha | T_\ell E) \tag{38}$$

where $T_\ell E$ is the Tate module $T_\ell E = \varprojlim E(k)_{\ell^n}$ of E.

When Λ is a free module over some ring R and $\alpha\colon \Lambda \to \Lambda$ is R-linear, $\mathrm{Tr}(\alpha|\Lambda)$ denotes the trace (sum of diagonal terms) of the matrix of α relative to some basis for Λ — it is independent of the choice of basis.

PROPOSITION 7.5 *Let E be an elliptic curve over $\mathbb{F}_p$. Then the trace of the Frobenius endomorphism φ_p on $T_\ell E$,*

$$\mathrm{Tr}(\varphi_p|T_\ell E) = a_p \stackrel{\text{def}}{=} p + 1 - N_p.$$

PROOF. For any 2×2 matrix A, $\det(A - I_2) = \det A - \mathrm{Tr}\, A + 1$. On applying this to the matrix of φ_p acting on $T_\ell E$, and using (38), we find that

$$\deg(\varphi_p - 1) = \deg(\varphi_p) - \mathrm{Tr}(\varphi_p|T_\ell E) + 1.$$

As we noted in IV, Section 9, $\deg(\varphi_p - 1) = N_p$ and $\deg(\varphi_p) = p$. □

As we noted above, a correspondence $T: X \vdash X$ defines a map $J(X) \to J(X)$. When E is an elliptic curve, $E(k) = J(E)$, and so T acts on $E(k)$, and hence also on $T_\ell(E)$.

COROLLARY 7.6 *Let E be an elliptic curve over $\mathbb{F}_p$. Then*

$$\mathrm{Tr}(\varphi_p^{\mathrm{tr}}|T_\ell E) = \mathrm{Tr}(\varphi_p|T_\ell E).$$

PROOF. Because φ_p has degree p, $\varphi_p \circ \varphi_p^{\mathrm{tr}} = p$. Therefore, if α, β are the eigenvalues of φ_p, so that in particular $\alpha\beta = \deg \varphi = p$, then

$$\mathrm{Tr}(\varphi_p^{\mathrm{tr}}|T_\ell E) = p/\alpha + p/\beta = \beta + \alpha.$$ □

The action of the Hecke operators on $H_1(E, \mathbb{Z})$

Again, we first need an elementary result from linear algebra.

Let V be a real vector space and suppose that we are given the structure of a complex vector space on V. This means that we are given an $\mathbb{R}$-linear map $J: V \to V$ such that $J^2 = -1$. The map J extends by linearity to $V \otimes_{\mathbb{R}} \mathbb{C}$, and $V \otimes_{\mathbb{R}} \mathbb{C}$ splits as a direct sum

$$V \otimes_{\mathbb{R}} \mathbb{C} = V^+ \oplus V^-,$$

with $V^\pm$ the ± 1 eigenspaces of J.

PROPOSITION 7.7 *(a) The map*

$$V \xrightarrow{v \mapsto v \otimes 1} V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\text{project}} V^+$$

is an isomorphism of complex vector spaces.

(b) The map $v \otimes z \mapsto v \otimes \bar{z}: V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$-linear involution of $V \otimes_{\mathbb{R}} \mathbb{C}$ interchanging V^+ and V^-.

PROOF. Easy exercise. □

COROLLARY 7.8 *Let α be an endomorphism of V which is $\mathbb{C}$-linear. Write A for the matrix of α regarded as an $\mathbb{R}$-linear endomorphism of V, and A_1 for the matrix of α as a $\mathbb{C}$-linear endomorphism of V. Then*

$$A \sim A_1 \oplus \bar{A}_1.$$

(By this I mean that the matrix A is equivalent to the matrix $\begin{pmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{pmatrix}$.)

PROOF. Follows immediately from the above Proposition.[13] □

COROLLARY 7.9 *For any $p \nmid N$,*

$$\mathrm{Tr}(T(p) \,|\, H_1(X_0(N),\mathbb{Z})) = \mathrm{Tr}(T(p) \,|\, \Omega^1(X_0(N))) + \overline{\mathrm{Tr}(T(p) \,|\, \Omega^1(X_0(N)))}.$$

PROOF. To say that $H_1(X_0(N),\mathbb{Z})$ is a lattice in $\Omega^1(X_0(N))^\vee$ means that

$$H_1(X_0(N),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} = \Omega^1(X_0(N))^\vee$$

(as real vector spaces). Clearly

$$\mathrm{Tr}(T(p) \mid H_1(X_0(N),\mathbb{Z})) = \mathrm{Tr}(T(p) \mid H_1(X_0(N),\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}),$$

and so we can apply the preceding corollary. □

The proof that $c(p) = a_p$

THEOREM 7.10 *Consider an $f = \sum c(n)q^n$ and a map $X_0(N) \to E$, as in (6.3). For all $p \nmid N$,*

$$c(p) = a_p \stackrel{\text{def}}{=} p + 1 - N_p(E).$$

PROOF. We assume initially that $X_0(N)$ has genus 1. Then $X_0(N) \to E$ is an isogeny, and we can take $E = X_0(N)$. Let p be a prime not dividing N. Then E has good reduction at p, and for any $\ell \neq p$, the reduction map $T_\ell E \to T_\ell \tilde{E}$ is an isomorphism. The Eichler-Shimura relation states that

$$\tilde{T}(p) = \varphi_p + \varphi_p^{\mathrm{tr}}.$$

[13] When V has dimension 2, which is the only case we are interested in, we can identify V (as a real or complex vector space) with $\mathbb{C}$. For the map "multiplication by $\alpha = a + ib$" the statement becomes,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \sim \begin{pmatrix} a+ib & 0 \\ 0 & a-ib \end{pmatrix},$$

which is true because the two matrices are semisimple and have the same trace and determinant.

On taking traces on $T_\ell \tilde{E}$, we find (using 7.5, 7.6, 7.9) that

$$2c(p) = a_p + a_p.$$

The proof of the general case is very similar except that, at various places in the argument, an elliptic curve has to be replace either by a curve or the jacobian variety of a curve. Ultimately, one uses that $T_\ell E$ is the largest quotient of $T_\ell \operatorname{Jac}(X_0(N))$ on which $T(p)$ acts as multiplication by $c(p)$ for all $p \nmid N$ (perhaps after tensoring with $\mathbb{Q}_\ell$). $\square$

NOTES Let X be a Riemann surface. The map $[P] - [P_0] \mapsto \int_{P_0}^{P} \omega$ extends by linearity to map $\operatorname{Div}^0(X) \to \operatorname{Jac}(X)$. The famous theorem of Abel-Jacobi says that this induces an isomorphism $J(X) \to \operatorname{Jac}(X)$ (Fulton 1995, 21d). The jacobian variety $\operatorname{Jac}(X)$ of a curve X over a field k (constructed in general by Weil) has the property that $\operatorname{Jac}(X)(k) = J(X)$, at least when $J(k) \neq \emptyset$. For more on jacobian and abelian varieties over arbitrary fields, see Milne 1986a,c.

8 Wiles's proof

Somebody with an average or even good mathematical background might feel that all he ends up with after reading [. . .]'s paper is what he suspected before anyway: The proof of Fermat's Last Theorem is indeed very complicated.

M.Flach

In this section, I explain the strategy of Wiles's proof of the Taniyama conjecture for semistable elliptic curves over $\mathbb{Q}$ (i.e., curves with at worst nodal reduction).

Recall, that if S denotes the sphere, then $\pi \stackrel{\text{def}}{=} \pi_1(S \smallsetminus \{P_1, \ldots, P_s\}, O)$ is generated by loops $\gamma_1, \ldots, \gamma_s$ around each of the points $P_1, \ldots, P_s$, and that π classifies the coverings of S unramified except over $P_1, \ldots, P_s$.

Something similar is true for $\mathbb{Q}$. Let K be a finite extension of $\mathbb{Q}$, and let $\mathcal{O}_K$ be the ring of integers in K. In $\mathcal{O}_K$, the ideal $p\mathcal{O}_K$ factors into a product of powers of prime ideals: $p\mathcal{O}_K = \prod \mathfrak{p}^{e_\mathfrak{p}}$. The prime p is said to be ***unramified*** in K if no $e_\mathfrak{p} > 1$.

Now assume $K/\mathbb{Q}$ is Galois with Galois group G. Let p be prime, and choose a prime ideal $\mathfrak{p}$ dividing $p\mathcal{O}_K$ (so that $\mathfrak{p} \cap \mathbb{Z} = (p)$). Let $G(\mathfrak{p})$ be the subgroup of G of σ such that $\sigma\mathfrak{p} = \mathfrak{p}$. One shows that the action of $G(\mathfrak{p})$ on $\mathcal{O}_K/\mathfrak{p} = k(\mathfrak{p})$ defines a surjection $G(\mathfrak{p}) \to \operatorname{Gal}(k(\mathfrak{p})/\mathbb{F}_p)$ which is an isomorphism if and only if p is unramified in K. The element $F_\mathfrak{p} \in G(\mathfrak{p}) \subset G$ mapping to the Frobenius element $x \mapsto x^p$ in $\operatorname{Gal}(k(\mathfrak{p})/\mathbb{F}_p)$ is called the

Frobenius element at $\mathfrak{p}$. Thus $F_{\mathfrak{p}} \in G$ is characterised by the conditions:

$$\begin{cases} F_{\mathfrak{p}}\mathfrak{p} &= \mathfrak{p}, \\ F_{\mathfrak{p}}x &\equiv x^p \mod \mathfrak{p}, \text{ for all } x \in \mathcal{O}_K. \end{cases}$$

If $\mathfrak{p}'$ also divides $p\mathcal{O}_K$, then there exists a $\sigma \in G$ such that $\sigma\mathfrak{p} = \mathfrak{p}'$, and so $F_{\mathfrak{p}'} = \sigma F_{\mathfrak{p}} \sigma^{-1}$. Therefore, the conjugacy class of $F_{\mathfrak{p}}$ depends on p — I'll often write F_p for any one of the $F_{\mathfrak{p}}$. The analogue of π being generated by the loops γ_i is that G is generated by the $F_{\mathfrak{p}}$ (varying p).

The above discussion extends to infinite extensions. Fix a finite nonempty set S of prime numbers, and let K_S be the union of all $K \subset \mathbb{C}$ that are of finite degree over $\mathbb{Q}$ and unramified outside S — it is an infinite Galois extension of $\mathbb{Q}$. For each $p \in S$, there is an element $F_p \in \mathrm{Gal}(K_S/\mathbb{Q})$, well-defined up to conjugation, called the ***Frobenius element*** at p.

PROPOSITION 8.1 *Let E be an elliptic curve over $\mathbb{Q}$. Let ℓ be a prime, and let*

$$S = \{p \mid E \text{ has bad reduction at } p\} \cup \{\ell\}.$$

Then all points of order ℓ^n on E have coordinates in K_S, i.e., $E(K_S)_{\ell^n} = E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$ for all n.

PROOF. Let $P \in E(\mathbb{Q}^{\mathrm{al}})$ be a point of ℓ-power order, and let K be a finite Galois extension of $\mathbb{Q}$ such that $P \in E(K)$. Let H be the subgroup of $G \stackrel{\text{def}}{=} \mathrm{Gal}(K/\mathbb{Q})$ of elements fixing P. Then H is the kernel of $G \to \mathrm{Aut}(\langle P\rangle)$, and so is normal. After replacing K with K^H, we may suppose that G acts faithfully on $\langle P\rangle$. Let $p \in S$, and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ dividing (p). The reduction map $E(K)_{\ell^n} \to E(k(\mathfrak{p}))_{\ell^n}$ is injective, and so if σ lies in the kernel of $G(\mathfrak{p}) \to \mathrm{Gal}(k(\mathfrak{p})/k)$, it must fix P, and so be trivial. This shows that K is unramified at p. Since this is true for all $p \in S$, we have that $K \subset K_S$. □

EXAMPLE 8.2 The smallest field containing the coordinates of the points of order 2 on the curve $E : Y^2Z = X^3 + aXZ^2 + bZ^3$ is the splitting field of $X^3 + aX + b$. Those who know a little algebraic number theory will recognize that this field is unramified at the primes not dividing the discriminant Δ of $X^3 + aX + b$, i.e., at the primes where E has good reduction (ignoring 2).

For an elliptic curve over a field k, we define the Tate module $T_\ell E$ to be the Tate module of $E_{k^{\mathrm{al}}}$. Thus, for E over $\mathbb{Q}$ and S as in the proposition, $T_\ell E$ is the free $\mathbb{Z}_\ell$-module of rank 2 such that

$$T_\ell E/\ell^n T_\ell E = E(K_S)_{\ell^n} = E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$$

for all n. The action of G_S on the quotients defines a continuous action of G_S on $T_\ell E$, i.e., a continuous homomorphism (also referred to as a representation)

$$\rho_\ell \colon G_S \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell E) \approx \mathrm{GL}_2(\mathbb{Z}_\ell).$$

PROPOSITION 8.3 *Let E, ℓ, S be as in the previous proposition. For all $p \notin S$,*

$$\operatorname{Tr}(\rho_\ell(F_p) \mid T_\ell E) = a_p \stackrel{\text{def}}{=} p + 1 - N_p(E).$$

PROOF. Because $p \notin S$, E has good reduction to an elliptic curve E_p over $\mathbb{F}_p$, and the reduction map $P \mapsto \bar{P}$ induces an isomorphism $T_\ell E \to T_\ell E_p$. By definition F_p maps to the Frobenius element in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$, and the two have the same action on $T_\ell E$. Therefore the proposition follows from (7.5). □

DEFINITION 8.4 A continuous homomorphism $\rho\colon G_S \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ is said to be ***modular*** if $\operatorname{Tr}(\rho(F_p)) \in \mathbb{Z}$ for all $p \notin S$ and there exists a cusp form $f = \sum c(n)q^n$ in $\mathcal{S}_{2k}(\Gamma_0(N))$ for some k and N such that

$$\operatorname{Tr}(\rho(F_p)) = c(p)$$

for all $p \notin S$.

Thus, in order to prove that E is modular one must prove that $\rho_\ell\colon G_S \to \mathrm{Aut}(T_\ell E)$ is modular for some ℓ. Note that then ρ_ℓ will be modular for all ℓ.

Similarly, one says that a continuous homomorphism $\rho\colon G_S \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is modular if there exists a cusp form $f = \sum c(n)q^n$ in $\mathcal{S}_{2k}(\Gamma_0(N))$ for some k and N such that

$$\operatorname{Tr}(\rho(F_p)) \equiv c(p) \mod \ell$$

for all $p \notin S$. There is the following remarkable conjecture.

CONJECTURE 8.5 (SERRE) *Every odd irreducible representation $\rho : G_S \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is modular.*

"Odd" means that $\det \rho(c) = -1$, where c is complex conjugation. "Irreducible" means that there is no one-dimensional subspace of $\mathbb{F}_\ell^2$ stable under the action of G_S. Let $E_\ell = E(\mathbb{Q}^{\mathrm{al}})_\ell$. The Weil pairing (Silverman 1986, III.8), shows that $\bigwedge^2 E_\ell \simeq \mu_\ell$ (the group of ℓ-roots of 1 in $\mathbb{Q}^{\mathrm{al}}$). Since $c\zeta = \zeta^{-1}$, this shows that the representation of G_S on E_ℓ is odd. It need not be irreducible; for example, if E has a point of order ℓ with coordinates in $\mathbb{Q}$, then it won't be.

As we shall discuss in the next section, Serre in fact gave a recipe for defining the level N and weight $2k$ of modular form.

By the early 1990s, there as much numerical evidence supporting Serre's conjecture, but few theorems.[14] The most important of these was the following.

THEOREM 8.6 (LANGLANDS, TUNNELL) *If $\rho : G_S \to \mathrm{GL}_2(\mathbb{F}_3)$ is odd and irreducible, then it is modular.*

[14]There has been much progress on the conjecture since then.

Note that $\mathrm{GL}_2(\mathbb{F}_3)$ has order $8 \cdot 6 = 48$. The action of $\mathrm{PGL}_2(\mathbb{F}_3)$ on the projective plane over $\mathbb{F}_3$ identifies it with S_4, and so $\mathrm{GL}_2(\mathbb{F}_3)$ is a double cover $\tilde{S}_4$ of S_4.

The theorem of Langlands and Tunnell in fact concerned representations $G_S \to \mathrm{GL}_2(\mathbb{C})$. In the nineteenth century, Klein classified the finite subgroups of $\mathrm{GL}_2(\mathbb{C})$: their images in $\mathrm{PGL}_2(\mathbb{C})$ are cyclic, dihedral, A_4, S_4, or A_5. Langlands constructed candidates for the modular forms, and verified they had the correct property in the A_4 case. Tunnell verified this in the S_4 case, and, since $\mathrm{GL}_2(\mathbb{F}_3)$ embeds into $\mathrm{GL}_2(\mathbb{C})$, this verifies Serre's conjecture for $\mathbb{F}_3$.

Fix a representation $\rho_0 \colon G_S \to \mathrm{GL}_2(\mathbb{F}_\ell)$. In future, R will always denote a complete local Noetherian ring with residue field $\mathbb{F}_\ell$, for example, $\mathbb{F}_\ell$, $\mathbb{Z}_\ell$, or $\mathbb{Z}_\ell[[X]]$. Two homomorphism $\rho_1, \rho_2 \colon G_S \to \mathrm{GL}_2(R)$ will be said to be ***strictly equivalent*** if

$$\rho_1 = M\rho_2 M^{-1}, \quad M \in \mathrm{Ker}(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k)).$$

A ***deformation*** of ρ_0 is a strict equivalence class of homomorphisms $\rho \colon G_S \to \mathrm{GL}_2(R)$ whose composite with $\mathrm{GL}_2(R) \to \mathrm{GL}_2(\mathbb{F}_p)$ is ρ_0.

Let $*$ be a set of conditions on representations $\rho \colon G_S \to \mathrm{GL}(R)$. Mazur showed that, for certain $*$, there is a universal $*$-deformation of ρ_0, i.e., a ring $\tilde{R}$ and a deformation $\tilde{\rho} \colon G_S \to \mathrm{GL}_2(\tilde{R})$ satisfying $*$ such that for any other deformation $\rho \colon G_S \to \mathrm{GL}_2(R)$ satisfying $*$, there is a unique homomorphism $\tilde{R} \to R$ for which the composite $G_S \xrightarrow{\tilde{\rho}} \mathrm{GL}_2(\tilde{R}) \to \mathrm{GL}_2(R)$ is ρ.

Now assume ρ_0 is modular. Work of Hida and others show that, for certain $*$, there exists a deformation $\rho_{\mathbb{T}} \colon G_S \to \mathrm{GL}_2(\mathbb{T})$ that is universal for *modular* deformations satisfying $*$. Because $\tilde{\rho}$ is universal for *all* $*$-representations, there exists a unique homomorphism $\delta \colon \tilde{R} \to \mathbb{T}$ carrying $\tilde{\rho}$ into $\rho_{\mathbb{T}}$. It is onto, and it is injective if and only if *every* $*$-representation is modular.

It is now possible to explain Wiles's strategy. First, state conditions $*$ as strong as possible but which are satisfied by the representation of G_S on $T_\ell E$ for E a semistable elliptic curve over $\mathbb{Q}$. Fixing a modular ρ_0 we get a homomorphism $\delta \colon \tilde{R} \to \mathbb{T}$.

THEOREM 8.7 (WILES) *The homomorphism $\delta \colon \tilde{R} \to \mathbb{T}$ is an isomorphism (and so every $*$-representation lifting ρ_0 is modular).*

Now let E be an elliptic curve over $\mathbb{Q}$, and assume initially that the representation of G_S on E_3 is irreducible. By the theorem of Langlands and Tunnell, the representation $\rho_0 \colon G_S \to \mathrm{Aut}(E(K_S)_3)$ is modular, and so, by the theorem of Wiles, $\rho_3 \colon G_S \to \mathrm{Aut}(T_3 E)$ is modular, which implies that E is modular.

What if the representation of G_S on $E(K_S)_3$ is not irreducible, for example, if $E(\mathbb{Q})$ contains a point of order three? It is not hard to show that the representations of G_S on $E(K_S)_3$ and $E(K_S)_5$ can't both be reducible, because otherwise either E or a curve isogenous to E will have rational points of order

3 and 5, hence a point of order 15, which is impossible. Unfortunately, there is no Langlands-Tunnell theorem for 5. Instead, Wiles uses the following elegant argument.

He shows that, given E, there exists a semistable elliptic curve E' over $\mathbb{Q}$ such that:

(a) $E'(K_S)_3$ is irreducible;

(b) $E'(K_S)_5 \approx E(K_S)_5$ as G_S-modules.

Because of (a), the preceding argument applies to E' and shows it to be modular. Hence the representation $\rho_5\colon G_S \to \mathrm{Aut}(T_5E')$ is modular, which implies that $\rho_0\colon G_S \to \mathrm{Aut}(E'(K_S)_5) \approx \mathrm{Aut}(E(K_S)_5)$ is modular. Now, Wiles can apply his original argument with 3 replaced by 5.

9 Fermat, at last

Fix a prime number ℓ, and let E be an elliptic curve over $\mathbb{Q}$. For a prime p it is possible to decide whether or not E has good reduction at p purely by considering the action of $G = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$ on the modules $E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$, for all $n \geq 1$.

Let M be a finite abelian group, and let $\rho\colon G \to \mathrm{Aut}(M)$ be a continuous homomorphism (discrete topology on $\mathrm{Aut}(M)$). The kernel H of ρ is an open subgroup of G, and therefore its fixed field $\mathbb{Q}^{\mathrm{al}H}$ is a finite extension of $\mathbb{Q}$. We say that ρ is ***unramified*** at p if p is unramified in $\mathbb{Q}^{\mathrm{al}H}$. With this terminology, we can now state a converse to Proposition 8.1.

THEOREM 9.1 *Let ℓ be a prime. The elliptic curve E has good reduction at p if and only if the representation of G on $E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$ is unramified for all n.*

The proof makes use of the theory of Néron models.

There is a similar criterion for $p = \ell$.

THEOREM 9.2 *Let ℓ be a prime. The elliptic curve E has good reduction at ℓ if and only if the representation of G on E_{ℓ^n} is flat for all n.*

For the experts, the representation of G on $E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$ is flat if there is a finite flat group scheme H over $\mathbb{Z}_\ell$ such that $H(\mathbb{Q}_\ell^{\mathrm{al}}) \approx E(\mathbb{Q}_\ell^{\mathrm{al}})_{\ell^n}$ as G-modules. Some authors say “finite” or “crystalline” instead of flat.

These criteria show that it is possible to detect whether E has bad reduction at p, and hence whether p divides the conductor of E, from knowing how G acts on $E(\mathbb{Q}^{\mathrm{al}})_{\ell^n}$ for *all* n — it may not be possible to detect bad reduction simply by looking at $E(\mathbb{Q}^{\mathrm{al}})_\ell$ for example.

Recall that Serre conjectured that every odd irreducible representation $\rho\colon G \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is modular, i.e., that there exists an $f = \sum c(n)q^n \in \mathcal{S}_{2k}(\Gamma_0(N))$, some k and N, such that

$$\operatorname{Tr}(\rho(F_p)) = c(n) \mod \ell$$

whenever ρ is unramified at p.

CONJECTURE 9.3 (REFINED SERRE) *Every odd irreducible representation $\rho: G \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is modular for a specific k and N. For example, a prime $p \neq \ell$ divides N if and only if ρ is ramified at p, and p divides N if and only if ρ is not flat.*

THEOREM 9.4 (RIBET AND OTHERS) *If $\rho: G \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is modular, then it is possible to choose the cusp form to have the weight $2k$ and level N predicted by Serre.*

This proof is difficult.

Now let E be the curve defined in (4.21) corresponding to a solution to $X^\ell + Y^\ell = Z^\ell$, $\ell > 3$. It is not hard to verify, using nontrivial facts about elliptic curves, that the representation ρ_0 of G on $E(\mathbb{Q}^{\mathrm{al}})_\ell$ is irreducible; moreover, that it is unramified for $p \neq 2, \ell$, and that it is flat for $p = \ell$. The last statement follows from the facts that E has at worst nodal reduction at p, and if it does have bad reduction at p, then $p^\ell | \Delta$.

Now

$$\begin{aligned} E \text{ modular} &\implies \rho_0 \text{ modular} \\ &\overset{\text{Ribet}}{\implies} \rho_0 \text{ modular for a cusp form of weight } 2, \text{ level } 2. \end{aligned}$$

But $X_0(N)$ has genus 0, and so there is no such cusp form. Wiles's theorem proves that E doesn't exist.

ASIDE 9.5 Great problems are important because, like the Riemann hypothesis, they have important applications, or, like Fermat's Last Theorem, they reveal our ignorance and inspire great mathematics. Fermat's Last Theorem has certainly inspired great mathematics but it needs to be said that, even after its solution, our ignorance of the rational solutions of polynomial equations over $\mathbb{Q}$ remains almost as profound as before because the method applies only to Fermat's equation (or very similar equations). We don't even know, for example, whether it is possible for there to exist an algorithm for deciding whether a polynomial equation with coefficients in $\mathbb{Q}$ has a solution with coordinates in $\mathbb{Q}$.

NOTES Among the many works inspired by the proof of Fermat's Last Theorem, I mention only the book Mozzochi 2000, which gives an engaging eyewitness account of the events surrounding the proof, the book Diamond and Shurman 2005, which gives a much more detailed description of the modularity theorem and its background than that in this chapter, and the book of the 1995 instructional conference (Cornell et al. 1997) devoted to explaining the work of Ribet and others on Serre's conjecture and of Wiles and others on the modularity conjecture .

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